CORRESPONDENCE.

THE EXPRESSION FOR THE SPURIOUS SELECTION INTRODUCED BY AMALGAMATING A NUMBER OF EXPERIENCES.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—In the discussion which followed the reading of Mr. Elderton's paper on "Spurious Selection", the opinion was expressed (*J.I.A.*, xl, 237, 240) that, while it had been satisfactorily established that the effect in question took place when selection was exhausted, "the case still required to be investigated, as to the precise effect of mixing the data, where selection was still in operation." In a letter which appeared in the *Journal* (xl, 304), Mr. Bacon returned to the question and gave an expression (A) for the measure of selection in an amalgamation of two tables, in which selection had not yet ceased to operate. This expression may be greatly simplified by observing that the last four terms of the numerator contain a factor $E_t + E'_t$, which is common to the denominator, so that we have

$$Q_{t+1} - Q_t = \frac{E_{t+1}\Delta_t + E'_{t+1}\Delta'_t}{E_{t+1} + E'_{t+1}} + \frac{(q'_t - q_t)(E'_{t+1}E_t - E'_tE_{t+1})}{(E'_t + E_t)(E'_{t+1} + E_{t+1})}$$

$$= \overline{\Delta}_t \qquad\qquad + (q'_t - q_t)K_{xt} \quad . \quad (1)$$

that is,

$$\left\{\begin{matrix}\text{the measure of selection}\\ \text{in combined experience}\end{matrix}\right\} = \text{True Selection} \quad + \text{Spurious Selection}$$

https://doi.org/10.1017/S0020268100024537 Published online by Cambridge University Press

the "true selection" being obtained by the ordinary centre of gravity formula*; and the "spurious selection" being of the form given by Mr. Elderton.

There seems to be good reason to believe that selection is more persistent in the "new" assurances than in the "old"; so that an interesting case is that in which

$$\Delta'_t = 0, \text{ and } \Delta_t \text{ not} = 0.$$

We have then

$$Q_{t+1} - Q_t = \frac{E_{t+1}\Delta_t}{E_{t+1} + E'_{t+1}} + (q'_t - q_t)K_{xt} \quad . \quad . \quad . \qquad (2)$$

that is, the "true selection" of the amalgamation is the "new" selection reduced in the ratio $E_{t+1} : E_{t+1} + E'_{t+1}$. This reduction will therefore compensate the amount of "spurious selection" represented by the second term.

An examination of the annuitant experience seems to show that the *difference* $q'_t - q_t$ is relatively small; so that in this experience spurious selection is probably very small, if not practically non-existent.

The form of equation (1) given above suggests that it might be extended to the amalgamation of any number of experiences. It is convenient, however, to slightly alter the notation—using small letters ϵ, q, δ for the component experiences, and capital letters E, Q, Δ for the amalgamated experiences. We should then have

$$Q_t = \frac{\epsilon_t q_t + \epsilon'_t q'_t}{\epsilon_t + \epsilon'_t}; \quad Q_{t+1} = \frac{\epsilon_{t+1} q_{t+1} + \epsilon'_{t+1} q'_{t+1}}{\epsilon_{t+1} + \epsilon'_{t+1}}$$

and

$$Q_{t+1} - Q_t = \frac{\epsilon_{t+1}\delta_t + \epsilon'_{t+1}\delta'_t}{\epsilon_{t+1} + \epsilon'_{t+1}} + (q'_t - q_t) \cdot \frac{\epsilon'_{t+1}\epsilon_t - \epsilon'_t\epsilon_{t+1}}{(\epsilon_t + \epsilon'_t)(\epsilon_{t+1} + \epsilon'_{t+1})}$$

or, writing E for $\epsilon + \epsilon'$,

$$E_{t+1}\Delta_t = \epsilon_{t+1}\delta_t + \epsilon'_{t+1}\delta'_t + (q'_t - q_t) \cdot \frac{\epsilon'_{t+1}\epsilon_t - \epsilon'_t\epsilon_{t+1}}{E_t}$$

Now, add a third experience ϵ'', q'', δ'': and let Q' represent the resulting rate, then

$$Q'_{t+1} - Q'_t = \frac{E_{t+1}\Delta_t + \epsilon''_{t+1}\delta''_t}{E_{t+1} + \epsilon''_{t+1}} + (q''_t - Q_t) \cdot \frac{\epsilon''_{t+1}E_t - \epsilon''_t E_{t+1}}{(E_t + \epsilon''_t)(E_{t+1} + \epsilon''_{t+1})}$$

* *See* Elderton on "Frequency Curves and Correlation", p. 9.

https://doi.org/10.1017/S0020268100024537 Published online by Cambridge University Press

or, writing E′ for $E+\epsilon''$, *i.e.*, $\epsilon+\epsilon'+\epsilon''$, we have

$$\Delta'_t = \frac{\epsilon_{t+1}\delta_t + \epsilon'_{t+1}\delta'_t + \epsilon''_{t+1}\delta''_t}{E'_{t+1}} + (q'_t - q_t)\cdot\frac{\epsilon'_{t+1}\epsilon_t - \epsilon'_t\epsilon_{t+1}}{E_tE'_{t+1}}$$

$$+ (q''_t - Q_t)\cdot\frac{\epsilon''_{t+1}E_t - \epsilon''_tE_{t+1}}{E'_tE'_{t+1}}$$

$$= {}_1\bar{\delta}_t + (q'_t - q_t)K'_{xt} + (q''_t - Q_t)K''_x \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

${}_1\bar{\delta}_t$, representing the "true selection", is still obtained by the centre of gravity formula ; and the addition of the third experience introduces a further element of "spurious selection" of the same form as before.

It is easy to show that the last two terms of (3) may be written in the form

$$\frac{1}{E'_tE'_{t+1}}\left[q_t(\epsilon_{t+1}E'_t - \epsilon_tE'_{t+1}) + q'_t(\epsilon'_{t+1}E'_t - \epsilon'_tE'_{t+1}) + q''_t(\epsilon''_{t+1}E' - \epsilon''_tE'_{t+1})\right]$$

which is equal to

$$\frac{\Sigma(\epsilon_{t+1}q_t)}{E'_{t+1}} - \frac{\Sigma(\epsilon_tq_t)}{E'}$$

or

$${}_1\bar{q}_t - \bar{q}_t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This leads us back to, and is derivable directly from, the original expressions for Q'_{t+1} and Q'_t, namely :

$$\frac{\Sigma(\epsilon_{t+1}q_{t+1})}{E'_{t+1}} \text{ and } \frac{\Sigma(\epsilon_tq_t)}{E'_t}$$

It also suggests another form of (3), giving alternative expressions for the measures of "true" and "spurious selection", which can be written down at once :

$$\Delta'_t = \frac{\Sigma(\epsilon_t\delta_t)}{E'_t} + \left[\frac{\Sigma(\epsilon_{t+1}q_{t+1})}{E'_{t+1}} - \frac{\Sigma(\epsilon_tq_{t+1})}{E'}\right]$$

$$= \bar{\delta}_t + \left[{}_1\bar{q}_{t+1} - \bar{q}_{t+1}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The two measures of "spurious selection" given in (4) and (5) are obviously identical, when selection has ceased to operate, that is, when $q_{t+1} = q_t$. They are also symmetrical with respect to ϵ, ϵ', ϵ'' and q, q', q''; so that they can evidently be extended to the amalgamation of any number of experiences.

The point of practical interest rests in the fact that the measures of both "true" and "spurious" selection in any amalgamation of experiences can be written down by the ordinary centre of gravity formula. The "new" assurances are themselves an amalgamation of the experiences of a large number of calendar years of entry; and the select curve corresponding to the combination will, apart from the element of spurious selection introduced, occupy the centre of mean position of a series of curves weighted in proportion to the values ϵ, ϵ', ϵ''. . . , starting possibly from different points, rising with different degrees of steepness, and running into the ultimate curve at different points. The effect would apparently be to make it run up to the ultimate curve more gradually than the component curves.

A curve, whose ordinates represent the true selection of the combined table, would follow a similar course and would approach the line of no selection more gradually than the component curves; so that, in spite of the element of spurious selection introduced, it is possible that the duration of selection might appear to be shorter than it really is.

I am, Dear Sir,
Yours faithfully,

JAMES BUCHANAN.

9, St. Andrew Square,
Edinburgh,
15 *November* 1907.

https://doi.org/10.1017/S0020268100024537 Published online by Cambridge University Press