MULTIPLIER SYSTEMS FOR HILBERT'S AND SIEGEL'S MODULAR GROUPS

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Dedicated to Prof. Robert A. Rankin on the occasion of his 70th birthday

1. Introduction. The classical generalizations (already investigated in the second half of last century) of the modular group $\mathrm{SL}(2,\mathbb{Z})$ are the groups $\Gamma_K = \mathrm{SL}(2,\mathfrak{o})$ ($\mathfrak{o}$ the principal order of a totally real number field K, $[K:\mathbb{Q}]=n$), operating, originally, on a product $\mathfrak{H}$ of n upper half-planes or, for $n=2$, on the product $\mathfrak{H}_1 \times \mathfrak{H}_{-1}$ of an upper and a lower half-plane by

$$\tau = (\tau^{(1)}, \ldots, \tau^{(n)}) \mapsto L(\tau) = \left(\frac{a^{(1)}\tau^{(1)}+b^{(1)}}{c^{(1)}\tau^{(1)}+d^{(1)}}, \ldots, \frac{a^{(n)}\tau^{(n)}+b^{(n)}}{c^{(n)}\tau^{(n)}+d^{(n)}}\right) \quad \text{for} \quad L = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{1.1}$$

(where $\nu^{(j)}$, for $\nu \in K$, denotes the jth conjugate of ν), and $\Gamma_n = \mathrm{Sp}(n,\mathbb{Z})$, operating on $\mathfrak{H}_n = \{Z \mid Z = X + iY \in \mathbb{C}^{(n,n)}, {}^tZ = Z, Y > 0\}$ by

$$Z \mapsto M\langle Z\rangle = (AZ+B)(CZ+D)^{-1} \quad \text{for} \quad M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{1.2}$$

Nowadays Γ_K is called Hilbert's modular group of K and Γ_n Siegel's modular group of degree (or genus) n. For $n=1$ we have $\Gamma_{\mathbb{Q}} = \Gamma_1 = \mathrm{SL}(2,\mathbb{Z})$. The functions corresponding to modular forms and modular functions for $\mathrm{SL}(2,\mathbb{Z})$ and its subgroups are holomorphic (or meromorphic) functions with an invariance property of the form

$$f(L(\tau)) = J(L,\tau)f(\tau) \quad \text{for} \quad L \in \Gamma_K \quad \text{or} \quad f(M\langle Z\rangle) = J(M,Z)f(Z) \quad \text{for} \quad M \in \Gamma_n, \tag{1.3}$$

$J(L,\tau)$ for fixed L (or $J(M,Z)$ for fixed M) denoting a holomorphic function without zeros on $\mathfrak{H}$ (or on $\mathfrak{H}_n$). A function J, defined on $\Gamma_K \times \mathfrak{H}$ or $\Gamma_n \times \mathfrak{H}_n$, to be able to appear in (1.3) with $f \not\equiv 0$, has to satisfy certain functional equations (see below, (2.3)–(2.5) for Γ_K, (5.7)–(5.9) for Γ_n) and is called an automorphic factor (AF) then. In close analogy to the case $n=1$, mainly AFs of the following kind have been used:

$$J(L,\tau) = \nu(L)\mathcal{N}(c\tau+d)^r = \nu(L)\prod_{j=1}^{n}(c^{(j)}\tau^{(j)}+d^{(j)})^r \quad \text{for} \quad \Gamma_K, \tag{1.4}$$

$$J(M,Z) = \nu(M)\det(CZ+D)^r \quad \text{for} \quad \Gamma_n, \tag{1.5}$$

with a complex number r, the weight of J, and complex numbers $\nu(L)$, $\nu(M)$. AFs of this kind are called classical automorphic factors (CAF) in the sequel. If $r \notin \mathbb{Z}$, the values of the function ν on Γ_K (or Γ_n) depend on the branch of $(\ldots)^r$. For a fixed choice of the branch (for each $L \in \Gamma_K$ or $M \in \Gamma_n$) the functional equations for J, by (1.4), (1.5), correspond to functional equations for ν. A function ν satisfying those equations is called a multiplier system (MS) of weight r for Γ_K (or Γ_n).

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For SL(2, $\mathbb{Z}$) and its subgroups. MSs of any complex weight exist. If $n>1$, conditions, however, are different. In 1941 Maass [**10**] showed that for $K=\mathbb{Q}(\sqrt{5})$ the Hilbert modular group Γ_K has MSs of integral weights only ($r\in\mathbb{Z}$) and its theta sub-group has MSs of integral and half integral weights ($2r\in\mathbb{Z}$). In 1962, Christian [**1**] proved that the weight r of a MS for a subgroup of finite index in Γ_K or Γ_n has to be a rational number if $n>1$, in particular $r\in\mathbb{Z}$ for Γ_n itself. Two more results for the weight r of a MS are available. In [**6**], $r\in\mathbb{Z}$ was shown for $\Gamma_{\mathbb{Q}(\sqrt{2})}$ and for a certain extension of degree 2 of $\Gamma_{\mathbb{Q}(\sqrt{3})}$. In 1982, Endres [**2**] proved $2r\in\mathbb{Z}$ for the theta subgroup of Γ_n, $n>1$.

The method of proving $r\in\mathbb{Q}$ and deriving an upper bound for the denominator of the weight r of a MS in all cases mentioned above was as follows. On the subgroup Δ with $c=0$ in (1.1) (or $C=0$ in (1.2)), J does not depend on τ (or Z), as a consequence J is an abelian character on Δ. Owing to the existence of certain units if $n>1$ ([**10**, §1], [**1**, Chapter III, §3]), the commutator subgroup of Δ is of finite index in Δ; hence $J^k\equiv 1$ on Δ, $k\in\mathbb{N}$ depending on the unit used. Secondly, there are relations involving matrices $T_1, T_2, \ldots$ of finite order in Γ_K (or Γ_n) and elements $L_1, L_2, \ldots\in\Delta$. These relations, together with the functional equations for ν, imply $r\in\mathbb{Q}$ and supply a number $g\in\mathbb{N}$, depending on k, such that $gr\in\mathbb{Z}$. This method does work satisfactorily (i.e. ends with a reasonably small g) only in special cases, because a unit is needed leading to a small value for k, and fails for most subgroups, requiring the existence of suitable matrices of finite order (compare [**2**, pp. 285, 287], where a conjugate of the theta subgroup $\Gamma_{2,\theta}$ of Γ_2 has to be used, because $\Gamma_{2,\theta}$ itself does not contain a matrix of the special form necessary).

The method employed in this paper is applicable to any subgroup Γ of finite index in Γ_K (or Γ_n), $n>1$, and works as follows. Injections can be constructed of the upper half-plane $\mathfrak{H}_1$ into $\mathfrak{H}$ (or $\mathfrak{H}_n$) and associated embeddings of groups Λ, conjugate in SL(2, $\mathbb{R}$) to congruence subgroups of SL(2, $\mathbb{Z}$), into Γ such that restriction of a MS ν of weight r for Γ yields a MS $\tilde{\nu}_\Lambda$ of weight nr for Λ. While there are MSs of any complex weight for Λ, the values are not arbitrary, $\tilde{\nu}_\Lambda$ must satisfy a congruence derived by Petersson [**12**, (70)] (see (3.5) Section 3) connecting nr, the volume of the fundamental domain of Λ, and the values of $\tilde{\nu}_\Lambda$ for certain generators of Λ. From this congruence, for fixed Λ, $r\in\mathbb{Q}$ can be derived (actually, to show $r\in\mathbb{Q}$, only the cases $\Gamma=\Gamma_K$ and $\Gamma=\Gamma_n$ need to be considered, compare Theorems 3.1, 5.2). To find an upper bound for the power of a prime q dividing the denominator of r one has to select several embeddings leading to groups $\Lambda_1, \Lambda_2, \ldots$ such that from the respective set of congruences the values $\tilde{\nu}_{\Lambda_1}, \tilde{\nu}_{\Lambda_2}, \ldots$ can be eliminated to some extent ending in the result that the denominator of $q^l r$ (for an explicitly given l) is prime to q. Selecting the embeddings and the elimination process require a certain amount of elementary algebraic number theory. The method, being a reduction to $n=1$, relies heavily on the knowledge of subgroups of SL(2, $\mathbb{Z}$) and the rules for calculations with MSs for these groups. For this information the reference is Rankin's book on modular functions and forms [**13**].

The paper is organized as follows. Section 2 contains (for the Hilbert modular groups) the necessary definitions, the basic facts about MSs, and the construction of the embeddings of the groups Λ, mentioned above, into Γ_K. The main general result for Hilbert's modular groups is derived in Section 3 (Theorem 3.3). One has to distinguish two cases

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for Γ_K operating on a product of n half-planes, some of them upper half-planes, some lower half-planes, namely $\delta = 1$ (an even number of lower half-planes in the product) and $\delta = -1$ (if the number of lower half-planes is odd). For $\delta = 1$ and even n the result is $2nr \in \mathbb{Z}$. In the other cases the denominator of nr can contain only prime factors q with $(q-1) \mid (n-1)$ for odd n and with $(q-1) \mid n$ for even n; an upper bound for the exponent of q in the denominator of nr is given in Theorem 3.3. For $n = 2$ better results are proved in Section 4, e.g. $r \in \mathbb{Z}$ for $\delta = 1$ and fields K with discriminant $d_K \equiv 0, 5 \bmod (8)$ (Theorem 4.1) and, independent of δ, $2r \in \mathbb{Z}$ for all symmetric Hilbert modular groups of real quadratic number fields, $r \in \mathbb{Z}$ if $d_K \equiv 5 \bmod (8)$ (Theorem 4.2). These results have to depend on the value of the discriminant, since, for $d_K \equiv 1 \bmod (8)$, MSs of weight $\frac{1}{2}$ for the modular group and for the symmetric modular group do exist. In Section 5 the application of the method to Γ_n and its subgroups is presented.

Another result concerns the modulus of ν. Most methods for constructing modular forms work, for reasons of convergence, only if all values of ν are of modulus 1, which is generally introduced as an extra assumption ([**1**], [**9**], [**13**, (3.1.4, 11)]). While this is necessary for $n = 1$ (subgroups of $\mathrm{SL}(2, \mathbb{Z})$ of genus $p_0 > 0$ have MSs violating $|\nu| = 1$ even for $r \in \mathbb{Z}$), as an easy byproduct of the proof of the rationality of r, it is proved here (Theorems 3.2, 5.3) that for $n > 1$ the values of ν are roots of unity. This was asserted in 1977 by Grosche [**4**, Satz p. 192], but his proof is not valid, relying on his Lemma 3 [**4**, p. 191] stating that ν is an abelian character, which in general is false (see the counter-example at the end of Section 5).

In view of the experience with subgroups of $\mathrm{SL}(2, \mathbb{Z})$, it is doubtful whether MSs of weights r with $2r \notin \mathbb{Z}$ will have arithmetical applications. The knowledge that, in certain cases, such MSs do not exist, can, however, be very useful (see [**2**], where this fact for the theta subgroup $\Gamma_{n,\theta}$ of Γ_n, $n > 1$, is used to show that for $n \geq 8$ the zero divisor of the classical theta function is irreducible).

2. Hilbert's modular groups and multiplier systems. Let

(i) K be a totally real number field, $[K:\mathbb{Q}] = n$,
(ii) $\mathfrak{o}$ the ring of algebraic integers of K,
(iii) d_K the discriminant of K, $\mathfrak{d}$ the different,
(iv) (ν), for $\nu \in K$, the ideal generated by ν,
(v) $\mathfrak{H}_1$ the upper half-plane, $\mathfrak{H}_{-1}$ the lower half-plane, in $\mathbb{C}$,
(vi) E the unit matrix in $K^{(2,2)}$, $E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Hilbert's modular group for K is the group

$$\Gamma = \Gamma_K = \left\{ L \mid L = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, a, b, c, d \in \mathfrak{o}, \det L = 1 \right\} \subset \mathrm{SL}(2, K).$$

The n different injections of K into $\mathbb{R}$ map K onto the conjugates $K^{(1)}, \ldots, K^{(n)} \subset \mathbb{R}$. To each $K^{(j)}$ one assigns a complex variable $\tau^{(j)}$, the jth conjugate of $\tau = (\tau^{(1)}, \ldots, \tau^{(n)})$. The canonical isomorphisms of $K(\tau)$ onto $K^{(j)}(\tau^{(j)})$ (with $\tau \to \tau^{(j)}$), for $j = 1, \ldots, n$, map a

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rational function $R(\tau)\in K(\tau)$ onto its conjugates $R^{(j)}(\tau^{(j)})$. Calculation with elements from $K(\tau)$ always stand for simultaneous calculations with the conjugates in $K^{(j)}(\tau^{(j)})$, $1\le j\le n$. For $R(\tau)\in K(\tau)$, trace and norm are defined by

$$\mathcal{S}R(\tau)=\sum_{j=1}^{n}R^{(j)}(\tau^{(j)}),\qquad \mathcal{N}(R(\tau))=\prod_{j=1}^{n}R^{(j)}(\tau^{(j)}).$$

To each $L\in \mathrm{SL}(2, K)$, one assigns a transformation

$$\tau\mapsto L(\tau)=(a\tau+b)(c\tau+d)^{-1}\qquad \left(\text{for } L=\begin{bmatrix} a & b\\ c & d\end{bmatrix}\right), \tag{2.1}$$

i.e. a simultaneous transformation

$$\tau^{(j)}\mapsto L^{(j)}(\tau^{(j)})=(a^{(j)}\tau^{(j)}+b^{(j)})(c^{(j)}\tau^{(j)}+d^{(j)})^{-1}\qquad (1\le j\le n).$$

By (2.1), a subgroup $\Lambda\subset \mathrm{SL}(2, K)$ commensurable with Γ (i.e. $\Gamma\cap\Lambda$ has finite index in Γ and in Λ) acts as a group of analytic automorphisms on a product

$$\mathfrak{H}_{\mathrm{e}}=\mathfrak{H}_{e_1}\times\mathfrak{H}_{e_2}\times\ldots\times\mathfrak{H}_{e_n}\qquad (\mathrm{e}=(e_1,\ldots,e_n)) \tag{2.2}$$

of half-planes $\mathfrak{H}_{e_j}$, $e_j=\pm1$, $1\le j\le n$.

An *automorphic factor* (*AF*) of Λ on $\mathfrak{H}_{\mathrm{e}}$ is a mapping

$$J:\Lambda\times\mathfrak{H}_{\mathrm{e}}\to\mathbb{C}$$

such that

(2.3) $J(L,\tau)$, for fixed $L\in\Lambda$, is holomorphic without zeros on $\mathfrak{H}_{\mathrm{e}}$,

(2.4) $J(LM,\tau)=J(L,M(\tau))J(M,\tau)$ for L, $M\in\Lambda$, $\tau\in\mathfrak{H}_{\mathrm{e}}$,

(2.5) $J(-L,\tau)=J(L,\tau)$ if L, $-L\in\Lambda$, $\tau\in\mathfrak{H}_{\mathrm{e}}$.

An AF is called a *classical automorphic factor* (*CAF*) if

$$J(L,\tau)=\nu(L)\mathcal{N}(c\tau+d)^r\quad\text{for}\quad L=\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in\Lambda,\ \tau\in\mathfrak{H}_{\mathrm{e}} \tag{2.6}$$

with a complex number r, the weight of J, and complex numbers $\nu(L)$, $L\in\Lambda$, the value $\nu(L)$, for each $L\in\Lambda$, of course, depending on the choice of the branch of $\log(c^{(j)}\tau^{(j)}+d^{(j)})$ on $\mathfrak{H}_{e_j}$, $1\le j\le n$. ν is called the associated multiplier system (for the chosen branch of the logarithms).

The automorphic factor, defined in [**13**, 3, 1] for $n=1$, is the CAF, as defined here, with the additional restrictions that the weight r (k in [**13**, 3.1]) is real and $|\nu(L)|=1$ [**13**, (3.1.4, 11)]. For $n>1$, however, $r\in\mathbb{Q}$ and $|\nu(L)|=1$ can be proved from (2.3)–(2.6) (see Theorems 3.1, 2). The term CAF is used, because some other automorphic factors have found applications lately. (For a discussion of all possible automorphic factors for $n\ge2$, see [**3**]).

From (2.3), (2.4), it follows that $J(-E,\tau)=\pm1$; (2.5) is equivalent to

$$J(-E,\tau)=1,\quad\text{if}\quad -E\in\Lambda. \tag{2.7}$$

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A suitable choice of the branch of the above mentioned logarithm on $\mathfrak{H}_e$ is

$$\log(\alpha z+\beta)=\log|\alpha z+\beta|+i\arg_e(\alpha z+\beta)\quad\text{for}\quad \alpha,\beta\in\mathbb{R}, \alpha z+\beta\neq 0,$$

with

$$-\pi<\arg_1(\alpha z+\beta)\leq\pi\quad\text{for}\quad z\in\mathfrak{H}_1, -\pi\leq\arg_{-1}(\alpha z+\beta)<\pi\quad\text{for}\quad z\in\mathfrak{H}_{-1}\tag{2.8}$$

(see [**5**]). As usual, for matrices

$$M=\begin{bmatrix} * & * \\ m_1 & m_2\end{bmatrix},\qquad L=\begin{bmatrix} * & * \\ \alpha & \beta\end{bmatrix},\qquad ML=\begin{bmatrix} * & * \\ n_1 & n_2\end{bmatrix}$$

from $\mathrm{SL}(2,\mathbb{R})$ and $z\in\mathfrak{H}_e$, one puts

$$2\pi w_e(M,L)=\arg_e(m_1L(z)+m_2)+\arg_e(\alpha z+\beta)-\arg_e(n_1z+n_2)\tag{2.9}$$

(in [**12**], [**13**], $w(M,L)=w_1(M,L)$). $w(M,L)$ takes only the values -1, 0, 1 and

$$w_1(M,L)+w_{-1}(M,L)=0.\tag{2.10}$$

Using, for $r\in\mathbb{C}$, $\tau\in\mathfrak{H}_e$ and $L\in\mathrm{SL}(2,K)$, the notation

$$\mu_r(L,\tau)=\mathcal{N}(c\tau+d)^r=\exp(r\mathcal{S}\log(c\tau+d))\qquad\left(L=\begin{bmatrix} a & b\\ c & d\end{bmatrix}\right),\tag{2.11}$$

from (2.9), for L_1, $L_2\in\mathrm{SL}(2,K)$, we have (as in [**13**, (3.1.15)] in the case $n=1$)

$$\sigma_e^{(r)}(L_1,L_2)=\frac{\mu_r(L_1,L_2(\tau))\mu_r(L_2,\tau)}{\mu_r(L_1L_2,\tau)}=e^{2\pi i r\mathcal{S}w_e(L_1,L_2)}.\tag{2.12}$$

$\sigma_e^{(r)}(L_1,L_2)$ depends on L_1, L_2, r, e, but not on τ, and is 1 if $r\in\mathbb{Z}$.

A *multiplier system* (*MS*) of weight r for Λ on $\mathfrak{H}_e$ can now be defined as a mapping

$$\nu:\Lambda\to\mathbb{C}\backslash\{0\}$$

such that

(2.13) $\quad\nu(L_1L_2)=\sigma_e^{(r)}(L_1,L_2)\nu(L_1)\nu(L_2)\quad$ for $\quad L_1,L_2\in\Lambda$,

(2.14) $\quad\nu(-E)=\exp(-\pi i r\mathcal{S}e)\quad$ if $\quad -E\in\Lambda$

($\mathcal{S}e=e_1+e_2+\ldots+e_n$). Then

$$J(L,\tau)=\nu(L)\mathcal{N}(c\tau+d)^r,\quad\text{for}\quad L\in\Lambda,\tau\in\mathfrak{H}_e,$$

is a CAF of weight r for Λ on $\mathfrak{H}_e$ if and only if $\nu:\Lambda\to\mathbb{C}\backslash\{0\}$ is a MS of weight r for Λ on $\mathfrak{H}_e$.

LEMMA 2.1. *Let* Λ *be a subgroup of* $\mathrm{SL}(2,K)$ *and*

$$J(L,\tau)=\nu(L)\mu_r(L,\tau)\qquad(L\in\Lambda,\tau\in\mathfrak{H}_e)$$

a CAF of weight r *for* Λ *on* $\mathfrak{H}_e$, $S\in\mathrm{SL}(2,K)$. *Then*

$$J_S(S^{-1}LS,\tau):=\mu_r(S,\tau)\mu_r(S,S^{-1}LS(\tau))^{-1}J(L,S(\tau))\qquad(L\in\Lambda,\tau\in\mathfrak{H}_e)\tag{2.15}$$

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is a CAF of weight r for $S^{-1}\Lambda S$ *on* $\mathfrak{H}_e$,

$$J_S(S^{-1}LS, \tau) = \nu_S(S^{-1}LS)\mu_r(S^{-1}LS, \tau) \tag{2.16}$$

with

$$\nu_S(S^{-1}LS) = \sigma_e^{(r)}(L, S)\sigma_e^{(r)}(S, S^{-1}LS)^{-1}\nu(L). \tag{2.17}$$

That (2.15) defines a CAF is well known. (2.3)–(2.5) for J_S are easily checked (as in the proof of [**13**, (3.1.17)]). Expressing $\sigma_e^{(r)}(L, S)$ and $\sigma_e^{(r)}(S, S^{-1}LS)$ in (2.17) in terms of values of μ_r according to (2.12) immediately gives us (2.16). For special values of ν_S we have the following lemma.

Lemma 2.2. *Under the conditions of Lemma* 2.1,

$$\nu_S(S^{-1}LS) = \nu(L) \text{ if } L \text{ or } S^{-1}LS = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}, \qquad d \gg 0$$

($d \gg 0$ *meaning* $d^{(j)} > 0$, $1 \leq j \leq n$).

This follows from [**13**, (3.2.17, 21)] by which, for such matrices, $\sigma_e^{(r)}(L, S) = \sigma_e^{(r)}(S, S^{-1}LS) = 1$.

Lemma 2.3. *Let* Λ *be a subgroup of* $\mathrm{SL}(2, K)$, Λ_0 *a subgroup of* Λ, $[\Lambda : \Lambda_0] = h < \infty$,

$$\Lambda = \bigcup_{j=1}^{h} \Lambda_0 L_j, \qquad \Lambda_0 L_j M = \Lambda_0 L_{\theta(M,j)} \quad \textit{for} \quad M \in \Lambda \qquad (\theta(M, j) \in \{1, \ldots, h\})$$

and J_0 *a CAF of weight r for* Λ_0 *on* $\mathfrak{H}_e$. *Then*

$$J(M, \tau) = \prod_{j=1}^{h} \mu_r(L_j, \tau)\mu_r(L_j, M(\tau))^{-1} J_0(L_j M L_{\theta(M,j)}^{-1}, L_{\theta(M,j)}(\tau)),$$

for $M \in \Lambda$, $\tau \in \mathfrak{H}_e$, *is a CAF of weight hr for* Λ *on* $\mathfrak{H}_e$.

This is well known [**1**, (122)]. Using

$$\mu_r(M_1 M_2, \tau) = (*)\mu_r(M_1, M_2(\tau))\mu_r(M_2, \tau) \qquad (M_1, M_2 \in \mathrm{SL}(2, K)),$$

where $(*)$ denotes a factor which does not depend on τ (see (2.12)), we can easily check (2.6) for J. (2.3)–(2.5) follow exactly as in the proof of [**13**, (3.1.17)].

Remark 2.1. *The definition of* J_S *in Lemma* 2.1 *and of* J *in Lemma* 2.3 *is independent of the choice of the branch of* $\mu_r(L, \tau) = \mathcal{N}(c\tau + d)^r$.

This follows from the fact that another choice of the branch of $\mathcal{N}(c\tau + d)^r$ results in the multiplication of $\mu_r(L, \tau)$ by a factor which is independent of τ.

For $\xi \in \mathfrak{o}$, $\xi \neq 0$, put

$$p = |\mathcal{N}(\xi)|, \qquad \delta = \operatorname{sign} \mathcal{N}(\xi), \qquad \xi^* = \mathcal{N}(\xi)/\xi. \tag{2.18}$$

If ${}_+\sqrt{p}$ denotes the positive square root from p then

$$\mathfrak{A}(\xi) = \{\tau \mid \tau = \frac{\xi}{{}_+\sqrt{p}} z, z \in \mathfrak{H}_1\} \subset \mathfrak{H}_e, \qquad e = (\operatorname{sign} \xi^{(1)}, \ldots, \operatorname{sign} \xi^{(n)}) \tag{2.19}$$

$\left(\tau = \frac{\xi}{{}_{+}\sqrt{p}} z\right.$ meaning, of course, $\left.\tau^{(j)} = \frac{\xi^{(j)}}{{}_{+}\sqrt{p}} z, 1 \le j \le n\right)$ is an analytic subvariety of $\mathfrak{H}_e$. If

$$L_0 = \begin{bmatrix} a_0 & b_0 \, {}_{+}\sqrt{p} \\ c_0\delta \, {}_{+}\sqrt{p} & d_0 \end{bmatrix}, \qquad a_0, b_0, c_0, d_0 \in \mathbb{Z}, \qquad \det L_0 = 1 \tag{2.20}$$

then

$$L = \begin{bmatrix} a_0 & b_0\xi \\ c_0\xi^* & d_0 \end{bmatrix} \in \Gamma, \qquad \frac{\xi}{{}_{+}\sqrt{p}} L_0(z) = L\left(\frac{\xi}{{}_{+}\sqrt{p}} z\right) \quad \text{for} \quad z \in \mathfrak{H}_1 \tag{2.21}$$

and, therefore, $L(\mathfrak{A}(\xi)) = \mathfrak{A}(\xi)$. The next theorem is proved exactly as in [**5**, §5] (for $n = 2$).

THEOREM 2.1. *Let* $\xi \in \mathfrak{o}$, $\xi \neq 0$, $e = (\operatorname{sign} \xi^{(1)}, \ldots, \operatorname{sign} \xi^{(n)})$, *let* p, δ, ξ^* *be defined by* (2.18),

$$\Gamma_{\mathfrak{A}(\xi)} := \left\{L_0 \mid L_0 = \begin{bmatrix} a_0 & b_0 \, {}_{+}\sqrt{p} \\ c_0\delta \, {}_{+}\sqrt{p} & d_0 \end{bmatrix}, \qquad a_0, b_0, c_0, d_0 \in \mathbb{Z}, \qquad \det L_0 = 1\right\}.$$

$\Gamma_{\mathfrak{A}(\xi)}$ *is a group conjugate to the congruence subgroup* $\Gamma^0_{\mathbb{Q}}(p)$ *of* $\Gamma_{\mathbb{Q}} = \mathrm{SL}(2, \mathbb{Z})$ *in* $\mathrm{SL}(2, \mathbb{R})$. *If* J *is a CAF of weight* r *for* Γ *on* $\mathfrak{H}_e$ *then*

$$\tilde{J}(L_0, z) := J\left(L, \frac{\xi}{{}_{+}\sqrt{p}} z\right) \quad (L \textit{ given by } (2.21))$$

is a CAF of weight nr *for* $\Gamma_{\mathfrak{A}(\xi)}$ *on* $\mathfrak{H}_1$. *The associated MS is given by*

$$\tilde{\nu}(L_0) = \begin{cases} \nu(L) & \textit{for} \quad c_0 \neq 0 \quad \textit{or} \quad d_0 > 0, \\ \nu(L)\exp(-\pi i r \mathscr{S}(1-e)) & \textit{for} \quad c_0 = 0 \quad \textit{and} \quad d_0 < 0. \end{cases} \tag{2.22}$$

That $\Gamma_{\mathfrak{A}(\xi)}$ is conjugate in $\mathrm{SL}(2, \mathbb{R})$ to

$$\Gamma^0_{\mathbb{Q}}(p) = \left\{M \mid M = \begin{bmatrix} a & bp \\ c & d \end{bmatrix}, a, b, c, d \in \mathbb{Z}, \det M = 1\right\}$$

is trivial. (2.3)–(2.5) for $\tilde{J}$ are easily verified. (2.6) for $\tilde{J}$ follows from

$$\nu(L)\mathcal{N}\left(c_0\xi^*\left(\frac{\xi}{{}_{+}\sqrt{p}} z\right) + d_0\right)^r = \nu(L) \prod_{j=1}^{n} (c_0\delta \, {}_{+}\sqrt{p} z + d_0)^r,$$

where, because of the choice of the branch of $\log(c_0\xi^*\tau + d_0)$ for $\tau = \frac{\xi}{{}_{+}\sqrt{p}} z$ according to (2.8) on the left hand side, the principal value of $\log(c_0\delta \, {}_{+}\sqrt{p} z + d_0)$ has to be chosen for $c_0 \neq 0$ or $d_0 > 0$, which is in accordance with (2.8) for $z \in \mathfrak{H}_1$, whereas for $c_0 = 0$, $d_0 < 0$,

$$\mathcal{N}(d_0)^r = \prod_{\substack{j=1 \\ e_j = 1}}^{n} (e^{\pi i} |d_0|)^r \prod_{\substack{j=1 \\ e_j = -1}}^{n} (e^{-\pi i} |d_0|)^r = e^{-\pi i r \mathscr{S}(1-e)} e^{\pi i n r} |d_0|^{nr}$$

(with $\mathscr{S}(1-e) = 1 - e_1 + 1 - e_2 + \ldots + 1 - e_n$), which gives (2.22).

3. Weight and modulus of multiplier systems for Hilbert's modular groups.

THEOREM 3.1. *For a subgroup Λ of* SL(2, *K*), *commensurable with Hilbert's modular group Γ of a totally real number field K of degree $n>1$, acting on $\mathfrak{H}_\mathfrak{e}$, there exists a (minimal) number $\mathfrak{g}(\Lambda, \mathfrak{e}) \in \mathbb{N}$ with the following property: if J is a CAF of weight r for Λ on $\mathfrak{H}_\mathfrak{e}$ then*

$$r \in \mathbb{Q}, \qquad \mathfrak{g}(\Lambda, \mathfrak{e})r \in \mathbb{Z}$$

and if Λ_0 is a subgroup of finite index in Λ and J_0 a CAF of weight r_0 for Λ_0 on $\mathfrak{H}_\mathfrak{e}$ then

$$\mathfrak{g}(\Lambda, \mathfrak{e})[\Lambda : \Lambda_0]r_0 \in \mathbb{Z}.$$

The second part is a consequence of Lemma 2.3, stating that, from a CAF of weight r_0 for Λ_0 on $\mathfrak{H}_\mathfrak{e}$, one can construct a CAF of weight $[\Lambda:\Lambda_0]r_0$ for Λ on $\mathfrak{H}_\mathfrak{e}$. The restriction of J to $\Gamma \cap \Lambda$ is a CAF of weight r for $\Gamma \cap \Lambda$ on $\mathfrak{H}_\mathfrak{e}$. For $n>1$, $\Gamma \cap \Lambda$ has to be a congruence subgroup of Γ, so one can restrict J to a principal congruence subgroup $\Gamma(\mathfrak{a}) \subset \Gamma \cap \Lambda$ for some integral ideal $\mathfrak{a} \neq (0)$. Lemma 2.3 yields a CAF of weight hr, $h=[\Gamma:\Gamma(\mathfrak{a})]$, for Γ. The existence of $\mathfrak{g}(\Gamma, \mathfrak{e})$ has been proved by Christian [**1**, Satz 1] for $\mathfrak{e}=(1, \ldots, 1)$. In fact, the proof does not depend on the special value of $\mathfrak{e}$. The existence of $\mathfrak{g}(\Gamma, \mathfrak{e})$ can also be proof along the lines of [5], the proof for $n=2$ given there [**5**, Satz 10] does not depend on the value of $n>1$, as is shown below (3.8). Hence $\mathfrak{g}(\Gamma, \mathfrak{e})hr \in \mathbb{Z}$, q.e.d.

THEOREM 3.2. *Under the conditions of Theorem* 3.1, *the MS ν, associated with a CAF of Λ, is of modulus* 1 *(i.e. $|\nu(L)|=1$ for all $L \in \Lambda$) with roots of unity as values.*

By $\tilde{\nu}(L)=(\nu(L))^{2\mathfrak{g}(\Lambda,\mathfrak{e})}$, $L \in \Lambda$, a MS of even integral weight $\tilde{r}=2\mathfrak{g}(\Lambda, \mathfrak{e})r$ of Λ is defined, which, because of (2.13), (2.14) and

$$\sigma_\mathfrak{e}^{(\tilde{r})}(L_1, L_2)=1, \qquad \exp(-\pi i \tilde{r} \mathcal{S} e)=1 \quad \text{for} \quad \tilde{r} \in \mathbb{Z}, 2 \mid \tilde{r},$$

is an abelian character on Λ. As mentioned above, there is a principal congruence subgroup $\Gamma(\mathfrak{a}) \subset \Lambda$. The commutator subgroup of $\Gamma(\mathfrak{a})$ is of finite index in Γ (see [**8**]); hence $\tilde{\nu}(L)$, $L \in \Gamma(\mathfrak{a})$, is a root of unity, but, for $L_0 \subset \Lambda$, a suitable power, say $L_0^k \in \Gamma(\mathfrak{a})$, thus $\tilde{\nu}(L_0)^k=\tilde{\nu}(L_0^k)$ is a root of unity.

It does not, however, follow that ν is trivial on a suitable principal congruence subgroup (i.e. $\nu(L)=1$ for all $L \in \Gamma(\mathfrak{a})$), as claimed in [**4**, Korollar 2]. A counter-example can easily be constructed. Take $n=2$, d_K a prime congruent to 1 mod (8). There exists a MS ν of weight $\frac{1}{2}$, namely the multiplier system of a certain theta series for Γ on $\mathfrak{H}_{(1,-1)}$ [**7**, p. 30]. Let ε_0 be the fundamental unit of K with $\varepsilon_0^{(1)}>1$ (and $\varepsilon_0^{(2)}<0$), take $m \in \mathbb{N} \cap \mathfrak{a}$ and put

$$L_1=\begin{bmatrix} 1+m\varepsilon_0 & -m\varepsilon_0^2 \\ m & 1-m\varepsilon_0 \end{bmatrix}, \qquad L_2=\begin{bmatrix} 1-m\varepsilon_0 & -m\varepsilon_0^2 \\ m & 1+m\varepsilon_0 \end{bmatrix}.$$

Then L_1, $L_2 \in \Gamma(\mathfrak{a})$, from [**13**, (3.2.6)], $w(L_1^{(1)}, L_2^{(1)})=1$, $w(L_1^{(2)}, L_2^{(2)})=0$, and, consequently, from (2.12),

$$\sigma_{(1,-1)}^{(1/2)}(L_1, L_2)=\exp(\pi i(w_1(L_1^{(1)}, L_2^{(1)})+w_{-1}(L_1^{(2)}, L_2^{(2)})))=-1.$$

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(2.12) yields

$$\nu(L_1L_2) = -\nu(L_1)\nu(L_2).$$

At least one of the values $\nu(L_1L_2)$, $\nu(L_1)$, $\nu(L_2)$ has to be different from 1.

In order to calculate $g(\Gamma, \mathfrak{e})$ or, at least, a small multiple of $g(\Gamma, \mathfrak{e})$ for $n>1$, one proceeds as follows. For $q \in \mathbb{N}$, $q \neq 1$, put

$$\mathbb{Z}_q = \{xy^{-1} \mid x, y \in \mathbb{Z}, y \text{ prime to } q\}.$$

Then $\mathbb{Z}_0 = \mathbb{Z}$ and

$$a \equiv b \bmod \mathbb{Z}_q \qquad (a, b \in \mathbb{C})$$

means that $a-b \in \mathbb{Z}_q$, i.e. a and b differ only by a rational number which is integral for q.

Let ν be a MS of weight r for Γ on $\mathfrak{H}_{\mathfrak{e}}$. There is a $\kappa = (\kappa^{(1)}, \ldots, \kappa^{(n)})$ such that [**9**, p. 543]

$$\nu\left(\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}\right) = e^{2\pi i \mathcal{S}\kappa\alpha} \quad \text{for all} \quad \alpha \in \mathfrak{o}. \tag{3.1}$$

For L, $S \in \Gamma$, by (2.13),

$$\nu(SLS^{-1}S) = \sigma_{\mathfrak{e}}^{(r)}(SLS^{-1}, S)\nu(SLS^{-1})\nu(S),$$
$$\nu(SL) = \sigma_{\mathfrak{e}}^{(r)}(S, L)\nu(S)\nu(L).$$

If $L = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$, by [**13**, (3.2.17, 21)], both σ-factors are 1; hence

$$\nu(SLS^{-1}) = \nu(L) \quad \text{for} \quad L = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}. \tag{3.2}$$

Taking

$$S = \begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{bmatrix}, \ \varepsilon \text{ a unit in } \mathfrak{o},\ \varepsilon \neq \pm 1$$

(such a unit exists for $n>1$, k totally real), (3.2) together with (3.1) gives us

$$e^{2\pi i \mathcal{S}\kappa\varepsilon^2\alpha} = \nu(SLS^{-1}) = \nu(L) = e^{2\pi i \mathcal{S}\kappa\alpha};$$

thus

$$\mathcal{S}\kappa(\varepsilon^2 - 1)\alpha \in \mathbb{Z} \quad \text{for all} \quad \alpha \in \mathfrak{o} \tag{3.3}$$

and consequently

$$\kappa \in K, \qquad (\varepsilon^2 - 1)\kappa \in \mathfrak{d}^{-1}. \tag{3.4}$$

From Theorem 2.1, for $\xi \in \mathfrak{o}$, $\mathcal{N}(\xi) = p\delta$, $p \in \mathbb{N}$, $\delta = \pm 1$, $\operatorname{sign}\xi^{(j)} = e_j$, $1 \leq j \leq n$, we obtain a MS $\tilde{\nu}$, associated with ν, of weight nr for the group $\Gamma_{\mathfrak{A}(\xi)}$ on $\mathfrak{H}_1$, which is conjugate to $\Gamma_{\mathbb{Q}}^0(p)$ in $\mathrm{SL}(2, \mathbb{R})$. In order to use condition [**12**, (70)]

$$\sum_{h=1}^{\sigma_0} \eta_h + \sum_{m=1}^{e_0} \frac{c_m}{l_m} \equiv nr\left(p_0 - 1 + \frac{q_0}{2}\right) \bmod \mathbb{Z} \tag{3.5}$$

for the existence of $\tilde{\nu}$, we have to calculate the terms in (3.5). $\left(p_0 - 1 + \dfrac{q_0}{2}\right)$ is $\dfrac{1}{4\pi}$ times the

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volume of the fundamental domain of $\Gamma_{\mathfrak{A}(\xi)}$, which is $\frac{1}{12}$ for $p=1$ ($\Gamma_{\mathfrak{A}(\xi)}=\Gamma_{\mathbb{Q}}$), and $\frac{1}{12}(p+1)$, if p is a prime. $l_1,\ldots,l_{e_0}$ are the orders of the elliptic fixed points ($l_1=2$, $l_2=3$, $e_0=2$ for $p=1$, $l_m\in\{2,3\}$, $1\le m\le e_0$, for $p\in\mathbb{N}$), $0\le c_m\le l_m-1$, $c_m\in\mathbb{Z}$, $1\le m\le e_0$. $\exp(2\pi i\eta_h)$ are the values of $\tilde{\nu}$ for the standard generators of the parabolic subgroups of $\Gamma_{\mathfrak{A}(\xi)}$ for the cusps. For the cusp ∞ of $\Gamma_{\mathfrak{A}(\xi)}$ we have, by (2.22), (3.1),

$$e^{2\pi i\eta_1}=\tilde{\nu}\left(\begin{bmatrix}1 & {}_{+}\sqrt{p}\\ 0 & 1\end{bmatrix}\right)=\nu\left(\begin{bmatrix}1 & \xi\\ 0 & 1\end{bmatrix}\right)=e^{2\pi i\mathcal{S}\kappa\xi}.$$

If $p=1$, we have only one cusp and (3.5) is

$$\mathcal{S}\kappa\xi+\tfrac{1}{2}c_1+\tfrac{1}{3}c_2\equiv\tfrac{1}{12}nr \bmod \mathbb{Z}. \tag{3.6}$$

If p is a prime, we have another cusp at 0; using (2.22), (3.1), (3.2), we find

$$e^{2\pi i\eta_2}=\tilde{\nu}\left(\begin{bmatrix}1 & 0\\ -{}_{+}\sqrt{p} & 1\end{bmatrix}\right)=\nu\left(\begin{bmatrix}1 & 0\\ -\delta\xi^* & 1\end{bmatrix}\right)=\nu\left(\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}1 & \delta\xi^*\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\right)$$
$$=\nu\left(\begin{bmatrix}1 & \delta\xi^*\\ 0 & 1\end{bmatrix}\right)=e^{2\pi\mathcal{S}\kappa\delta\xi^*}.$$

(3.5) is

$$\mathcal{S}\kappa(\xi+\delta\xi^*)+\tfrac{1}{2}t_2+\tfrac{1}{3}t_3\equiv\tfrac{1}{12}nr(p+1) \bmod \mathbb{Z} \tag{3.7}$$

with

$$t_2=\sum_{\substack{m=1\\ l_m=2}}^{e_0}c_m,\qquad t_3=\sum_{\substack{m=1\\ l_m=3}}^{e_0}c_m.$$

Let $\tilde{g}\in\mathbb{N}$ be a multiple of 6 and of (ε^2-1) in $\mathfrak{o}$ for a unit ε of $\mathfrak{o}$, $\varepsilon\neq\pm1$ (or, better, the smallest multiple of 6 which is in the ideal generated by $\varepsilon_1^2-1,\ldots,\varepsilon_{n-1}^2-1$ for a set $\varepsilon_1,\ldots,\varepsilon_{n-1}$ of fundamental units of $\mathfrak{o}$). Take any $\xi\in\mathfrak{o}$ such that sign $\xi^{(j)}=e_j$, $1\le j\le n$, and $|\mathcal{N}(\xi)|=p$ is a prime (such ξ always exists). Then, from (3.3) and (3.7), we get

$$\tfrac{1}{12}\tilde{g}(p+1)nr\in\mathbb{Z}, \tag{3.8}$$

i.e. $r\in\mathbb{Q}$, the denominator of r divides $\frac{1}{12}\tilde{g}(p+1)n$. If there is a unit ε in $\mathfrak{o}$ with sign $\varepsilon^{(j)}=e_j$, $1\le j\le n$ (e.g. for $\mathfrak{e}=(1,\ldots,1)$ or $(-1,\ldots,-1)$) we can take $\xi=\varepsilon$ and use (3.6) instead of (3.7), obtaining

$$\tfrac{1}{12}\tilde{g}nr\in\mathbb{Z}\quad\text{(for }\xi\text{ a unit).} \tag{3.9}$$

If we take $n=2$, ε_0 the fundamental unit with $\varepsilon_0^{(1)}>1$, we have the following examples from (3.9):

$$\begin{array}{lllll}
d_K=5, & \varepsilon_0=\frac{1}{2}(1+\sqrt{5}), & \varepsilon_0^2-1=\varepsilon, & \tilde{g}=6, & r\in\mathbb{Z}\ \text{for all }\mathfrak{e},\\
d_K=8, & \varepsilon_0=1+\sqrt{2}, & \varepsilon_0^2-1=2\varepsilon, & \tilde{g}=6, & r\in\mathbb{Z}\ \text{for all }\mathfrak{e},\\
d_K=12, & \varepsilon_0=2+\sqrt{3}, & \varepsilon_0^2-1=2\sqrt{3}\varepsilon, & \tilde{g}=6, & r\in\mathbb{Z}\ \text{for }\mathfrak{e}=\pm(1,1),\\
d_K=13, & \varepsilon_0=\frac{1}{2}(3+\sqrt{13}), & \varepsilon_0^2-1=3\varepsilon, & \tilde{g}=6, & r\in\mathbb{Z}\ \text{for all }\mathfrak{e}.
\end{array} \tag{3.10}$$

For other discriminants, however, $\tilde{g}$ may be quite large, as the following examples show:

$$\begin{array}{llll} d_K = 4\,.\,6, & \varepsilon_0 = 5+2\sqrt{6}, & \varepsilon_0^2 - 1 = 4\sqrt{6}\varepsilon_0, & \tilde{g} = 24, \\ d_K = 4\,.\,14, & \varepsilon_0 = 15+4\sqrt{14}, & \varepsilon_0^2 - 1 = 8\sqrt{14}\varepsilon_0, & \tilde{g} = 2^4\,.\,3\,.\,7, \\ d_K = 4\,.\,66, & \varepsilon_0 = 65+8\sqrt{66}, & \varepsilon_0^2 - 1 = 16\sqrt{66}\varepsilon_0, & \tilde{g} = 2^5\,.\,3\,.\,11. \end{array}$$

To obtain an estimate of the denominator of r, valid for fixed n and $\mathfrak{e}$ for all totally real number fields of degree n, one has to proceed more subtly than just use (3.6) or (3.7) for a single value of ξ.

For a prime $q \in \mathbb{N}$, choose $k \in \mathbb{N}$, $k \geq 3$, and $m_0 \in \mathbb{N}$, $q \nmid m_0$, such that

$$m_0 q^k \kappa \in \mathfrak{d}^{-1}, \qquad \tfrac{1}{12} n r q^k \in \mathbb{Z}_q. \tag{3.11}$$

For $\zeta \in \mathfrak{o}$, ζ prime to q, by Dirichlet's prime number theorem we can choose a number $\xi \in \mathfrak{o}$ such that (ξ) is a prime ideal of degree 1 in $\mathfrak{o}$, sign $\xi^{(j)} = e_j$, and

$$\xi \equiv \zeta \bmod (q^k) \quad \text{in} \quad \mathfrak{o}. \tag{3.12}$$

If we take $\zeta = a \in \mathbb{N}$, $q \nmid a$, we have

$$\xi \equiv a \bmod (q^k), \qquad \xi^* \equiv a^{n-1} \bmod (q^k), \qquad p = \delta \mathcal{N}(\xi) \equiv \delta a^n \bmod (q^k) \tag{3.13}$$

and $p = \delta a^n + \hat{p} q^k$, $\hat{p} \in \mathbb{Z}$. From (3.7), (3.11), (3.13), we obtain

$$(a + \delta a^{n-1}) \mathcal{S}_K + \tfrac{1}{2} t_2 + \tfrac{1}{3} t_3 \equiv \tfrac{1}{12} n r (1 + \delta a^n) \bmod \mathbb{Z}_q. \tag{3.14}$$

If $2 \mid n$, for $a = \pm b$, $b \in \mathbb{N}$, multiplying (3.14) by 6, we get

$$\pm 6 (b + \delta b^{n-1}) \mathcal{S}_K \equiv \tfrac{1}{2} n r (1 + \delta b^n) \bmod \mathbb{Z}_q.$$

and, by adding the congruences for b and $-b$,

$$0 \equiv n r (1 + \delta b^n) \bmod \mathbb{Z}_q, \tag{3.15}$$

If $\delta = 1$, we can take $b = 1$ and obtain $2nr \in \mathbb{Z}_q$ for every prime q; hence

$$2nr \in \mathbb{Z} \quad \text{for} \quad 2 \mid n, \delta = 1. \tag{3.16}$$

If $\delta = -1$, a number $b \in \mathbb{Z}$, $q \nmid b$, can be chosen such that

$$b^n \not\equiv 1 \begin{cases} \bmod (q) & \text{for} \quad (q-1) \nmid n, \\ \bmod (q^{l+2}) & \text{for} \quad n = (q-1) q^l m, q \nmid m, q \neq 2, \\ \bmod (q^{l+3}) & \text{for} \quad n = (q-1) q^l m, q \nmid m, q = 2. \end{cases} \tag{3.17}$$

For $q \neq 2$, (3.15) with this choice of b gives

$$nr \in \mathbb{Z}_q \quad \text{for} \quad (q-1) \nmid n, \qquad n r q^{l+1} \in \mathbb{Z}_q \quad \text{for} \quad n = (q-1) q^l m, \qquad q \nmid m. \tag{3.18}$$

For $q = 2$, $p \equiv -b^n \equiv -1 \bmod (4)$ ((3.13) with $a = \pm b$, $q = 2$, $k \geq 3$). For such a prime p, however, $\Gamma_{\mathbb{Q}}^{(0)}(p)$ and, therefore $\Gamma_{\mathfrak{A}(\xi)}$, has no elliptic fixed point of order 2, $t_2 = 0$ in (3.14), and by multiplying (3.14) by 3 instead of 6 we get

$$\pm 3(b-b^{n-1})\mathcal{S}\kappa \equiv \tfrac{1}{4}nr(1-b^n) \bmod \mathbb{Z}_2,$$

$\frac{1}{2}nr(1-b^n)\in\mathbb{Z}_2$ instead of (3.15), which, with b from (3.17), gives

$$nr2^{l+1}\in\mathbb{Z}_2 \quad \text{for} \quad n=(q-1)q^l m, \qquad q \nmid m, \qquad q=2. \tag{3.19}$$

If $2\nmid n$, for $a=\delta$, b, $b\in\mathbb{N}$, multiplying (3.14) by 6, we get

$$12\delta\mathcal{S}\kappa \equiv nr, \qquad 6(b+\delta b^{n-1})\mathcal{S}\kappa \equiv \tfrac{1}{2}nr(1+\delta b^n) \bmod \mathbb{Z}_q,$$

and from these congruences ($\frac{1}{2}(b+\delta b^{n-1})\in\mathbb{Z}$!)

$$0\equiv \tfrac{1}{2}nr(b^{n-1}-1)(b-\delta) \bmod \mathbb{Z}_q. \tag{3.20}$$

$b\in\mathbb{Z}$, $q\nmid b$, can be chosen such that

$$b^{n-1}\not\equiv 1\begin{cases}\bmod(q) & \text{for} \quad (q-1)\nmid(n-1),\\ \bmod(q^{l+2}) & \text{for} \quad n-1=(q-1)q^l m,\ q\nmid m,\ q\neq 2,\\ \bmod(q^{l+3}) & \text{for} \quad n-1=(q-1)q^l m,\ q\nmid m,\ q=2.\end{cases} \tag{3.21}$$

As $n-1$ is even, b can be chosen such $b\not\equiv\delta$ $(=\pm 1)\bmod(q)$ if $q\neq 2$, and $b\not\equiv\delta \bmod (4)$ if $q=2$. For $q\neq 2$, (3.20) with this choice of b gives

$$nr\in\mathbb{Z}_q \text{ for } (q-1)\nmid(n-1), \quad nrq^{l+1}\in\mathbb{Z}_q \text{ for } n-1=(q-1)q^l m,\ q\nmid n,\ q\neq 2. \tag{3.22}$$

For $q=2$, we get

$$nr2^{l+2}\in\mathbb{Z}_2 \quad \text{for} \quad n-1=(q-1)q^l m, \quad q\nmid m, \quad q=2. \tag{3.23}$$

Collecting our results (3.16, 18, 19, 22, 23) we obtain the following theorem.

THEOREM 3.3. *Let v be a MS of weight r for Hilbert's modular group Γ of a totally real number field K of degree $n>1$ on $\mathfrak{H}_e$, $e=(e_1,\dots,e_n)$, $\delta=e_1\dots e_n$.*

(a) *If $2\mid n$, $\delta=1$, then $2nr\in\mathbb{Z}$.*

(b) *If $2\mid n$, $\delta=-1$, the denominator of nr has only prime factors q with $(q-1)\mid n$. We have*

$$nr\prod_{(q-1)\mid n} q^{l(q)+1}\in\mathbb{Z} \text{ with } n=(q-1)q^{l(q)}m_q,\ q\nmid m_q, \text{ for } q \text{ prime}.$$

(c) *If $2\nmid n$, the denominator of nr has only prime factors q with $(q-1)\mid(n-1)$. We have*

$$2nr\prod_{(q-1)\mid(n-1)} q^{l(q)+1}\in\mathbb{Z} \text{ with } n-1=(q-1)q^{l(q)}m_q,\ q\nmid m_q, \text{ for } q \text{ prime}.$$

For special values of n, from Theorem 3.3 we have:

(3.24) if $n=2$ then $4r\in\mathbb{Z}$ for $\delta=1$, $2^3\,.\,3r\in\mathbb{Z}$ for $\delta=-1$;

(3.25) if $n=3$ then $2^3\,.\,3^2r\in\mathbb{Z}$;

(3.26) if $n=4$ then $2^3r\in\mathbb{Z}$ for $\delta=1$, $2^5\,.\,3\,.\,5r\in\mathbb{Z}$ for $\delta=-1$.

For fixed n, by a more detailed investigation, depending on the value of n, improvements of the general results of Theorem 3.3 are possible (see Section 4 for $n=2$).

Our method can easily be directly applied to subgroups of Γ, yielding better results than by simply multiplying $g(\Gamma, \mathfrak{e})$ by the index of the subgroup in Γ as in Theorem 3.1. An example is given by the next theorem.

THEOREM 3.4 (Maass [**10**]). *Put* $K=\mathbb{Q}(\sqrt{5})$. *The Hilbert modular group* Γ_K *has MSs of integral weight only, the MSs of the theta subgroup*

$$\Gamma_{K,\theta}=\left\{L \mid L=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_K,\ a \equiv d \equiv 0 \text{ or } b \equiv c \equiv 0 \bmod (2)\right\}$$

are of integral and half-integral weight.

The assertion for Γ is the example $d_K=5$ in (3.10). Put $\xi=\pm 1$ for $\mathfrak{e}=\pm(1,1)$, $\xi=\pm\varepsilon_0$ for $\mathfrak{e}=\pm(1,-1)$ $(\varepsilon_0=\frac{1}{2}(1+\sqrt{5}),\ \varepsilon_0^{(1)}>1,\ \varepsilon_0^{(2)}<0)$. Then $\Gamma_{\mathfrak{A}(\xi)}=\Gamma_{\mathbb{Q}}$ (Theorem 2.1), the restriction of $\Gamma_{K,\theta}$ to $\mathfrak{A}(\xi)$ yields the theta-subgroup $\Gamma_{\mathbb{Q},\theta}$ of $\Gamma_{\mathbb{Q}}=\mathrm{SL}(2, \mathbb{Z})$. The volume of the fundamental domain of $\Gamma_{\mathbb{Q},\theta}$ is π, the right-hand side of (3.5) is $\frac{1}{2}r$. $\Gamma_{\mathbb{Q},\theta}$ has two cusps and one elliptic fixed point of order 2. Multiplying (3.5) by 4, we get

$$4\eta_1+4\eta_2 \equiv 2r \bmod \mathbb{Z}. \tag{3.27}$$

$\exp(2\pi i\eta_1)$ and $\exp(2\pi i\eta_2)$ are values of $\tilde{\nu}$, the MS associated with the MS ν of $\Gamma_{K,\theta}$, for parabolic matrices $L_0 \in \Gamma_{\mathbb{Q},\theta}$ which are conjugate to a matrix of the form $\begin{bmatrix} 1 & N \\ 0 & 1 \end{bmatrix}$ in $\Gamma_{\mathbb{Q}}$; hence $d_0>0$ if $c_0=0$ in the notation of Theorem 2.1, and (by Theorem 2.1, Lemma 2.2)

$$\tilde{\nu}(L_0)=\nu(L)=\nu\left(S\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}S^{-1}\right)=\nu_S\left(\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}\right)=e^{2\pi i \mathcal{S}_{K_S}\alpha} \tag{3.28}$$

for suitable $\alpha \in \mathfrak{o}$, $S \in \Gamma_K$, with $L \in \Gamma_{K,\theta}$. We have

$$\varepsilon_0^3 \equiv 1 \bmod (2), \qquad (\varepsilon_0^3)^2-1=4\varepsilon_0^3. \tag{3.29}$$

As the principal congruence subgroup $\Gamma_K(2)=\{L \mid L \in \Gamma_K,\ L \equiv E \bmod (2)\} \subset \Gamma_{K,\theta}$ is a normal subgroup of Γ_K,

$$\begin{bmatrix} \varepsilon_0^3 & 0 \\ 0 & \varepsilon_0^{-3} \end{bmatrix}, \quad \begin{bmatrix} 1 & \varepsilon_0^{-3}\alpha-\alpha \\ 0 & 1 \end{bmatrix} \in \Gamma_K(2) \subset S^{-1}\Gamma_{K,\theta}S, \quad \begin{bmatrix} 1 & \varepsilon_0^{-3}\alpha \\ 0 & 1 \end{bmatrix} \in S^{-1}\Gamma_{K,\theta}S, \tag{3.30}$$

(α from (3.28)). By the same reasoning as in (3.3) with $\varepsilon=\varepsilon_0^3$,

$$4\mathcal{S}_{K_S}\varepsilon_0^3\beta \in \mathbb{Z} \quad \text{for all } \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix} \in S^{-1}\Gamma_{K,\theta}S.$$

Taking $\beta=\varepsilon_0^{-3}\alpha$ (3.30), we obtain $4\mathcal{S}_{K_S}\alpha \in \mathbb{Z}$ and, from (3.28), (3.27),

$$0 \equiv 2r \bmod \mathbb{Z}.$$

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4. Multiplier systems for Hilbert's modular groups of real quadratic fields. For $n=2$, let D be the square-free kernel of the discriminant d_K. Then $K=\mathbb{Q}(\sqrt{D})$ and ξ^* (2.18) is given by

$$\xi^*=u_1-u_2\sqrt{D} \quad \text{for} \quad \xi=u_1+u_2\sqrt{D}\in K \qquad (u_1, u_2\in\mathbb{Q}). \tag{4.1}$$

The result of Theorem 3.3 for $n=2$ (3.24) can be improved.

First the factor 3 for $\delta=-1$ in (3.24) can be removed. If $D\equiv 1 \bmod(3)$, choose $\zeta=\pm\sqrt{D}$ in (3.12). For

$$\xi\equiv\pm\sqrt{D} \bmod(3^k) \text{ in } \mathfrak{o},$$

we have

$$\xi-\xi^*\equiv\pm 2\sqrt{D} \bmod(3^k), \qquad p=-\mathcal{N}(\xi)\equiv D \bmod(3^k), \qquad p\equiv 1 \bmod(3)$$

instead of (3.13), and, multiplying (3.7) by 6, we get

$$\pm 6\mathcal{S}(\kappa 2\sqrt{D})\equiv r(1+D) \bmod \mathbb{Z}_3 \text{ and hence } \quad 0\equiv 2r(1+D) \bmod \mathbb{Z}_3,$$

i.e. $r\in\mathbb{Z}_3$, as $3\nmid 2(1+D)$.

If $D\equiv 2 \bmod(3)$, choose

$$\xi\equiv\pm b\sqrt{D} \bmod(3^k), \qquad b\in\mathbb{Z}, \qquad b^2D\not\equiv 8 \bmod(9), \qquad 3\nmid b.$$

Then

$$\xi-\xi^*\equiv\pm 2b\sqrt{D} \bmod(3^k), \qquad p=-\mathcal{N}(\xi)\equiv b^2D \bmod(3^k), \qquad p\equiv 2 \bmod(3).$$

For $p\equiv 2 \bmod(3)$, $\Gamma^0_{\mathbb{Q}}(p)$ has no elliptic fixed points of order 3, for otherwise there would be a matrix

$$\begin{bmatrix} x_1 & * \\ * & x_2 \end{bmatrix}\in\Gamma^0_{\mathbb{Q}}(p) \quad \text{with} \quad x_1x_2\equiv 1 \bmod(p), \qquad x_1+x_2=\pm 1,$$

which would result in a solution for

$$x^2\pm x+1\equiv 0 \bmod(p) \quad \text{or} \quad y^2\pm 2y+4\equiv 0 \bmod(p) \quad \text{for} \quad y=2x, p\neq 2.$$

But, for $p=2$, there is no solution, and for $p\neq 2$, $(y\pm 1)^2+3\equiv 0 \bmod(p)$ would imply $\left(\dfrac{-3}{p}\right)=1$, which is impossible for $p\equiv 2 \bmod(3)$. Thus $t_3=0$ in (3.7) and, multiplying (3.7) by 2, we obtain

$$\pm 2\mathcal{S}(\kappa 2b\sqrt{D})\equiv\tfrac{1}{3}r(1+b^2D) \bmod \mathbb{Z}_3 \quad \text{and hence} \quad r\tfrac{2}{3}(1+b^2D)\equiv 0 \bmod \mathbb{Z}_3,$$

i.e. $r\in\mathbb{Z}_3$, since $3^2\nmid 2(1+b^2D)$.

If $D\equiv 0 \bmod(3)$, choose

$$\xi\equiv\pm b(1+\sqrt{D}) \bmod(3^k), \quad b\in\mathbb{Z}, \qquad b^2(D-1)\not\equiv 8 \bmod(9), \quad 3\nmid b.$$

Then

$$\xi-\xi^*\equiv\pm 2b\sqrt{D} \bmod(3^k), \qquad p=-\mathcal{N}(\xi)\equiv b^2(D-1) \bmod(3^k), \qquad p\equiv 2 \bmod(3).$$

From here we proceed exactly as in the case $D\equiv 2 \bmod(3)$ and get $r\in\mathbb{Z}_3$. Thus we have

$$r\in\mathbb{Z}_3 \quad \text{for} \quad n=2, \delta=-1. \tag{4.2}$$

Next, a smaller power of 2 can be taken in (3.24). If $\delta = 1$, in (3.7) we have (by (4.1))

$$\mathcal{S}\kappa(\xi + \delta\xi^*) = \mathcal{S}\kappa(\xi + \xi^*) = \mathcal{S}\kappa\mathcal{S}\xi. \tag{4.3}$$

If $D \equiv 3 \bmod(4)$, choose

$$\xi \equiv \sqrt{D} \bmod(2^k), \quad \text{then} \quad \mathcal{S}\xi \equiv 0 \bmod(2^k), \qquad p = \mathcal{N}(\xi) \equiv -D \bmod(2^k).$$

From (3.7), multiplying by 6, we have

$$0 \equiv r(1-D) \bmod \mathbb{Z}_2, \qquad 1 - D \equiv 2 \bmod(4)$$

and hence $2r \in \mathbb{Z}_2$.

If $D \equiv 2 \bmod(4)$, choose

$$\xi \equiv 1 + \sqrt{D} \bmod(2^k), \quad \text{then} \quad \mathcal{S}\xi \equiv 2 \bmod(2^k), \qquad p = \mathcal{N}(\xi) \equiv 1 - D \bmod(2^k)$$

and $p \equiv 3 \bmod(4)$ $(k \geq 3)$. There are no elliptic fixed points of order 2, $t_2 = 0$ in (3.7) and, by multiplying (3.7) by 3, we get

$$6\mathcal{S}\kappa \equiv \tfrac{1}{2}r(2-D) \bmod \mathbb{Z}_2. \tag{4.4}$$

By (3.24), $4r \in \mathbb{Z}$. Thus, multiplying (4.4) by 2 yields

$$12\mathcal{S}\kappa \equiv r(2-D) \equiv 0 \bmod \mathbb{Z}_2. \tag{4.5}$$

For $\delta = 1$, $n = 2$, we have $\mathfrak{e} = (1, 1)$ or $(-1, -1)$ and can, therefore, put $\xi = 1$ or -1, resulting in $p = 1$. Multiplying (3.6) by 6, we obtain

$$r \equiv 6\mathcal{S}\kappa \quad \text{or} \quad r \equiv -6\mathcal{S}\kappa \bmod \mathbb{Z}_2. \tag{4.6}$$

Applying (4.5), we find $2r \in \mathbb{Z}_2$. Using this result in (4.4), we get $6\mathcal{S}\kappa \equiv 0 \bmod \mathbb{Z}_2$ which in turn from (4.6) gives $r \in \mathbb{Z}_2$.

If $D \equiv 5 \bmod(8)$, choose

$$\xi \equiv \tfrac{1}{2}(b + \sqrt{D}) \bmod(2^k), \quad b \in \mathbb{Z}, \quad 2 \nmid b, \qquad b^2 - D \equiv -4 \bmod(32).$$

Then

$$\mathcal{S}\xi \equiv b \bmod(2^k), \qquad p = \mathcal{N}(\xi) \equiv \tfrac{1}{4}(b^2 - D) \bmod(2^k), \quad p \equiv -1 \bmod(4).$$

There are no elliptic fixed points of order 2, $t_2 = 0$ in (3.7), thus, multiplying (3.7) by 3 and using $4r \in \mathbb{Z}$ (3.24), we get

$$3b\mathcal{S}\kappa \equiv \tfrac{1}{2}r(1 + \tfrac{1}{4}(b^2 - D)) \equiv 0 \bmod \mathbb{Z}_2, \qquad \mathcal{S}\kappa \equiv 0 \bmod \mathbb{Z}_2.$$

Putting $\xi = 1$ or -1, as in the case $D \equiv 2 \bmod(4)$, from (3.6) we obtain

$$r \equiv \pm 6\mathcal{S}\kappa \equiv 0 \bmod \mathbb{Z}_2, \quad r \in \mathbb{Z}_2.$$

If $D \equiv 1 \bmod(8)$, numbers from $\mathfrak{o}$ with odd trace are not prime to 2 and numbers prime to 2 with trace not divisible by 4 lead to $p \equiv 3 \bmod(8)$; so the procedure used for $D \equiv 5 \bmod(8)$ does not work here. Thus we have

$$r \in \mathbb{Z} \text{ for } D \equiv 2 \bmod(4), \qquad D \equiv 5 \bmod(8), \quad 2r \in \mathbb{Z} \text{ for } D \equiv 3 \bmod(4). \tag{4.7}$$

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If $\delta=-1$, in (3.7) we have (by (4.1))

$$\mathcal{S}_K(\xi+\delta\xi^*)=\mathcal{S}_K(\xi-\xi^*)=2u_2\mathcal{S}_K\sqrt{D}\quad \text{for}\quad \xi=u_1+u_2\sqrt{D}.$$

If $D\equiv 3 \bmod(4)$, choose $b\in\mathbb{Z}$, $2\mid b$, such that $D-b^2\equiv 3\bmod(8)$ and

$$\xi\equiv\pm(b+\sqrt{D})\bmod(2^k),\quad \text{then}\quad p=-\mathcal{N}(\xi)\equiv D-b^2\bmod(2^k),\qquad p\equiv 3\bmod(4).$$

There are no elliptic fixed points of order 2, $t_2=0$ in (3.7) and thus, multiplying (3.7) by 3, we obtain

$$\pm 6\mathcal{S}_K\sqrt{D}\equiv\tfrac{1}{2}r(1+D-b^2)\bmod\mathbb{Z}_2,\qquad 0\equiv r(1+D-b^2)\bmod\mathbb{Z}_2,$$

and from this, because of $D-b^2\equiv 3\bmod(8)$, $4r\in\mathbb{Z}_2$.

If $D\equiv 2\bmod(4)$, choose

$$\xi\equiv\pm(1+\sqrt{D})\bmod(2^k),\quad \text{then}\quad p=-\mathcal{N}(\xi)\equiv D-1\bmod(2^k).$$

Multiplying (3.7) by 6, we get

$$\pm 12\mathcal{S}_K\sqrt{D}\equiv rD\bmod\mathbb{Z}_2,\qquad 0\equiv 2rD\bmod\mathbb{Z}_2,\quad 4r\in\mathbb{Z}_2.$$

If $D\equiv 1\bmod(4)$, choose

$$\xi\equiv\pm\sqrt{D}\bmod(2^k),\quad \text{then}\quad p=-\mathcal{N}(\xi)\equiv D\bmod(2^k).$$

Multiplying (3.7) by 6, we find

$$\pm 12\mathcal{S}_K\sqrt{D}\equiv r(D+1)\bmod\mathbb{Z}_2,\qquad 0\equiv 2r(D+1)\bmod\mathbb{Z}_2,\quad 4r\in\mathbb{Z}_2.$$

If $D\equiv 5\bmod(8)$, put

$$\xi\equiv\tfrac{1}{2}(1+\sqrt{D})\bmod(2^k),\quad \text{then}\quad p=-\mathcal{N}(\xi)\equiv\tfrac{1}{4}(D-1)\bmod(2^k).$$

Multiplying (3.7) by 12 and using $4r\in\mathbb{Z}_2$, as just shown for $D\equiv 1\bmod(4)$, we obtain

$$12\mathcal{S}_K\sqrt{D}\equiv 2r(1+\tfrac{1}{4}(D-1))\equiv 0\bmod\mathbb{Z}_2.$$

For $\xi\equiv\sqrt{D}$, we now get

$$r(D+1)\equiv 12\mathcal{S}_K\sqrt{D}\equiv 0\bmod\mathbb{Z}_2,\quad 2r\in\mathbb{Z}_2.$$

Thus we have

$$2r\in\mathbb{Z}\quad\text{for}\quad D\equiv 5\bmod(8)\quad\text{and}\quad\delta=-1,\quad 4r\in\mathbb{Z}\quad\text{for}\quad\delta=-1. \tag{4.8}$$

Collecting our results (4.7) and (4.8) and noting that $D\equiv 2\bmod(4)$ is equivalent to $d_K\equiv 0\bmod(8)$ and $D\equiv 3\bmod(4)$ is equivalent to $d_K\equiv 4\bmod(8)$ we obtain the next theorem.

THEOREM 4.1. *Let v be a MS of weight r for Hilbert's modular group Γ of a real quadratic field K on $\mathfrak{H}_e$, $e=(e_1,e_2)$, $\delta=e_1e_2$. Then $4r\in\mathbb{Z}$.*

For special values of the discriminant d_K we have:

(a) *if $\delta=1$, $d_K\equiv 0, 5\bmod(8)$ then $r\in\mathbb{Z}$;*

(b) *if $\delta=1$, $d_K\equiv 4\bmod(8)$ then $2r\in\mathbb{Z}$;*

(c) *if $\delta=-1$, $d_K\equiv 5\bmod(8)$ then $2r\in\mathbb{Z}$.*

For the symmetric modular group $\hat{\Gamma}_e$ of a real quadratic number field K an even better result can be proved. $\hat{\Gamma}_e$ is an extension of degree 2 of Γ (for a detailed discussion see [**5**]),

$$\hat{\Gamma}_e = \Gamma \cup \Gamma L_*, \qquad L_*^2 = E, \qquad L_* L = L_\delta L^* L_\delta L_* \tag{4.9}$$

with

$$L^* = \begin{bmatrix} a^* & b^* \\ c^* & d^* \end{bmatrix} \quad \text{for} \quad L = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma, \qquad L_\delta = \begin{bmatrix} \delta & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$L_*(\tau) = \delta\tau^* = (\delta\tau^{(2)}, \delta\tau^{(1)}) \quad \text{for} \quad \tau = (\tau^{(1)}, \tau^{(2)}) \in \mathfrak{H}_e.$$

For $\xi \in \mathfrak{o}$, sign $\xi^{(j)} = e_j$, $|\mathcal{N}(\xi)| = p$, a prime number, and by restriction of $\hat{\Gamma}_e$ to $\mathfrak{A}(\xi)$, we obtain an extension of $\Gamma_{\mathfrak{A}(\xi)}$, namely (see [**5**, §3])

$$(\hat{\Gamma}_e)_{\mathfrak{A}(\xi)} = \Gamma_{\mathfrak{A}(\xi)} \cup \Gamma_{\mathfrak{A}(\xi)} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

A CAF for $\hat{\Gamma}_e$ is defined as usual by (2.3), (2.4) for $L, M \in \hat{\Gamma}_e$, (2.5), (2.6) for $L \in \Gamma$ and

$$J(L_*, \tau) = \nu(L_*) \text{ independent of } \tau.$$

Then

$$J(L_* L, \tau) = J(L_*, L(\tau)) J(L, \tau) = \nu(L_*) J(L, \tau).$$

From (4.9), we have

$$\nu(L_*)^2 = 1, \qquad \nu(L_*) J(L, \tau) = J(L_\delta L^* L_\delta, L_*(\tau)) \nu(L_*)$$

which results in the restriction

$$\nu(L_\delta L^* L_\delta) = \begin{cases} \nu(L) & \text{for} \quad L \in \Gamma, c \neq 0, \\ \nu(L) \exp(-\pi i \mathcal{S}\tfrac{1}{2}(e - e^*) \operatorname{sign} d) & \text{for} \quad L \in \Gamma, c = 0 \end{cases} \tag{4.10}$$

(with $(e_1^*, e_2^*) = (e_2, e_1)$) for the associated MS, taking into account the choice of the branch of $\log(c^{(j)}\tau^{(j)} + d^{(j)})$ for $\tau^{(j)} \in \mathfrak{H}_{e_j}$. On the other hand, a MS of weight r for Γ on $\mathfrak{H}_e$ which satisfies (4.10) can be extended to a MS of weight r for $\hat{\Gamma}_e$ by putting $J(L_*, \tau) = \nu(L_*) = 1$ or -1 (see [**5**, §1]). $\tilde{J}$ as defined in Theorem 2.1 is a CAF of weight $2r$ for $(\hat{\Gamma}_e)_{\mathfrak{A}(\xi)}$. Here L, for $L_0 \in \Gamma_{\mathfrak{A}(\xi)}$, is given in (2.21), whereas

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = (L_* T)_0, \qquad T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \Gamma.$$

$(\hat{\Gamma}_e)_{\mathfrak{A}(\xi)}$ has only one cusp (at ∞) and elliptic fixed points of order 2, 3 only if $p \neq 2, 3$ (there is an elliptic fixed point of order 4 if $p = 2$, and of order 6 if $p = 3$). The volume of the fundamental domain is, of course, half the volume for $\Gamma_{\mathfrak{A}(\xi)}$. Thus, instead of (3.7), by multiplying (3.5) by 6, we get

$$6\mathcal{S}\kappa\xi \equiv \tfrac{1}{2}r(p+1) \bmod \mathbb{Z} \quad \text{for} \quad p \neq 2, 3. \tag{4.11}$$

For $\delta = 1$, choose

$$\xi \equiv \pm 1 \bmod (2^k), \quad \text{then} \quad p = \mathcal{N}(\xi) \equiv 1 \bmod (2^k),$$

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and, from (4.11), we have

$$\pm 6\mathscr{S}\kappa \equiv r \bmod \mathbb{Z}_2, \qquad 0 \equiv 2r \bmod \mathbb{Z}_2, \qquad 2r \in \mathbb{Z}_2 \quad \text{for} \quad \delta = 1. \tag{4.12}$$

For $\delta = -1$, first choose

$$\xi \equiv 1 \bmod (2^k), \quad \text{then} \quad p = -\mathcal{N}(\xi) \equiv -1 \bmod (2^k)$$

and from (4.11) we have

$$6\mathscr{S}\kappa \equiv \tfrac{1}{2}r(-1+1) \equiv 0 \bmod \mathbb{Z}_2. \tag{4.13}$$

If $D \equiv 2 \bmod (4)$, $\delta = -1$, choose

$$\xi \equiv \pm(1+\sqrt{D}) \bmod (2^k), \quad \text{then} \quad p = -\mathcal{N}(\xi) \equiv \tfrac{1}{4}(D-1) \bmod (2^k).$$

and (4.11) yields

$$\pm 6\mathscr{S}\kappa(1+\sqrt{D}) \equiv \tfrac{1}{2}rD \bmod \mathbb{Z}_2, \qquad 0 \equiv rD \bmod \mathbb{Z}_2, \quad 2r \in \mathbb{Z}_2. \tag{4.14}$$

If $D \equiv 3 \bmod (4)$, $\delta = -1$, choose ξ with $|\mathcal{N}(\xi)| \neq 3$,

$$\xi \equiv 2+\sqrt{D},\ \sqrt{D} \bmod (2^k), \text{ giving } p \equiv D-4,\ D \bmod (2^k),$$

which, from (4.11), yields

$$6\,.\,2\mathscr{S}\kappa + 6\mathscr{S}\kappa\sqrt{D} \equiv \tfrac{1}{2}r(1+D-4), \quad 6\mathscr{S}\kappa\sqrt{D} \equiv \tfrac{1}{2}r(1+D) \bmod (2^k).$$

These congruences, together with (4.13), imply

$$0 \equiv \tfrac{1}{2}r(-4) \equiv -2r \bmod \mathbb{Z}_2, \quad 2r \in \mathbb{Z}_2. \tag{4.15}$$

If $D \equiv 1 \bmod (4)$, $\delta = -1$, choose

$$\xi \equiv \pm\sqrt{D} \bmod (2^k), \quad \text{then} \quad p = -\mathcal{N}(\xi) \equiv D \bmod (2^k),$$

and, by (4.11), we get

$$\pm 6\mathscr{S}\kappa\sqrt{D} \equiv \tfrac{1}{2}r(D+1) \bmod \mathbb{Z}_2, \qquad 0 \equiv r(D+1) \bmod \mathbb{Z}_2, \quad 2r \in \mathbb{Z}_2. \tag{4.16}$$

If $D \equiv 5 \bmod (8)$, $\delta = -1$, for $\xi \equiv 2+\sqrt{D} \bmod (2^k)$ we find $p \equiv D-4 \bmod (2^k)$ and

$$12\mathscr{S}\kappa + 6\mathscr{S}\kappa\sqrt{D} \equiv \tfrac{1}{2}(1+D-4) \bmod \mathbb{Z}_2, \tag{4.17}$$

for $\xi \equiv \tfrac{1}{2}(1+\sqrt{D}) \bmod (2^k)$ we find $p \equiv \tfrac{1}{4}(D-1) \bmod (2^k)$ and

$$3\mathscr{S}\kappa + 3\mathscr{S}\kappa\sqrt{D} \equiv \tfrac{1}{2}r(1+\tfrac{1}{4}(D-1)) \bmod \mathbb{Z}_2. \tag{4.18}$$

(4.17), (4.18), together with (4.13) yield

$$0 \equiv \tfrac{1}{2}(-1-D+4+2+\tfrac{1}{2}(D-1))r \bmod \mathbb{Z}_2.$$

Since $-1-D+4+2 \equiv 0 \bmod (4)$, $\tfrac{1}{2}(D-1) \equiv 2 \bmod (4)$, we have

$$r \in \mathbb{Z}_2 \quad \text{for} \quad D \equiv 5 \bmod (8), \qquad \delta = -1. \tag{4.19}$$

Collecting our results (4.12, 14, 15, 16, 19) and taking into consideration Theorem 4.1(a) we obtain the following theorem.

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THEOREM 4.2. *Let ν be a MS of weight r for the symmetric Hilbert modular group $\hat{\Gamma}_e$ of a real quadratic field K on $\mathfrak{H}_e$, $e=(e_1, e_2)$, $\delta=e_1e_2$. Then $2r\in\mathbb{Z}$. For special values of the discriminant d_K we have:*

(a) *if $\delta=1$, $d_K\equiv 0, 5 \bmod(8)$ then $r\in\mathbb{Z}$;*

(b) *if $\delta=-1$, $d_K\equiv 5 \bmod(8)$ then $r\in\mathbb{Z}$.*

REMARK 4.1. *If $\delta=-1$, $d_K\equiv 1 \bmod(8)$, there exist MSs of weight $\frac{1}{2}$ for $\hat{\Gamma}_e$, e.g. the MS belonging to a certain theta series* [**7**, p. 30].

5. Multiplier systems for Siegel's modular group. Siegel's modular group of degree (or genus) n is the group

$$\Gamma=\Gamma_n=\mathrm{Sp}(n,\mathbb{Z})=\mathrm{Sp}(n,\mathbb{R})\cap\mathrm{GL}(2n,\mathbb{Z}), \tag{5.1}$$

$\mathrm{Sp}(n,\mathbb{R})$ consisting of the matrices

$$M=\begin{bmatrix} A & B\\ C & D\end{bmatrix},\ A, B, C, D\in\mathbb{R}^{(n,n)}, \qquad {}^tM\begin{bmatrix} 0 & E\\ -E & 0\end{bmatrix}M=\begin{bmatrix} 0 & E\\ -E & 0\end{bmatrix}. \tag{5.2}$$

The theta subgroup of Γ is defined by

$$\Gamma_\theta=\Gamma_{n,\theta}=\{M\mid M\in\Gamma_n,\ A{}^tC,\ B{}^tD \text{ have even diagonal elements}\}. \tag{5.3}$$

A subgroup Λ of $\mathrm{Sp}(n,\mathbb{R})$ operates on the Siegel upper half space

$$\mathfrak{H}_n=\{Z\mid Z=X+iY\in\mathbb{C}^{(n,n)},\ {}^tZ=Z,\ Y>0\} \tag{5.4}$$

by

$$Z\mapsto M\langle Z\rangle=(AZ+B)(CZ+D)^{-1}. \tag{5.5}$$

An *automorphic factor* (*AF*) of Λ is a mapping

$$J:\Lambda\times\mathfrak{H}_n\to\mathbb{C} \tag{5.6}$$

such that

(5.7) $J(M, Z)$, for fixed $M\subset\Lambda$, is holomorphic without zeros on $\mathfrak{H}_n$,

(5.8) $J(MN, Z)=J(M, N\langle Z\rangle)J(N, Z)$ for $M, N\in\Lambda$, $Z\in\mathfrak{H}_n$,

(5.9) $J(-M, Z)=J(M, Z)$ if $M, -M\in\Lambda$, $Z\in\mathfrak{H}_n$.

An automorphic factor J is called a *classical automorphic factor* (*CAF*) if

$$J(M, Z)=\nu(M)\det(CZ+D)^r \text{ for } M\in\Lambda,\ Z\in\mathfrak{H}_n, \tag{5.10}$$

with a complex number r, the weight of J, and complex numbers $\nu(M)$, depending, of course, on the branch of $\log\det(CZ+D)$. ν is called the associated multiplier system. Usually that branch of $\log\det(CZ+D)$ is chosen, which, at $Z=iE$, coincides with the principal value, i.e.

$$-\pi<\operatorname{Im}\log\det(Ci+D)\le\pi. \tag{5.11}$$

LEMMA 5.1. *If J is a CAF of weight r on a subgroup Λ of $\mathrm{Sp}(n,\mathbb{R})$,*

$$\mu_r(M, Z)=\det(CZ+D)^r \text{ for } M\in\mathrm{Sp}(n,\mathbb{R}),\ Z\in\mathfrak{H}_n,$$

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then, for $S \in \mathrm{Sp}(n, \mathbb{R})$,

$$J_S(S^{-1}MS, Z) := \mu_r(S, Z)\mu_r(S, S^{-1}MS\langle Z\rangle)J(M, S\langle Z\rangle), \qquad M \in \Lambda,$$

is a CAF of weight r for $S^{-1}\Lambda S$. *The definition of* J_S *does not depend on the choice of the branch of* $\log\det(CZ+D)$.

The lemma is well known ([**1**], [**2**]) and as easily checked as in the case of the Hilbert modular group (Lemma 2.1).

In [**2**, 1.1 Definition], the condition (5.9) is omitted. For Γ_n, $\Gamma_{n,\theta}$ and even n, (5.9) is a consequence of (5.7), (5.8), (5.10). By (5.10), J is independent of Z for $C=0$, and, therefore, by (5.8) a character on the subgroup of elements with $C=0$. Because of

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad (\text{for } n=2),$$

$-E_{2n}$ is a commutator in this subgroup; hence $J(-E_{2n}, Z) = 1$.

THEOREM 5.1. *Let J be a CAF of weight r (with or without condition* (5.9)) *for* Γ_n (*or* $\Gamma_{n,\theta}$), *and* $n>1$. *Then* $r \in \mathbb{Z}$ (*or* $2r \in \mathbb{Z}$).

This theorem is due to Christian [**1**, p. 285] for Γ_n and Endres [**2**, Theorem 1] for $\Gamma_{n,\theta}$. For $n>2$ put $m = n-2$ and

$$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & E_m \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{D} = \begin{bmatrix} D & 0 \\ 0 & E_m \end{bmatrix} \quad \text{for} \quad M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Then

$$J\left(\begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}, \begin{bmatrix} Z & 0 \\ 0 & iE_m \end{bmatrix}\right) \quad \text{for} \quad Z \in \mathfrak{H}_2, M \in \Gamma_2 \; (\text{or } \Gamma_{2,\theta})$$

is a CAF of weight r for Γ_2 (or $\Gamma_{2,\theta}$). We can assume, therefore, that $n=2$ (and (5.9) is satisfied, as mentioned above). With $E = E_2$,

$$\hat{J}(L, z) := J\left(\begin{bmatrix} aE & bE \\ cE & dE \end{bmatrix}, \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}\right) \quad \text{for} \quad z \in \mathfrak{H}_1, L = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_1 \; (\text{or } \Gamma_{1,\theta}) \tag{5.12}$$

is an AF for Γ_1 (or $\Gamma_{1,\theta}$). Because of

$$\det(cEz + dE) = (cz+d)^2, \qquad \hat{J}(-E, z) = J(-E_4, zE) = 1,$$

$\hat{J}$ is a CAF of weight $2r$ (with condition (5.9)) for Γ_1 (or $\Gamma_{1,\theta}$). Put

$$\tilde{S} = \begin{bmatrix} \alpha E & \beta E \\ \gamma E & \delta E \end{bmatrix} \quad \text{for} \quad S = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \Gamma_1, \qquad \tilde{Z} = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} \quad \text{for} \quad z \in \mathfrak{H}_1. \tag{5.13}$$

With the notation of Lemma 5.1, we have

$$\hat{J}_S(S^{-1}LS, z) = J_{\tilde{S}}(\tilde{S}^{-1}\tilde{L}\tilde{S}, \tilde{Z}) \quad \text{for} \quad L \in \Gamma_1 \; (\text{or } \Gamma_{1,\theta}). \tag{5.14}$$

Put

$$T(W)=\begin{bmatrix} E & W \\ 0 & E \end{bmatrix} \quad \text{for} \quad W={}^tW\in\mathbb{Z}^{(2,2)}, \qquad M(V)=\begin{bmatrix} V & 0 \\ 0 & {}^tV^{-1} \end{bmatrix} \quad \text{for} \quad V\in \mathrm{GL}(2,\mathbb{Z}). \tag{5.15}$$

Then

$$M(V)T(bE)M(V)^{-1}T(bE)^{-1}=T(b(V{}^tV-E)).$$

As already mentioned, a CAF is a character on the subgroup of elements with $C=0$. Hence

$$J_{\tilde{S}}(T(b(V{}^tV-E)),Z)=1 \quad \text{if} \quad M(V),\ T(bE)\in \tilde{S}^{-1}\Gamma_2\tilde{S}\ (\text{or } \tilde{S}^{-1}\Gamma_{2,\theta}\tilde{S}).$$

We have

$$V{}^tV-E=\begin{bmatrix} a^2 & a \\ a & 0 \end{bmatrix} \quad \text{for} \quad V=\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, \qquad V{}^tV-E=\begin{bmatrix} 0 & -a \\ -a & a^2 \end{bmatrix} \quad \text{for} \quad V=\begin{bmatrix} 1 & 0 \\ -a & 1 \end{bmatrix},$$

and consequently

$$J_{\tilde{S}}(T(a^2bE),Z)=J_{\tilde{S}}\left(T\left(b\begin{bmatrix} a^2 & a \\ a & 0 \end{bmatrix}+b\begin{bmatrix} 0 & -a \\ -a & a^2 \end{bmatrix}\right),Z\right)=1 \tag{5.16}$$

if

$$T(bE),\ M\left(\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\right), \qquad M\left(\begin{bmatrix} 1 & 0 \\ -a & 1 \end{bmatrix}\right)\in \tilde{S}^{-1}\Gamma_2\tilde{S}\ (\text{or } \tilde{S}^{-1}\Gamma_{2,\theta}\tilde{S}). \tag{5.17}$$

Γ_1 has one cusp (at ∞), one elliptic fixed point of order 2, one elliptic fixed point of order 3 and the volume of the fundamental domain is $\frac{1}{3}\pi$. From (3.5), we have

$$\eta_1+\tfrac{1}{2}c_1+\tfrac{1}{3}c_2\equiv 2r\,.\,\tfrac{1}{12} \bmod \mathbb{Z} \qquad (c_1,c_2\in\mathbb{Z}).$$

From (5.16), (5.17) with $\tilde{S}=E$, $a=b=1$, we have

$$e^{2\pi i\eta_1}=\hat{J}\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},z\right)=J(T(E),\tilde{Z})=1,$$

and hence $\eta_1=0$,

$$r\equiv 6\eta_1+3c_1+2c_2\equiv 0 \bmod \mathbb{Z}.$$

$\Gamma_{1,\theta}$ has two cusps, one elliptic fixed point of order 2 and the volume of the fundamental domain is π. From (3.5), we have

$$\eta_1+\eta_2+\tfrac{1}{2}c_1\equiv 2r\,.\,\tfrac{1}{4} \bmod \mathbb{Z} \quad (c_1\in\mathbb{Z}).$$

$\{N \mid N\in\Gamma_2,\ N\equiv E_4 \bmod 2\}$ is a normal subgroup of Γ_2 contained in $\Gamma_{2,\theta}$ and containing $M\left(\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\right)$ and $M\left(\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\right)$ (5.15). These matrices are, therefore, contained in $\tilde{S}^{-1}\Gamma_{2,\theta}\tilde{S}$ for $S\in\Gamma_1$. The cusps of $\Gamma_{1,\theta}$ are $S^{-1}\infty$, $S\in\Gamma_1$. From (5.16), (5.17), we have, for $a=2$,

$$\hat{J}_S\left(\begin{bmatrix} 1 & 4b \\ 0 & 1 \end{bmatrix},z\right)=J_{\tilde{S}}(T(4bE),\tilde{Z})=1 \quad \text{if} \quad \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\in S^{-1}\Gamma_{1,\theta}S$$

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and

$$(e^{2\pi i\eta})^4 = \hat{J}_S\left(\begin{bmatrix}1 & b\\ 0 & 1\end{bmatrix}, z\right)^4 = \hat{J}_S\left(\begin{bmatrix}1 & 4b\\ 0 & 1\end{bmatrix}, z\right) = 1;$$

hence

$$4\eta \in \mathbb{Z}, \qquad 2r \equiv 4\eta_1 + 4\eta_2 + 2c_1 \equiv 0 \bmod \mathbb{Z}.$$

Theorem 5.1 also is an easy consequence of Theorem 3.3. For a real quadratic number field K there exists an embedding of $\mathfrak{H}_{(1,-1)}$ into $\mathfrak{H}_2$ and a corresponding embedding of the Hilbert modular group Γ_K into Γ_2, taking $\Gamma_{K,\theta}$ into $\Gamma_{2,\theta}$. The CAF of weight r for Γ_2 (or $\Gamma_{2,\theta}$) yields a CAF of weight r for Γ_K (or $\Gamma_{K,\theta}$) on $\mathfrak{H}_{(1,-1)}$. Taking $K = \mathbb{Q}(\sqrt{5})$, from Theorem 3.3, we have $r \in \mathbb{Z}$ for Γ_K, $2r \in \mathbb{Z}$ for $\Gamma_{K,\theta}$. The details are as follows. In [**5**, Satz 2.2] with $K = \mathbb{Q}(\sqrt{5})$, put

$$\mathfrak{w} = \mathfrak{o}, \qquad \rho = \sqrt{5}^{-1}, \qquad \mathfrak{e} = (1, -1), \qquad \omega_1 = 1, \qquad \omega_2 = \tfrac{1}{2}(1 + \sqrt{5}).$$

Then

$$V_{\mathfrak{w}} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \qquad W = \begin{bmatrix}1 & 1\\ \frac{1}{2}(1 + {}_{+}\sqrt{5}) & \frac{1}{2}(1 - {}_{+}\sqrt{5})\end{bmatrix}, \qquad M = \begin{bmatrix}W & 0\\ 0 & {}^{t}W^{-1}\end{bmatrix}.$$

For $\tau \in \mathfrak{H}_{(1,-1)}$, in [**5**, Satz 2.2],

$$\hat{Z}(\tau) = M\langle \tilde{Z}(\tau)\rangle, \qquad \tilde{Z}(\tau) = \begin{bmatrix}\rho^{(1)}\tau^{(1)} & 0\\ 0 & \rho^{(2)}\tau^{(2)}\end{bmatrix} \in \mathfrak{H}_2.$$

For $\nu \in K$, put

$$\hat{\nu} = \begin{bmatrix}\nu^{(1)} & 0\\ 0 & \nu^{(2)}\end{bmatrix} \quad \text{and} \quad \tilde{L} = \begin{bmatrix}\hat{\alpha} & \hat{\rho}\hat{\beta}\\ \hat{\rho}^{-1}\hat{\gamma} & \hat{\delta}\end{bmatrix} \quad \text{for} \quad L = \begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix} \in \Gamma_K.$$

Then

$$M\tilde{L}M^{-1} \in \Gamma_2 \text{ (or } \Gamma_{2,\theta}) \quad \text{for} \quad L \in \Gamma_K \text{ (or } \Gamma_{K,\theta}).$$

If J is a CAF of weight r for Γ_2 (or $\Gamma_{2,\theta}$), J_M is a CAF of weight r for $M^{-1}\Gamma_2 M$ (or $M^{-1}\Gamma_{2,\theta}M$). Because of

$$\tilde{L}\left\langle\begin{bmatrix}\rho^{(1)}\tau^{(1)} & 0\\ 0 & \rho^{(2)}\tau^{(2)}\end{bmatrix}\right\rangle = \begin{bmatrix}\rho^{(1)}L^{(1)}(\tau^{(1)}) & 0\\ 0 & \rho^{(2)}L^{(2)}(\tau^{(2)})\end{bmatrix}$$

and

$$\det\left(\begin{bmatrix}\rho^{(1)^{-1}}\gamma^{(1)} & 0\\ 0 & \rho^{(2)^{-1}}\gamma^{(2)}\end{bmatrix}\begin{bmatrix}\rho^{(1)}\tau^{(1)} & 0\\ 0 & \rho^{(2)}\tau^{(2)}\end{bmatrix} + \begin{bmatrix}\delta^{(1)} & 0\\ 0 & \delta^{(2)}\end{bmatrix}\right) = \mathcal{N}(\gamma\tau + \delta),$$

by

$$J_0(L, \tau) = J_M(\tilde{L}, \tilde{Z}(\tau))$$

for $L \in \Gamma_K$ (or $\Gamma_{K,\theta}$) and $\tau \in \mathfrak{H}_{(1,-1)}$, a CAF of weight r is defined, q.e.d.

THEOREM 5.2. *For a subgroup Γ of* $\mathrm{Sp}(n, \mathbb{R})$, *commensurable with Siegel's modular group Γ_n of degree $n > 1$, there exists a (minimal) number $g(\Lambda) \in \mathbb{N}$ with the following property: if J is a CAF of weight r for Λ then*

$$r \in \mathbb{Q}, \qquad g(\Lambda) r \in \mathbb{Z}$$

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and if Λ_0 *is a subgroup of finite index in* Λ *and* J_0 *a CAF of weight* r_0 *for* Λ_0 *then*

$$g(\Lambda)[\Lambda:\Lambda_0]r_0 \in \mathbb{Z}.$$

This theorem is due to Christian [**2**, Satz 1] for congruence subgroups of Γ_n. It is proved exactly like Theorem 3.1 as a consequence of the analogue of Lemma 2.3 (which is as easily checked as in the Hilbert modular group case), the analogue of (3.9) (which is Theorem 5.1 for Γ_n) and the fact that

$$\mu_{\tilde{r}}(M, Z) = \det(CZ+D)^{\tilde{r}} \quad \text{for} \quad Z \in \mathfrak{H}_n, \qquad M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad \tilde{r} \in \mathbb{Z}, 2 \mid \tilde{r}$$

is a CAF of weight $\tilde{r}$ for any subgroup Λ of $\mathrm{Sp}(n, \mathbb{Z})$.

THEOREM 5.3. *Under the conditions of Theorem* 5.2, *the MS associated with a CAF J is of modulus* 1 *with roots of unity as values*

$$J(M, Z) = \nu(M)\det(CZ+D)^r, \qquad |\nu(M)| = 1, \quad \textit{for } Z \in \mathfrak{H}_n, \quad M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Lambda.$$

This theorem has been announced in [**4**, Satz]; the proof, however, depends on [**4**, Lemma 3], stating that a multiplier system ν of weight r for a congruence subgroup Ψ of Γ_n defines a homomorphism $\nu : \Psi \to \mathbb{C}^{\times}$, i.e. is an abelian character, which is false. It is not always possible, by a suitable choice of the branch of $\log(CZ+D)^r$ in

$$J(M, Z) = \nu(M)\det(CZ+D)^r$$

for each $M \in \Psi$, to assure that $\nu(M_1)\nu(M_2) = \nu(M_1M_2)$. E.g. $\Gamma_{2,\theta}$ has a CAF of weight $\frac{1}{2}$. Put

$$M = \begin{bmatrix} V & 0 \\ 0 & V \end{bmatrix}, \qquad V = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

J is a character on the subgroup of elements with $C = 0$. From $M \in \Gamma_{2,\theta}$, we have

$$J(M, Z)^2 = J(M^2, Z) = J(E_4, Z) = 1, \qquad J(M, Z) = \nu(M)(\det V)^{1/2};$$

hence

$$\nu(M)^2(\det(V)^{1/2})^2 = 1.$$

No matter, which branch of $(\det V)^{1/2}$ is chosen, $((\det V)^{1/2})^2 = \det V = -1$; whence $\nu(M)^2 = -1$, but $\nu(M^2) = \nu(E_4) = 1$. Theorem 5.3 is easily proved from Theorem 5.2 exactly as the corresponding result in Theorem 3.1.

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