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DETERMINATENESS AND THE PASCH AXIOM

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Let E be the (nonelementary) plane Euclidean geometry without the Pasch axiom. (The Pasch axiom says that a line cutting one side of a triangle must also cut another side. A full list of axoms for E is given in [5].) E satisfies in particular the full second-order continuity axiom.

Szczerba [5] has recently shown using a Hamel basis for the reals over the rationals that there exists a model of E not satisfying the Pasch axiom. It is natural to ask whether the axiom of choice plays an essential role in the proof. It will turn out that it does.

We describe briefly Szczerba's model. Let $f: R \to R$ be a solution of the functional equation $f(x+y)=f(x)+f(y)$, where f is onto and $0<f(1)$. Let $x <^* y$ iff $f(x)<f(y)$. Then $\langle R, +, \cdot, <^* \rangle$ is a semi-ordered field. (So $\langle R, +, <^* \rangle$ is an ordered group, but it is not necessarily the case that if $0 <^* x$ and $0 <^* y$ then $0 <^* xy$.) Szczerba now verifies that the Cartesian plane over $\langle R, +, \cdot, <^* \rangle$ satisfies the axioms of E. If f is not R-linear, $<^*$ is not the usual order, and the Cartesian plane does not satisfy the Pasch axiom.

We show that all models of E are isomorphic to models of the type constructed by Szczerba. Already for the much weaker *elementary* Euclidean geometry without Pasch axiom, every model is isomorphic to a Cartesian plane over a formally real Pythagorean semi-ordered field $\langle F, +, \cdot, <^* \rangle$, where we may take $0 <^* 1$ ([6]). For the nonelementary Euclidean geometry E, much more is true. For E one can show in the usual way that $\langle F, +, \cdot \rangle$ must be isomorphic to the field of real numbers. Furthermore, the order $<^*$ must have the least upper bound property (this is just the continuity axiom for E). The order $<^*$ must also be Archimedean (that is, for any y and any x such that $0 <^* x$, there is an integer n such that $y <^* nx$) since the Pasch axiom is not used in the proof ([1, p. 154]) that continuity implies Archimedeanness. So any model of E is isomorphic to the Cartesian plane over $\langle R, +, \cdot, <^* \rangle$ where $\langle R, +, \cdot \rangle$ is the field of real numbers and $\langle R, +, <^* \rangle$ is an ordered group such that $<^*$ has the least upper bound property and $0 <^* 1$. In the usual way it can be shown that the rationals are dense in R with respect to $<^*$. For let $x <^* y$. Let n be an integer such that $1 <^* n(y-x)$, and let m be the largest integer such that $m \cdot 1/n <^* y$. Then $x <^* m/n <^* y$. On the rationals, $<^*$ agrees with $<$, the usual order.

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For any $x \in R$, let Q_x be the set of rationals r such that $r <^* x$. Q_x is bounded above with respect to the standard ordering of R. For there is certainly a rational s such that $x <^* s$. But then for any $r \in Q_x$, $r <^* x$ and hence $r < s$. Let $f(x)$ be the sup of Q_x with respect to the standard ordering.

For any y, let x be the $<^*$ $-$sup of the rationals $<y$. It is easy to verify that $y = f(x)$, and so f is onto. If $x <^* y$, there are rationals r, s with $x <^* r < s <^* y$. So $r \leq f(x)$ and $s \leq f(y)$ and $f(x) < f(y)$. Similarly if $f(x) < f(y)$, $x <^* y$.

For any x, y we have $Q_{x+y} = Q_x + Q_y$. (It is trivial that $Q_x + Q_y \subseteq Q_{x+y}$. That $Q_{x+y} Q_x \supseteq Q_y$ follows from the denseness of the rationals with respect to $<^*$.) But then $\sup(Q_{x+y}) = \sup(Q_x) + \sup(Q_y)$, so $f(x+y) = f(x) + f(y)$.

If f is R-linear, the order $<^*$ is just the usual one and the Cartesian plane over $\langle R, +, \cdot, <^* \rangle$ is the usual one and satisfies the Pasch axiom. So if E has a model in which the Pasch axiom is false, there is a solution of the functional equation $f(x+y) = f(x) + f(y)$ which is not R-linear.

Recently a new axiom for set theory has been suggested [3], the axiom of determinateness (A.D.). Very roughly, A.D. says that for a certain class of two person infinite games, one of the two players has a winning strategy. A.D. contradicts the axiom of choice. A particularly interesting consequence of A.D. is that every subset of the real line is Lebesgue measurable [4]. But it is well-known [2] that any nonlinear solution of the functional equation $f(x+y) = f(x) + f(y)$ is not measurable. So A.D. implies that the functional equation has only linear solutions, and this in turn implies that every model of E is Paschian.

References

1. K. Borsuk and W. Szmielew, *Foundations of geometry*, North-Holland, Amsterdam, 1960.
2. H. Hahn and A. Rosenthal, *Set functions*, Univ. of New Mexico, 1948.
3. J. Mycielski, *On the axiom of determinateness*, Fund. Math. **53** (1963), 205–224.
4. J. Mycielski and S. Swierczkowski, *On the Lebesgue measurability and the axiom of determinateness*, Fund. Math. **54** (1964), 67–71.
5. L. W. Szczerba, *Independence of Pasch's axiom.*, Bull. Acad. Polon. Sci. Sér. Sci. Tech. (11), **18** (1970), 659–666.
6. L. W. Szczerba and W. Szmielew, *On the Euclidean geometry without the Pasch axiom*, Bull. Acad. Pol. Sci. (8) **18** (1970), 491–498.

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