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EXTENSION BETWEEN SIMPLE MODULES OF PRO-p-IWAHORI HECKE ALGEBRAS

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Abstract We calculate the extension groups between simple modules of pro-p-Iwahori Hecke algebras.

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1. Introduction

Let F be a non-Archimedean local field of residue characteristic p and G a connected reductive group over F. Motivated by the modulo p Langlands program, we study the modulo p representation theory of G. As in the classical (the representations over the field of complex numbers), Hecke algebras are useful tools for the study of modulo p representations. Especially, a pro-p-Iwahori Hecke algebra which is attached to a pro-p-Iwahori subgroup $I(1)$ has an important role in the study. (One reason is that any nonzero modulo p representation has a nonzero $I(1)$-fixed vector.) For example, this algebra is one of the most important tool for the proof of the classification theorem [5].

We focus on the representation theory of pro-p-Iwahori Hecke algebra. Since the simple modules are classified [3, 14, 18], we study its homological properties. The aim of this paper is to calculate the extension between simple modules. Note that such calculation was used to calculate the extension between irreducible modulo p representations of G when $G = \mathrm{GL}_2(\mathbb{Q}_p)$ [16]. As far as the author knows, a calculation of extensions was done only when $G = \mathrm{GL}_2$. Our calculation is in general, namely we do not assume anything about G. We also remark a related result in [1]. If π_1,π_2 are modulo p irreducible subquotients of principal series of G, by the main theorem of [1], then we have an embedding $\mathrm{Ext}^1_{\mathcal{H}}(\pi_1^{I(1)},\pi_2^{I(1)}) \hookrightarrow \mathrm{Ext}^1_G(\pi_1,\pi_2)$. Hence, the calculation in this paper should be helpful to calculate extensions between π_1 and π_2. When π_1,π_2 are principal series, some calculations are done by Hauseux [10, 11].

We explain our result. For each standard parabolic subgroup P, let $\mathcal{H}_P$ be the pro-p-Iwahori Hecke algebra of the Levi subgroup of P. Then for a module σ of $\mathcal{H}_P$, we can consider: the parabolic induction $I_P(\sigma)$ which is an $\mathcal{H}$-module, a certain parabolic subgroup $P(\sigma)$ containing P, a generalized Steinberg module $\mathrm{St}_Q^{P(\sigma)}(\sigma)$, where Q is a parabolic subgroup between P and $P(\sigma)$. By [3], each simple module is constructed by three steps: (1) starting with a supersingular module σ of $\mathcal{H}_P$, where P is a parabolic subgroup; (2) take a generalized Steinberg module $\mathrm{St}_Q^{P(\sigma)}(\sigma)$ (3) and take a parabolic induction $I_{P(\sigma)}(\mathrm{St}_Q^{P(\sigma)}(\sigma))$. (We do not explain the detail of notation here.) Our calculation follows these steps. Let $\pi_1 = I_{P(\sigma_1)}(\mathrm{St}_{Q_1}^{P(\sigma_1)}(\sigma_1))$ and $\pi_2 = I_{P(\sigma_2)}(\mathrm{St}_{Q_2}^{P(\sigma_2)}(\sigma_2))$ be two simple modules here σ_1 (resp. σ_2) is a simple supersingular module of $\mathcal{H}_{P_1}$ (resp. $\mathcal{H}_{P_2}$).

(1) By considering the central characters, the extension $\mathrm{Ext}^i_{\mathcal{H}}(\pi_1,\pi_2)$ is zero if $P_1 \neq P_2$ (Lemma 3.1). Hence, we may assume $P_1 = P_2$. Set $P = P_1$.

(2) We prove

$$\mathrm{Ext}^i_{\mathcal{H}}(I_{P(\sigma_1)}(\mathrm{St}_{Q_1}^{P(\sigma_1)}(\sigma_1)), I_{P(\sigma_2)}(\mathrm{St}_{Q_2}^{P(\sigma_2)}(\sigma_2))) \simeq \mathrm{Ext}^i_{\mathcal{H}_{P'}}(\mathrm{St}_{Q_1'}^{P'}(\sigma_1), \mathrm{St}_{Q_2'}^{P'}(\sigma_2))$$

for some Q_1', Q_2' and P' (Proposition 3.4). For the proof, we use the adjoint functors of parabolic induction and results in [4]. Hence, it is sufficient to calculate the extension groups between generalized Steinberg modules.

(3) We prove

$$\mathrm{Ext}^i_{\mathcal{H}}(\mathrm{St}_{Q_1}(\sigma_1), \mathrm{St}_{Q_2}(\sigma_2)) \simeq \mathrm{Ext}^{i-r}_{\mathcal{H}}(e(\sigma_1), e(\sigma_2))$$

for some (explicitly given) $r \in \mathbb{Z}_{\geq 0}$ or 0 (Theorem 3.8) using some involutions on $\mathcal{H}$ and results in [2]. Here, $e(\sigma)$ is the extension of σ to $\mathcal{H}$ (Definition 2.4).

(4) We prove

$$\mathrm{Ext}^i_{\mathcal{H}}(e(\sigma_1), e(\sigma_2)) \simeq \mathrm{Ext}^i_{\mathcal{H}_P/I}(\sigma_1, \sigma_2)$$

for some ideal $I \subset \mathcal{H}_P$ which acts on σ_1 and σ_2 by zero. We use results of Ollivier–Schneider [15] for the proof. The algebra $\mathcal{H}_P/I$ is not a pro-p-Iwahori Hecke algebra attached to a connected reductive group but a generic algebra in the sense of Vignéras [20, 4.3]. Hence, it is sufficient to calculate the extensions between supersingular simple modules of a generic algebra.

(5) Now let $\mathcal{H}$ be a generic algebra and π_1, π_2 be simple supersinglar modules. The algebra has the following decomposition as vector spaces: $\mathcal{H} = \mathcal{H}^{\mathrm{aff}} \otimes_{C[Z_\kappa]} C[\Omega(1)]$. Here, $\mathcal{H}^{\mathrm{aff}} \subset \mathcal{H}$ is an algebra called 'the affine subalgebra', $\Omega(1)$ is a certain commutative group acting on $\mathcal{H}^{\mathrm{aff}}$, Z_κ is a normal subgroup of $\Omega(1)$ and we have an embedding $C[Z_\kappa] \hookrightarrow \mathcal{H}^{\mathrm{aff}}$ which is compatible with the action of $\Omega(1)$ on $\mathcal{H}^{\mathrm{aff}}$. Set $\Omega = \Omega(1)/Z_\kappa$. By this decomposition and Hochschild–Serre type spectral sequence, we have an exact sequence

$$0 \to H^1(\Omega, \mathrm{Hom}_{\mathcal{H}^{\mathrm{aff}}}(\pi_1,\pi_2)) \to \mathrm{Ext}^1_{\mathcal{H}}(\pi_1,\pi_2) \to \mathrm{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\pi_1,\pi_2)^{\Omega}.$$

We prove that the last map is surjective (Theorem 4.5).

Therefore, it is sufficient to calculate two groups: $H^1(\Omega, \mathrm{Hom}_{\mathcal{H}^{\mathrm{aff}}}(\pi_1,\pi_2))$ and $\mathrm{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\pi_1,\pi_2)^{\Omega}$. By the classification result of supersingular simple modules [14, 18], the restriction of π_1, π_2 to $\mathcal{H}^{\mathrm{aff}}$ are the direct sum of characters of $\mathcal{H}^{\mathrm{aff}}$. Hence, $\mathrm{Hom}_{\mathcal{H}^{\mathrm{aff}}}(\pi_1,\pi_2)$ is easily described, and with this description we can calculate $H^1(\Omega, \mathrm{Hom}_{\mathcal{H}^{\mathrm{aff}}}(\pi_1,\pi_2))$

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using well-known calculation of group cohomologies. Note that Ω is commutative. We also calculate $\mathrm{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\Xi_1,\Xi_2)$, where Ξ_1,Ξ_2 are characters of $\mathcal{H}^{\mathrm{aff}}$ (Proposition 4.1) following the method of Fayers [8]. This is also calculated by Nadimpalli [13]. Using this description, we can calculate $\mathrm{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\pi_1,\pi_2)^{\Omega}$, and this finishes the calculation of extensions between simple $\mathcal{H}$-modules.

In the last two subsections, examples for GL_n are given.

2. Preliminaries

2.1. Pro-p-Iwahori Hecke algebra

Let $\mathcal{H}$ be a pro-p-Iwahori Hecke algebra over a commutative ring C [20]. We study modules over $\mathcal{H}$ in this paper. *In this paper, a module means a right module.* The algebra $\mathcal{H}$ is defined with combinatorial data $(W_{\mathrm{aff}},S_{\mathrm{aff}},\Omega,W,W(1),Z_\kappa)$ and a parameter (q,c).

We recall the definitions. The data satisfy the following:

- $(W_{\mathrm{aff}},S_{\mathrm{aff}})$ is a Coxeter system.
- Ω acts on $(W_{\mathrm{aff}},S_{\mathrm{aff}})$.
- $W = W_{\mathrm{aff}} \rtimes \Omega$.
- Z_κ is a finite commutative group.
- The group $W(1)$ is an extension of W by Z_κ, namely we have an exact sequence $1 \to Z_\kappa \to W(1) \to W \to 1$.

The subgroup Z_κ is normal in $W(1)$. Hence, the conjugate action of $w \in W(1)$ induces an automorphism of Z_κ, hence of the group ring $C[Z_\kappa]$. We denote it by $c \mapsto w\cdot c$.

Let $\mathrm{Ref}(W_{\mathrm{aff}})$ be the set of reflections in W_{aff} and $\mathrm{Ref}(W_{\mathrm{aff}}(1))$ the inverse image of $\mathrm{Ref}(W_{\mathrm{aff}})$ in $W(1)$. The parameter (q,c) is maps $q\colon S_{\mathrm{aff}} \to C$ and $c\colon \mathrm{Ref}(W_{\mathrm{aff}}(1)) \to C[Z_\kappa]$ with the following conditions. (Here, the image of s by q (resp. c) is denoted by q_s (resp. c_s).)

- For $w \in W$ and $s \in S_{\mathrm{aff}}$, if $wsw^{-1} \in S_{\mathrm{aff}}$, then $q_{wsw^{-1}} = q_s$.
- For $w \in W(1)$ and $s \in \mathrm{Ref}(W_{\mathrm{aff}}(1))$, $c_{wsw^{-1}} = w\cdot c_s$.
- For $s \in \mathrm{Ref}(W_{\mathrm{aff}}(1))$ and $t \in Z_\kappa$, we have $c_{ts} = tc_s$.

Let $S_{\mathrm{aff}}(1)$ be the inverse image of S_{aff} in $W(1)$. For $s \in S_{\mathrm{aff}}(1)$, we write q_s for $q_{\bar{s}}$, where $\bar{s} \in S_{\mathrm{aff}}$ is the image of s. The length function on W_{aff} is denoted by ℓ, and its inflation to W and $W(1)$ is also denoted by ℓ.

The C-algebra $\mathcal{H}$ is a free C-module and has a basis $\{T_w\}_{w\in W(1)}$. The multiplication is given by

- (Quadratic relations) $T_s^2 = q_sT_{s^2} + c_sT_s$ for $s \in S_{\mathrm{aff}}(1)$.
- (Braid relations) $T_{vw} = T_vT_w$ if $\ell(vw) = \ell(v)+\ell(w)$.

We extend $q\colon S_{\mathrm{aff}} \to C$ to $q\colon W \to C$ as follows. For $w \in W$, take $s_1,\dots,s_l$ and $u \in \Omega$ such that $w = s_1\cdots s_lu$ and $l = \ell(w)$. Then put $q_w = q_{s_1}\cdots q_{s_l}$. From the definition, we have $q_{w^{-1}} = q_w$. We also put $q_w = q_{\overline{w}}$ for $w \in W(1)$ with the image $\overline{w}$ in W.

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2.2. The data from a group

Let F be a non-Archimedean local field, κ its residue field, p its residue characteristic and G a connected reductive group over F. We can get the data in the previous subsection from G as follows. See [20], especially 3.9 and 4.2 for the details.

Fix a maximal split torus S, and denote the centralizer of S in G by Z. Let Z^0 be the unique parahoric subgroup of Z and $Z(1)$ its pro-p radical. Then the group $W(1)$ (resp. W) is defined by $W(1) = N_G(Z)/Z(1)$ (resp. $W = N_G(Z)/Z^0$), where $N_G(Z)$ is the normalizer of Z in G. We also let $Z_\kappa = Z^0/Z(1)$. Let G' be the group generated by the unipotent radical of parabolic subgroups [5, II.1] and W_{aff} the image of $G' \cap N_G(Z)$ in W. Then this is a Coxeter group. Fix a set of simple reflections S_{aff}. The group W has the natural length function, and let Ω be the set of length zero elements in W. Then we get the data $(W_{\mathrm{aff}}, S_{\mathrm{aff}}, \Omega, W, W(1), Z_\kappa)$.

Consider the apartment attached to S and an alcove surrounded by the hyperplanes fixed by S_{aff}. Let $I(1)$ be the pro-p-Iwahori subgroup attached to this alcove. Then with $q_s = \#(I(1)\widetilde{s}I(1)/I(1))$ for $s \in S_{\mathrm{aff}}$ with a lift $\widetilde{s} \in N_G(Z)$ and suitable c_s, the algebra $\mathcal{H}$ is isomorphic to the Hecke algebra attached to $(G, I(1))$ [20, Proposition 4.4].

When the data come from the group G, let $W_{\mathrm{aff}}(1)$ be the image of $G' \cap N_G(Z)$ in $W(1)$ and put $\mathcal{H}_{\mathrm{aff}} = \bigoplus_{w \in W_{\mathrm{aff}}(1)} CT_w$. This is a subalgebra of $\mathcal{H}$.

In this paper, except Section 4, we assume that the data come from a connected reductive group.

2.3. The root system and the Weyl groups

Let $W_0 = N_G(Z)/Z$ be the finite Weyl group. Then this is a quotient of W. Recall that we have the alcove defining $I(1)$. Fix a special point $\boldsymbol{x}_0$ from the border of this alcove. Then $W_0 \simeq \mathrm{Stab}_W \boldsymbol{x}_0$, and the inclusion $\mathrm{Stab}_W \boldsymbol{x}_0 \hookrightarrow W$ is a splitting of the canonical projection $W \to W_0$. Throughout this paper, we fix this special point and regard W_0 as a subgroup of W. Set $S_0 = S_{\mathrm{aff}} \cap W_0 \subset W$. This is a set of simple reflections in W_0. For each $w \in W_0$, we fix a representative $n_w \in W(1)$ such that $n_{w_1 w_2} = n_{w_1} n_{w_2}$ if $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$.

The group W_0 is the Weyl group of the root system Σ attached to (G, S). Our fixed alcove and special point give a positive system of Σ, denoted by Σ^+. The set of simple roots is denoted by Δ. As usual, for $\alpha \in \Delta$, let $s_\alpha \in S_0$ be a simple reflection for α.

The kernel of $W(1) \to W_0$ (resp. $W \to W_0$) is denoted by $\Lambda(1)$ (resp. Λ). Then $Z_\kappa \subset \Lambda(1)$, and we have $\Lambda = \Lambda(1)/Z_\kappa$. The group Λ (resp. $\Lambda(1)$) is isomorphic to Z/Z^0 (resp. $Z/Z(1)$). Any element in $W(1)$ can be uniquely written as $n_w \lambda$, where $w \in W_0$ and $\lambda \in \Lambda(1)$. We have $W = W_0 \ltimes \Lambda$.

2.4. The map ν

The group W acts on the apartment attached to S, and the action of Λ is by the translation. Since the group of translations of the apartment is $X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$, we have a group homomorphism $\nu \colon \Lambda \to X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$. The compositions $\Lambda(1) \to \Lambda \to X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ and $Z \to \Lambda \to X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ are also denoted by ν. The homomorphism $\nu \colon Z \to X_*(S) \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathrm{Hom}_{\mathbb{Z}}(X^*(S), \mathbb{R})$ is characterized by the following: For $t \in S$ and $\chi \in X^*(S)$, we have $\nu(t)(\chi) = -\mathrm{val}(\chi(t))$, where val is the normalized valuation of F.

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We call $\lambda \in \Lambda(1)$ dominant (resp. anti-dominant) if $\nu(\lambda)$ is dominant (resp. anti-dominant).

Since the group W_{aff} is a Coxeter system, it has the Bruhat order denoted by $\leq$. For $w_1,w_2 \in W$, we write $w_1 < w_2$ if there exists $u \in \Omega$ such that $w_1u,w_2u \in W_{\mathrm{aff}}$ and $w_1u < w_2u$. Moreover, for $w_1,w_2 \in W(1)$, we write $w_1 < w_2$ if $w_1 \in W_{\mathrm{aff}}(1)w_2$ and $\overline{w}_1 < \overline{w}_2$, where $\overline{w}_1,\overline{w}_2$ are the image of w_1,w_2 in W, respectively. We write $w_1 \leq w_2$ if $w_1 < w_2$ or $w_1 = w_2$.

2.5. Other basis

For $w \in W(1)$, take $s_1,\cdots,s_l \in S_{\mathrm{aff}}(1)$ and $u \in W(1)$ such that $l = \ell(w)$, $\ell(u) = 0$ and $w = s_1\cdots s_l u$. Set $T_w^* = (T_{s_1} - c_{s_1})\cdots(T_{s_l} - c_{s_l})T_u$. Then this does not depend on the choice, and $\{T_w^*\}_{w\in W(1)}$ is a basis of $\mathcal{H}$. In $\mathcal{H}[q_s^{\pm 1}]$, we have $T_w^* = q_w T_{w^{-1}}^{-1}$.

For a spherical orientation o, there is a basis $\{E_o(w)\}_{w\in W(1)}$ of $\mathcal{H}$ introduced in [20, Definition 5.22]. This satisfies the following product formula [20, Theorem 5.25].

$$E_o(w_1)E_{o\cdot w_1}(w_2) = q_{w_1w_2}^{-1/2}q_{w_1}^{1/2}q_{w_2}^{1/2}E_o(w_1w_2). \tag{2.1}$$

Remark 2.1. The term $q_{w_1w_2}^{-1/2}q_{w_1}^{1/2}q_{w_2}^{1/2}$ does not make sense in a usual way. See [4, Remark 2.2].

2.6. Parabolic induction

Since we have a positive system Σ^+, we have the minimal parabolic subgroup B with a Levi part Z. In this paper, *parabolic subgroups are always standard, namely containing B.* Note that such parabolic subgroups correspond to subsets of Δ.

Let P be a parabolic subgroup. Attached to the Levi part of P containing Z, we have the data $(W_{\mathrm{aff},P},S_{\mathrm{aff},P},\Omega_P,W_P,W_P(1),Z_\kappa)$ and the parameters (q_P,c_P). Hence, we have the algebra $\mathcal{H}_P$. The parameter c_P is given by the restriction of c; hence, we denote it just by c. The parameter q_P is defined as in [3, 4.1].

For the objects attached to this data, we add the suffix P. We have the set of simple roots Δ_P, the root system Σ_P and its positive system Σ_P^+, the finite Weyl group $W_{0,P}$, the set of simple reflections $S_{0,P} \subset W_{0,P}$, the length function ℓ_P and the base $\{T_w^P\}_{w\in W_P(1)}$, $\{T_w^{P*}\}_{w\in W_P(1)}$ and $\{E_o^P(w)\}_{w\in W_P(1)}$ of $\mathcal{H}_P$. Note that we have no Λ_P, $\Lambda_P(1)$ and $Z_{\kappa,P}$ since they are equal to Λ, $\Lambda(1)$ and Z_κ.

An element $n_w\lambda \in W_P(1)$, where $w \in W_{P,0}$ and $\lambda \in \Lambda(1)$ is called P-positive (resp. P-negative) if for any $\alpha \in \Sigma^+ \setminus \Sigma_P^+$ we have $\langle \alpha,\nu(\lambda)\rangle \leq 0$ (resp. $\langle \alpha,\nu(\lambda)\rangle \geq 0$). Set $\mathcal{H}_P^+ = \bigoplus_w CT_w^P$, where $w \in W_P(1)$ runs P-positive elements, and define $\mathcal{H}_P^-$ by the similar way. Then these are subalgebras of $\mathcal{H}_P$. The linear maps $j_P^{\pm}\colon \mathcal{H}_P^{\pm} \to \mathcal{H}$ and $j_P^{\pm *}\colon \mathcal{H}_P^{\pm} \to \mathcal{H}$ defined by $j_P^{\pm}(T_w^P) = T_w$ and $j_P^{\pm *}(T_w^{P*}) = T_w^*$ are algebra homomorphisms.

Proposition 2.2 ([19, Theorem 1.4]). *Let λ_P^+ (resp. λ_P^-) be in the center of $W_P(1)$ such that $\langle \alpha,\nu(\lambda_P^+)\rangle < 0$ (resp. $\langle \alpha,\nu(\lambda_P^-)\rangle > 0$) for all $\alpha \in \Sigma^+ \setminus \Sigma_P^+$. Then $T_{\lambda_P^+}^P = T_{\lambda_P^+}^{P*} = E_{o_{-,P}}^P(\lambda_P^+)$ (resp. $T_{\lambda_P^-}^P = T_{\lambda_P^-}^{P*} = E_{o_{-,P}}^P(\lambda_P^-)$) is in the center of $\mathcal{H}_P$, and we have $\mathcal{H}_P = \mathcal{H}_P^+ E_{o_{-,P}}^P(\lambda_P^+)^{-1}$ (resp. $\mathcal{H}_P = \mathcal{H}_P^- E_{o_{-,P}}^P(\lambda_P^-)^{-1}$).*

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Now, for an $\mathcal{H}_P$-module σ, we define the parabolically induced module $I_P(\sigma)$ by

$$I_P(\sigma) = \mathrm{Hom}_{(\mathcal{H}_P^-, j_P^{-*})}(\mathcal{H}, \sigma).$$

This satisfies:

- I_P is an exact functor.
- I_P has the left adjoint functor L_P. The functor L_P is exact.
- I_P has the right adjoint functor R_P.

For the existence and explicit descriptions of adjoint functors L_P, R_P, see [4, 5.1].

For parabolic subgroups $P \subset Q$, we also defines $\mathcal{H}_P^{Q\pm} \subset \mathcal{H}_P$ and $j_P^{Q\pm} \colon \mathcal{H}_P^{Q\pm} \to \mathcal{H}_Q$ and $j_P^{Q\pm *} \colon \mathcal{H}_P^{Q\pm} \to \mathcal{H}_Q$. This defines the parabolic induction I_P^Q from the category of $\mathcal{H}_P$-modules to the category of $\mathcal{H}_Q$-modules.

2.7. Twist by $n_{w_G w_P}$

For a parabolic subgroup P, let w_P be the longest element in $W_{0,P}$. In particular, w_G is the longest element in W_0. Let P' be a parabolic subgroup corresponding to $-w_G(\Delta_P)$; in other words, $P' = n_{w_G w_P} P^{\mathrm{op}} n_{w_G w_P}^{-1}$, where P^{op} is the opposite parabolic subgroup of P with respect to the Levi part of P containing Z. Set $n = n_{w_G w_P}$. Then the map $P^{\mathrm{op}} \to P'$ defined by $p \mapsto npn^{-1}$ is an isomorphism which preserves the data used to define the pro-p-Iwahori Hecke algebras. Hence, $T_w^P \mapsto T_{nwn^{-1}}^{P'}$ gives an isomorphism $\mathcal{H}_P \to \mathcal{H}_{P'}$. This sends T_w^{P*} to $T_{nwn^{-1}}^{P'*}$ and $E_{o_{+,P}\cdot v}^P(w)$ to $E_{o_{+,P'}\cdot nvn^{-1}}^{P'}(nwn^{-1})$, where $v \in W_{0,P}$.

Let σ be an $\mathcal{H}_P$-module. Then we define an $\mathcal{H}_{P'}$-module $n_{w_G w_P}\sigma$ via the pull-back of the above isomorphism: $(n_{w_G w_P}\sigma)(T_w^{P'}) = \sigma(T^P_{n_{w_G w_P}^{-1} w n_{w_G w_P}})$. For an $\mathcal{H}_{P'}$-module σ', we define $n_{w_G w_P}^{-1}\sigma'$ by $(n_{w_G w_P}^{-1}\sigma')(T_w^P) = \sigma'(T^{P'}_{n_{w_G w_P} w n_{w_G w_P}^{-1}})$.

2.8. The extension and the generalized Steinberg modules

Let P be the parabolic subgroup and σ an $\mathcal{H}_P$-module. For $\alpha \in \Delta$, let P_α be a parabolic subgroup corresponding to $\Delta_P \cup \{\alpha\}$. Then we define $\Delta(\sigma) \subset \Delta$ by

$$\Delta(\sigma) = \{\alpha \in \Delta \mid \langle \Delta_P, \alpha^\vee \rangle = 0,\ \sigma(T_\lambda^P) = 1 \text{ for any } \lambda \in W_{\mathrm{aff},P_\alpha}(1) \cap \Lambda(1)\} \cup \Delta_P.$$

Let $P(\sigma)$ be the parabolic subgroup corresponding to $\Delta(\sigma)$.

Proposition 2.3 ([6, Theorem 3.6]). *Let σ be an $\mathcal{H}_P$-module and Q a parabolic subgroup between P and $P(\sigma)$. Denote the parabolic subgroup corresponding to $\Delta_Q \setminus \Delta_P$ by P_2. Then there exists a unique $\mathcal{H}_Q$-module $e_Q(\sigma)$ acting on the same space as σ such that*

- *$e_Q(\sigma)(T_w^{Q*}) = \sigma(T_w^{P*})$ for any $w \in W_P(1)$.*
- *$e_Q(\sigma)(T_w^{Q*}) = 1$ for any $w \in W_{\mathrm{aff},P_2}(1)$.*

Definition 2.4. We call $e_Q(\sigma)$ the extension of σ to $\mathcal{H}_Q$.

A typical example of the extension is the trivial representation $\mathbf{1} = \mathbf{1}_G$. This is a one-dimensional $\mathcal{H}$-module defined by $\mathbf{1}(T_w) = q_w$, or equivalently $\mathbf{1}(T_w^*) = 1$. We have $\Delta(\mathbf{1}_P) = \{\alpha \in \Delta \mid \langle \Delta_P, \alpha^\vee \rangle = 0\} \cup \Delta_P$, and if Q is a parabolic subgroup between P and $P(\mathbf{1}_P)$, we have $e_Q(\mathbf{1}_P) = \mathbf{1}_Q$.

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Let $P(\sigma) \supset P_0 \supset Q_1 \supset Q \supset P$. Then as in [3, 4.5], we have $I_{Q_1}^{P_0}(e_{Q_1}(\sigma)) \subset I_Q^{P_0}(e_Q(\sigma))$. Define

$$\mathrm{St}_Q^{P_0}(\sigma) = \mathrm{Coker}\left(\bigoplus_{Q_1 \supsetneq Q} I_{Q_1}^{P_0}(e_{Q_1}(\sigma)) \to I_Q^{P_0}(e_Q(\sigma))\right).$$

When $P_0 = G$, we write $\mathrm{St}_Q(\sigma)$ and call it generalized Steinberg modules.

2.9. Supersingular modules

In this subsection, we assume that C is a field of characteristic p. Let $\mathcal{O}$ be a conjugacy class in $W(1)$ which is contained in $\Lambda(1)$. For a spherical orientation o, set $z_{\mathcal{O}} = \sum_{\lambda \in \mathcal{O}} E_o(\lambda)$. Then this does not depend on o and $z_{\mathcal{O}} \in \mathcal{Z}$, where $\mathcal{Z}$ is the center of $\mathcal{H}$[18, Theorem 5.1]. The length of $\lambda \in \mathcal{O}$ does not depend on λ. We denote it by $\ell(\mathcal{O})$. For $\lambda \in \Lambda(1)$ and $w \in W(1)$, we put $w \cdot \lambda = w\lambda w^{-1}$.

Definition 2.5. Let π be an $\mathcal{H}$-module. We call π supersingular if there exists $n \in \mathbb{Z}_{>0}$ such that $\pi z_{\mathcal{O}}^n = 0$ for any $\mathcal{O}$ such that $\ell(\mathcal{O}) > 0$.

Remark 2.6. Since $\pi z_{\mathcal{O}} \subset \pi$ is a submodule, if π is simple, then π is supersingular if and only if $\pi z_{\mathcal{O}} = 0$ for any $\mathcal{O}$ such that $\ell(\mathcal{O}) > 0$. Let $\lambda \in \Lambda(1)$. Then $\ell(\lambda) \neq 0$ if and only if $\langle \alpha, \nu(\lambda)\rangle \neq 0$ for some $\alpha \in \Sigma$ [4, Lemma 2.12]. Hence, a simple $\mathcal{H}$-module π is supersingular if and only if $\pi(z_{W(1)\cdot\lambda}) = 0$ for any λ such that $\langle \alpha, \nu(\lambda)\rangle \neq 0$ for some $\alpha \in \Sigma$.

The simple supersingular $\mathcal{H}$-modules are classified in [14, 18]. We recall their results. Let $W^{\mathrm{aff}}(1)$ be the inverse image of W_{aff} in $W(1)$.

Remark 2.7. When we do not assume that the data come from a group, we have no $W_{\mathrm{aff}}(1)$ but we have $W^{\mathrm{aff}}(1)$. Even though the data come from a group, $W_{\mathrm{aff}}(1)$ is not equal to $W^{\mathrm{aff}}(1)$. We have $Z_\kappa \subset W^{\mathrm{aff}}(1)$; however, $Z_\kappa \not\subset W_{\mathrm{aff}}(1)$ in general. Since we will not assume that the data come from a group, we do not use $W_{\mathrm{aff}}(1)$ here.

Put $\mathcal{H}^{\mathrm{aff}} = \bigoplus_{w \in W^{\mathrm{aff}}(1)} CT_w$. Let χ be a character of Z_κ and put $S_{\mathrm{aff},\chi} = \{s \in S_{\mathrm{aff}} \mid \chi(c_{\widetilde{s}}) \neq 0\}$, where $\widetilde{s} \in W(1)$ is a lift of $s \in S_{\mathrm{aff}}$. Note that, if $\widetilde{s}'$ is another lift, then $\widetilde{s}' = t\widetilde{s}$ for some $t \in Z_\kappa$. Hence, $\chi(c_{\widetilde{s}'}) = \chi(t)\chi(c_{\widetilde{s}})$. Therefore, the condition does not depend on a choice of a lift. Let $J \subset S_{\mathrm{aff},\chi}$. Then the character $\Xi = \Xi_{J,\chi}$ of $\mathcal{H}^{\mathrm{aff}}$ is defined by

$$\Xi_{J,\chi}(T_t) = \chi(t) \quad (t \in Z_\kappa),$$
$$\Xi_{J,\chi}(T_{\widetilde{s}}) = \begin{cases} \chi(c_{\widetilde{s}}) & (s \in S_{\mathrm{aff},\chi} \setminus J) \\ 0 & (s \notin S_{\mathrm{aff},\chi} \setminus J) \end{cases} = \begin{cases} \chi(c_{\widetilde{s}}) & (s \notin J), \\ 0 & (s \in J), \end{cases}$$

where $\widetilde{s} \in W^{\mathrm{aff}}(1)$ is a lift of s and the last equality easily follows from the definition of $S_{\mathrm{aff},\chi}$. Let $\Omega(1)_\Xi$ be the stabilizer of Ξ and V a simple $C[\Omega(1)_\Xi]$-module such that $V|_{Z_\kappa}$ is a direct sum of χ. Put $\mathcal{H}_\Xi = \mathcal{H}^{\mathrm{aff}} C[\Omega(1)_\Xi]$. This is a subalgebra of $\mathcal{H}$. For $X \in \mathcal{H}^{\mathrm{aff}}$ and $Y \in C[\Omega(1)_\Xi]$, we define the action of XY on $\Xi \otimes V$ by $x \otimes y \mapsto xX \otimes yY$. Then this defines a well-defined action of $\mathcal{H}_\Xi$ on $\Xi \otimes V$. Set $\pi_{\chi,J,V} = (\Xi \otimes V) \otimes_{\mathcal{H}_\Xi} \mathcal{H}$.

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Proposition 2.8 ([18, Theorem 1.6]). *The module $\pi_{\chi,J,V}$ is simple, and it is supersingular if and only if the groups generated by J and generated by $S_{\mathrm{aff},\chi} \setminus J$ are both finite. If C is an algebraically closed field, then any simple supersingular modules are given in this way.*

The construction of $\pi_{\chi,J,V}$ is still valid even if we do not assume that the data come from a group. In Section 4, we do not assume it, and we calculate the extension between the modules constructed as above.

2.10. Simple modules

Definition 2.9. We consider a triple (P,σ,Q) which satisfies the following:

- P is a parabolic subgroup of G.
- σ is a supersingular finite-dimensional $\mathcal{H}_P$-module.
- Q is a parabolic subgroup contained in $P(\sigma)$.

Then we define an $\mathcal{H}$-module $I(P,\sigma,Q)$ by

$$I(P,\sigma,Q) = I_{P(\sigma)}(\mathrm{St}_Q^{P(\sigma)}(\sigma)).$$

Theorem 2.10 ([3, Theorem 1.1]). *Assume that C is an algebraically closed field of characteristic p. The module $I(P,\sigma,Q)$ is simple, and any simple module has this form. Moreover, (P,σ,Q) is unique up to isomorphism.*

3. Reduction to supersingular representations

In the rest of this paper, we assume that C is a field of characteristic p. Let $(P_1,\sigma_1,Q_1),(P_2,\sigma_2,Q_2)$ be triples as in Definition 2.9. We calculate the extension group $\mathrm{Ext}^1_{\mathcal{H}}(I(P_1,\sigma_1,Q_1),I(P_2,\sigma_2,Q_2))$.

3.1. Central character

We prove the following lemma.

Lemma 3.1. *If $\mathrm{Ext}^i_{\mathcal{H}}(I(P_1,\sigma_1,Q_1),I(P_2,\sigma_2,Q_2)) \neq 0$ for some $i \in \mathbb{Z}_{\geq 0}$, then $P_1 = P_2$.*

To prove this lemma, we calculate the action of the center $\mathcal{Z}$ on simple modules. To do it, we need to calculate the action of $\mathcal{Z}$ on a parabolic induction.

Lemma 3.2. *Let P be a parabolic subgroup, σ a right $\mathcal{H}_P$-module. For $W(1)$-orbit $\mathcal{O}$ in $\Lambda(1)$, set $\mathcal{O}_P = \{\lambda \in \mathcal{O} \mid \lambda$ is P-negative$\}$. Then we have the following:*

(1) *The subset $\mathcal{O}_P \subset \Lambda(1)$ is $W_P(1)$-stable.*

(2) *Let $\mathcal{O}_P = \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_r$ be the decomposition into $W_P(1)$-orbits. The action of $z_{\mathcal{O}} \in \mathcal{Z}$ on $I_P(\sigma)$ is induced by the action of $\sum_i z^P_{\mathcal{O}_i}$ on σ.*

Proof. Since $\Sigma^+ \setminus \Sigma^+_P$ is stable under the action of $W_{0,P}$, (1) follows from the definition of P-negative.

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Let $\varphi \in I_P(\sigma) = \mathrm{Hom}_{(\mathcal{H}_P^-, j_P^{-*})}(\mathcal{H},\sigma)$. Then for $X \in \mathcal{H}$, we have

$$(\varphi z_{\mathcal{O}})(X) = \varphi(z_{\mathcal{O}}X) = \varphi(Xz_{\mathcal{O}})$$

since $z_{\mathcal{O}}$ is in the center of $\mathcal{H}$. Hence, by the definition of $z_{\mathcal{O}}$, we have

$$(\varphi z_{\mathcal{O}})(X) = \sum_{\lambda\in\mathcal{O}} \varphi(XE(\lambda)) = \sum_i \sum_{\lambda\in\mathcal{O}_i} \varphi(XE(\lambda)) + \sum_{\lambda\in\mathcal{O},\ \text{not } P\text{-negative}} \varphi(XE(\lambda))$$

We prove the vanishing of the second term.

Let $\lambda \in \mathcal{O}$, which is not P-negative. Then there exists $\alpha \in \Sigma^+ \setminus \Sigma_P^+$ such that $\langle \alpha, \nu(\lambda)\rangle < 0$. Let λ_P^- be as in Proposition 2.2. Then $\langle \alpha, \nu(\lambda_P^-)\rangle > 0$. Hence, $\nu(\lambda)$ and $\nu(\lambda_P^-)$ does not belongs to the same closed Weyl chamber. Therefore, we have $E(\lambda)E(\lambda_P^-) = 0$ in $\mathcal{H}_C$ by [4, (2.1), Lemma 2.11]. Hence, by [4, Lemma 2.6],

$$\begin{aligned}\varphi(XE(\lambda)) &= \varphi(XE(\lambda))E^P(\lambda_P^-)E^P(\lambda_P^-)^{-1}\\ &= \varphi(XE(\lambda)j_P^{-*}(E^P(\lambda_P^-)))E^P(\lambda_P^-)^{-1}\\ &= \varphi(XE(\lambda)E(\lambda_P^-))E^P(\lambda_P^-)^{-1} = 0.\end{aligned}$$

If $\lambda \in \mathcal{O}_i$, then $E(\lambda) \in \mathcal{H}_P^-$. Hence, we have $E(\lambda) = j_P^{-*}(E^P(\lambda))$ by [4, Lemma 2.6]. Therefore,

$$\sum_{\lambda\in\mathcal{O}_i} \varphi(XE(\lambda)) = \varphi(X) \sum_{\lambda\in\mathcal{O}_i} \sigma(E^P(\lambda)) = \varphi(X)\sigma(z_{\mathcal{O}_i}^P).$$

We get the lemma. □

Lemma 3.3. *Let (P,σ,Q) be a triple as in Definition 2.9. Let R be a parabolic subgroup and $\lambda = \lambda_R^-$ as in Proposition 2.2. Then $z_{\mathcal{O}_\lambda} \neq 0$ on $I(P,\sigma,Q)$ if and only if $P \subset R$.*

Proof. Set $\mathcal{O} = \mathcal{O}_\lambda$. Since $\Lambda(1) \subset W_R(1)$ and λ is in the center of $W_R(1)$, λ commutes with $\Lambda(1)$. Hence, $\mathcal{O} = \{n_w \cdot \lambda \mid w \in W_0\}$.

We prove that $W_P(1)$ acts transitively on $\mathcal{O}_P$. Let $\mu \in \mathcal{O}_P$, and take $w \in W_0$ such that $\mu = n_w \cdot \lambda$. Take $v \in W_{0,P}$ such that $v(\nu(\mu))$ is dominant with respect to Σ_P^+. Since $v^{-1}(\Sigma^+ \setminus \Sigma_P^+) = \Sigma^+ \setminus \Sigma_P^+$ and μ is P-negative, we have $\langle v(\nu(\mu)), \alpha\rangle \geq 0$ for any $\alpha \in \Sigma^+ \setminus \Sigma_P^+$. Hence, $v(\nu(\mu))$ is dominant. Now $\nu(\lambda)$ and $v(\nu(\mu)) = vw(\nu(\lambda))$ is both dominant. Hence, $vw \in \mathrm{Stab}_{W_0}(\nu(\lambda)) = W_{0,R}$. Since λ is in the center of $W_R(1)$, we have $(n_v n_w)\cdot\lambda = \lambda$. Hence, $\mu = n_v^{-1}\cdot\lambda$. Therefore, $W_P(1)$ acts transitively on $\mathcal{O}_P$.

By the definition, $I(P,\sigma,Q)$ is a quotient of $I_{P(\sigma)}(I_Q^{P(\sigma)}(e_Q(\sigma))) = I_Q(e_Q(\sigma))$. Moreover, by the definition of the extension, we have an embedding $e_Q(\sigma) \hookrightarrow I_P^Q(\sigma)$. Hence, we have $I_Q(e_Q(\sigma)) \hookrightarrow I_Q(I_P^Q(\sigma)) = I_P(\sigma)$. Let $\chi\colon \mathcal{Z}_P \to C$ be a central character of σ. By the above lemma and the fact that $\mathcal{O}_P$ is a single $W_P(1)$-orbit, on $I_P(\sigma)$, $z_{\mathcal{O}_\lambda}$ acts by $\chi(z_{\mathcal{O}_P}^P)$. Since $\nu(\lambda)$ is dominant, λ is P-negative. Hence, $\lambda \in \mathcal{O}_P$. By the definition of supersingular representations with Remark 2.6, $\chi(z_{\mathcal{O}_P}^P) = 0$ if and only if $\langle \alpha, \nu(\lambda)\rangle \neq 0$ for some $\alpha \in \Sigma_P^+$. The condition on $\lambda = \lambda_R^-$ tells that $\langle \alpha, \nu(\lambda)\rangle \neq 0$ if and only if $\alpha \in \Sigma^+ \setminus \Sigma_R^+$. Therefore, $\chi(z_{\mathcal{O}_P}^P) \neq 0$ if and only if $\Sigma_P^+ \cap (\Sigma^+ \setminus \Sigma_R^+) = \emptyset$ which is equivalent to $P \subset R$. □

Proof of Lemma 3.1. Assume that $P_1 \neq P_2$. Then we have $P_1 \not\subset P_2$ or $P_1 \not\supset P_2$. Assume $P_1 \not\subset P_2$, and take $\lambda = \lambda_{P_2}^-$ as in Proposition 2.2. Put $\mathcal{O} = \{w\cdot\lambda \mid w \in W(1)\}$. Then $z_{\mathcal{O}} = 0$

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on $I(P_1,\sigma_1,Q_1)$ and $z_{\mathcal{O}} \neq 0$ on $I(P_2,\sigma_2,Q_2)$. Hence, the vanishing follows from a standard argument since $z_{\mathcal{O}} \in \mathcal{H}$ is in the center. The case of $P_1 \not\supset P_2$ is proved by the same way. □

3.2. Reduction to generalized Steinberg modules

By Lemma 3.1, to calculate the extension between $I(P_1,\sigma_1,Q_1)$ and $I(P_2,\sigma_2,Q_2)$, we may assume $P_1 = P_2$. We prove the following proposition.

Proposition 3.4. *The extension group* $\operatorname{Ext}^i_{\mathcal{H}}(I(P,\sigma_1,Q_1),I(P,\sigma,Q_2))$ *is isomorphic to*

$$\operatorname{Ext}^i_{\mathcal{H}_{P(\sigma_1)\cap P(\sigma_2)}}(\operatorname{St}^{P(\sigma_1)\cap P(\sigma_2)}_{Q_1\cap P(\sigma_2)}(\sigma_1),\operatorname{St}^{P(\sigma_1)\cap P(\sigma_2)}_{Q_2}(\sigma_2)).$$

if $Q_2 \subset P(\sigma_1)$ *and* $\Delta(\sigma_1) \subset \Delta_{Q_1} \cup \Delta(\sigma_2)$. *Otherwise, the extension group is zero.*

Hence, for the calculation of the extension, it is sufficient to calculate the extensions between generalized Steinberg modules. For an $\mathcal{H}$-module π, set $\pi^* = \operatorname{Hom}_C(\pi,C)$. The right $\mathcal{H}$-module structure on π^* is given by $(fX)(v) = f(v\zeta(X))$ for $f \in \pi^*$, $v \in \pi$ and $X \in \mathcal{H}$. Here, the anti-involution $\zeta\colon \mathcal{H} \to \mathcal{H}$ is defined by $\zeta(T_w) = T_{w^{-1}}$.

Lemma 3.5. *We have* $\operatorname{Ext}^i_{\mathcal{H}}(\pi_1,\pi_2^*) \simeq \operatorname{Ext}^i_{\mathcal{H}}(\pi_2,\pi_1^*)$. *In particular, if* π_1 *or* π_2 *is finite-dimensional, then* $\operatorname{Ext}^i_{\mathcal{H}}(\pi_1,\pi_2) \simeq \operatorname{Ext}^i_{\mathcal{H}}(\pi_2^*,\pi_1^*)$.

Proof. We have the isomorphism for $i = 0$ since both sides are equal to $\{f\colon \pi_1 \times \pi_2 \to C \mid f(x_1X,x_2) = f(x_1,\zeta(X)x_2)\ (x_1 \in \pi_1, x_2 \in \pi_2, X \in \mathcal{H})\}$. Hence, in particular, if π is projective, then π^* is injective. Let $\cdots \to P_1 \to P_0 \to \pi_2 \to 0$ be a projective resolution. Then $\operatorname{Ext}^i_{\mathcal{H}}(\pi_2,\pi_1^*)$ is a i-th cohomology of the complex $\operatorname{Hom}(P_i,\pi_1^*) \simeq \operatorname{Hom}(\pi_1,P_i^*)$. Since $0 \to \pi_2^* \to P_0^* \to P_1^* \to \cdots$ is an injective resolution of π_2^*, this is $\operatorname{Ext}^i_{\mathcal{H}}(\pi_1,\pi_2^*)$.

If π_2 is finite-dimensional, then $\pi_2 \simeq (\pi_2^*)^*$. Hence, we have $\operatorname{Ext}^i_{\mathcal{H}}(\pi_1,\pi_2) \simeq \operatorname{Ext}^i_{\mathcal{H}}(\pi_1,(\pi_2^*)^*) \simeq \operatorname{Ext}^i_{\mathcal{H}}(\pi_2^*,\pi_1^*)$. By the same argument, we have $\operatorname{Ext}^i_{\mathcal{H}}(\pi_1,\pi_2) \simeq \operatorname{Ext}^i_{\mathcal{H}}(\pi_2^*,\pi_1^*)$ if π_1 is finite-dimensional. □

Proposition 3.6. *Let* P *be a parabolic subgroup,* π *an* $\mathcal{H}$*-module and* σ *an* $\mathcal{H}_P$*-module.*

(1) *We have* $\operatorname{Ext}^i_{\mathcal{H}}(\pi,I_P(\sigma)) \simeq \operatorname{Ext}^i_{\mathcal{H}_P}(L_P(\pi),\sigma)$.

(2) *We have* $\operatorname{Ext}^i_{\mathcal{H}}(I_P(\sigma),\pi^*) \simeq \operatorname{Ext}^i_{\mathcal{H}_P}(\sigma,R_P(\pi^*))$. *In particular, if* π *is finite-dimensional, then* $\operatorname{Ext}^i_{\mathcal{H}}(I_P(\sigma),\pi) \simeq \operatorname{Ext}^i_{\mathcal{H}_P}(\sigma,R_P(\pi))$.

Proof. The exactness of I_P and L_P implies (1).

Put $P' = n_{w_Gw_P}P^{\mathrm{op}}n_{w_Gw_P}^{-1}$ Define the functor $I'_{P'}$ by

$$I'_{P'}(\sigma') = \operatorname{Hom}_{(\mathcal{H}^-_{P'},j^-_{P'})}(\mathcal{H},\sigma')$$

for an $\mathcal{H}_{P'}$-module σ'. Then this has the left adjoint functor $L'_{P'}$ defined by $L'_{P'}(\pi) = \pi \otimes_{(\mathcal{H}^-_{P'},j^-_{P'})} \mathcal{H}_{P'}$. This is exact since $\mathcal{H}_{P'}$ is a localization of $\mathcal{H}^-_{P'}$ by Proposition 2.2. Set $\sigma_{\ell-\ell_P}(T^P_w) = (-1)^{\ell(w)-\ell_P(w)}\sigma(T_w)$ [4, 4.1]. Using [2, Proposition 4.2], for an $\mathcal{H}_P$-module σ, we have

$$\begin{aligned}\operatorname{Ext}^i_{\mathcal{H}}(I_P(\sigma),\pi^*) &\simeq \operatorname{Ext}^i_{\mathcal{H}}(\pi,I_P(\sigma)^*)\\ &\simeq \operatorname{Ext}^i_{\mathcal{H}}(\pi,I'_{P'}(n_{w_Gw_P}\sigma^*_{\ell-\ell_P}))\end{aligned}$$

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$$\begin{aligned}&\simeq \mathrm{Ext}^i_{\mathcal{H}_P}(n_{w_Gw_P}^{-1}L'_{P'}(\pi),\sigma^*_{\ell-\ell_P})\\&\simeq \mathrm{Ext}^i_{\mathcal{H}_P}(\sigma_{\ell-\ell_P},n_{w_Gw_P}^{-1}L'_{P'}(\pi)^*)\\&\simeq \mathrm{Ext}^i_{\mathcal{H}_P}(\sigma,(n_{w_Gw_P}^{-1}L'_{P'}(\pi))^*_{\ell-\ell_P}).\end{aligned}$$

Put $i=0$. Then we get $(n_{w_Gw_P}^{-1}L'_{P'}(\pi))^*_{\ell-\ell_P}\simeq R_P(\pi^*)$ by $\mathrm{Hom}_{\mathcal{H}}(I_P(\sigma),\pi^*)\simeq \mathrm{Hom}_{\mathcal{H}_P}(\sigma,R_P(\pi^*))$. Hence, we get (2).

If π is finite-dimensional, then $\pi=(\pi^*)^*$. Hence, we get $\mathrm{Ext}^i_{\mathcal{H}}(I_P(\sigma),\pi)\simeq \mathrm{Ext}^i_{\mathcal{H}_P}(\sigma,R_P(\pi))$ applying (3) to π^*. □

Proof of Proposition 3.4. Since $I(P,\sigma_2,Q_2)$ is finite-dimensional, we have

$$\mathrm{Ext}^i_{\mathcal{H}}(I(P,\sigma_1,Q_1),I(P,\sigma_2,Q_2))=\mathrm{Ext}^i_{\mathcal{H}_{P(\sigma_1)}}(\mathrm{St}^{P(\sigma_1)}_{Q_1}(\sigma_1),R_{P(\sigma_1)}(I(P,\sigma_2,Q_2))).$$

We have $R_{P(\sigma_1)}(I(P,\sigma_2,Q_2))=0$ if $Q_2\not\subset P(\sigma_1)$ by [4, Theorem 5.20]. If $Q_2\subset P(\sigma_1)$, then $R_{P(\sigma_1)}(I(P,\sigma_2,Q_2))=I_{P(\sigma)}(P,\sigma,Q_2)$. Hence, the extension group is isomorphic to

$$\begin{aligned}&\mathrm{Ext}^i_{\mathcal{H}_{P(\sigma_1)}}(\mathrm{St}^{P(\sigma_1)}_{Q_1}(\sigma_1),I_{P(\sigma_1)}(P,\sigma_2,Q_2))\\&=\mathrm{Ext}^i_{\mathcal{H}_{P(\sigma_1)}}(\mathrm{St}^{P(\sigma_1)}_{Q_1}(\sigma_1),I^{P(\sigma_1)}_{P(\sigma_2)\cap P(\sigma_1)}(\mathrm{St}_{Q_2}(\sigma_2)))\\&=\mathrm{Ext}^i_{\mathcal{H}_{P(\sigma_1)\cap P(\sigma_2)}}(L^{P(\sigma_1)}_{P(\sigma_2)\cap P(\sigma_1)}(\mathrm{St}^{P(\sigma_1)}_{Q_1}(\sigma_1)),\mathrm{St}^{P(\sigma_1)\cap P(\sigma_2)}_{Q_2}(\sigma_2))\end{aligned}$$

We have $L^{P(\sigma_1)}_{P(\sigma_2)\cap P(\sigma_1)}(\mathrm{St}^{P(\sigma_1)}_{Q_1}(\sigma_1))=0$ if $\Delta(\sigma_1)\neq\Delta(Q_1)\cup\Delta_{P(\sigma_1)\cap P(\sigma_2)}$ or $P\not\subset P(\sigma_1)\cap P(\sigma_2)$ by [4, Proposition 5.10, Proposition 5.18]. If it is not zero, then the extension group is isomorphic to

$$\mathrm{Ext}^i_{\mathcal{H}_{P(\sigma_1)\cap P(\sigma_2)}}(\mathrm{St}^{P(\sigma_1)\cap P(\sigma_2)}_{Q_1\cap P(\sigma_2)}(\sigma_1),\mathrm{St}^{P(\sigma_1)\cap P(\sigma_2)}_{Q_2}(\sigma_2)).$$

This holds if $Q_2\subset P(\sigma_1)$, $\Delta(\sigma_1)=\Delta(Q_1)\cup\Delta_{P(\sigma_1)\cap P(\sigma_2)}$ and $P\subset P(\sigma_1)\cap P(\sigma_2)$, and otherwise the extension group is zero. Note that we always have $P\subset P(\sigma_1)\cap P(\sigma_2)$ since both $P(\sigma_1)$ and $P(\sigma_2)$ contain P. Since $Q_1\subset P(\sigma_1)$, $\Delta(Q_1)\cup\Delta_{P(\sigma_1)\cap P(\sigma_2)}=(\Delta(\sigma_1)\cap\Delta(Q_1))\cup(\Delta(\sigma_1)\cap\Delta(\sigma_2))=\Delta(\sigma_1)\cap(\Delta_{Q_1}\cup\Delta(\sigma_2))$. (Recall that $P(\sigma_1)$ is the parabolic subgroup corresponding to $\Delta(\sigma_1)$.) Hence, we have $\Delta(\sigma_1)=\Delta(Q_1)\cup\Delta(P(\sigma_1)\cap P(\sigma_2))$ if and only if $\Delta(\sigma_1)\subset\Delta_{Q_1}\cup\Delta(\sigma_2)$. We get the proposition. □

Therefore, to calculate the extension groups, we may assume $P(\sigma_1)=P(\sigma_2)=G$.

3.3. Extensions between generalized Steinberg modules

We assume that $P(\sigma_1)=P(\sigma_2)=G$, and we continue the calculation of the extension groups.

Lemma 3.7. *Let Q_{11},Q_{12},Q_2 be parabolic subgroups and $\alpha\in\Delta_{Q_{12}}$ such that $\Delta_{Q_{11}}=\Delta_{Q_{12}}\setminus\{\alpha\}$. Then we have*

$$\mathrm{Ext}^i_{\mathcal{H}}(\mathrm{St}_{Q_{11}}(\sigma_1),\mathrm{St}_{Q_2}(\sigma_2))\simeq\begin{cases}\mathrm{Ext}^{i-1}_{\mathcal{H}}(\mathrm{St}_{Q_{12}}(\sigma_1),\mathrm{St}_{Q_2}(\sigma_2)) & (\alpha\in\Delta_{Q_2}),\\ \mathrm{Ext}^{i+1}_{\mathcal{H}}(\mathrm{St}_{Q_{12}}(\sigma_1),\mathrm{St}_{Q_2}(\sigma_2)) & (\alpha\notin\Delta_{Q_2}).\end{cases}$$

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Proof. Let P_1 be a parabolic subgroup corresponding to $\Delta \setminus \{\alpha\}$. First, we prove that there exists an exact sequence

$$0 \to \mathrm{St}_{Q_{12}}(\sigma_1) \to I_{P_1}(\mathrm{St}_{Q_{11}}^{P_1}(\sigma_1)) \to \mathrm{St}_{Q_{11}}(\sigma_1) \to 0. \tag{3.1}$$

We start with the following exact sequence.

$$0 \to \sum_{P_1 \supset Q \supsetneq Q_{11}} I_Q^{P_1}(e_Q(\sigma_1)) \to I_{Q_{11}}^{P_1}(e_{Q_{11}}(\sigma_1)) \to \mathrm{St}_{Q_{11}}^{P_1}(\sigma_1) \to 0,$$

Apply I_{P_1} to this exact sequence. Then we have

$$0 \to \sum_{P_1 \supset Q \supsetneq Q_{11}} I_Q(e_Q(\sigma_1)) \to I_{Q_{11}}(e_{Q_{11}}(\sigma_1)) \to I_{P_1}(\mathrm{St}_{Q_{11}}^{P_1}(\sigma_1)) \to 0.$$

Hence, we get the following commutative diagram with exact columns:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\sum_{P_1 \supset Q \supsetneq Q_{11}} I_Q(e_Q(\sigma_1)) & \hookrightarrow & \sum_{Q \supsetneq Q_{11}} I_Q(e_Q(\sigma)) \\
\downarrow & & \downarrow \\
I_{Q_{11}}(e_{Q_{11}}(\sigma_1)) & = & I_{Q_{11}}(e_{Q_{11}}(\sigma_1)) \\
\downarrow & & \downarrow \\
I_{P_1}(\mathrm{St}_{Q_{11}}^{P_1}(\sigma_1)) & \longrightarrow & \mathrm{St}_{Q_{11}}(\sigma_1) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Hence, $I_{P_1}(\mathrm{St}_{Q_{11}}^{P_1}(\sigma_1)) \to \mathrm{St}_{Q_{11}}(\sigma_1)$ is surjective, and the kernel is isomorphic to

$$\sum_{Q \supsetneq Q_{11}} I_Q(e_Q(\sigma)) \Big/ \sum_{P_1 \supset Q \supsetneq Q_{11}} I_Q(e_Q(\sigma_1))$$

by the snake lemma.

We prove:

(1) $I_{Q_{12}}(e_{Q_{12}}(\sigma_1)) + \sum_{P_1 \supset Q \supsetneq Q_{11}} I_Q(e_Q(\sigma_1)) = \sum_{Q \supsetneq Q_{11}} I_Q(e_Q(\sigma))$.

(2) $I_{Q_{12}}(e_{Q_{12}}(\sigma_1)) \cap \sum_{P_1 \supset Q \supsetneq Q_{11}} I_Q(e_Q(\sigma_1)) = \sum_{Q \supsetneq Q_{12}} I_Q(e_Q(\sigma))$.

We prove (1). Since $Q_{12} \supsetneq Q_{11}$, $I_{Q_{12}}(e_{Q_{12}}(\sigma_1))$ is contained in the right-hand side. Obviously, $\sum_{P_1 \supset Q \supsetneq Q_{11}} I_Q(e_Q(\sigma_1))$ is also contained in the right-hand side. Hence, $I_{Q_{12}}(e_{Q_{12}}(\sigma_1)) + \sum_{P_1 \supset Q \supsetneq Q_{11}} I_Q(e_Q(\sigma_1)) \subset \sum_{Q \supsetneq Q_{11}} I_Q(e_Q(\sigma))$. Take $Q \supsetneq Q_{11}$, and we prove that $I_Q(e_Q(\sigma_1)) \subset I_{Q_{12}}(e_{Q_{12}}(\sigma_1)) + \sum_{P_1 \supset Q \supsetneq Q_{11}} I_Q(e_Q(\sigma_1))$. If $P_1 \supset Q$, then it is obvious. We assume that $P_1 \not\supset Q$. Since $\Delta_{P_1} = \Delta \setminus \{\alpha\}$, this is equivalent to $\alpha \in \Delta_Q$. Hence, $\Delta_Q \supset \Delta_{Q_{11}} \cup \{\alpha\} = \Delta_{Q_{12}}$. Therefore, we have $Q \supset Q_{12}$. Hence, $I_Q(e_Q(\sigma_1)) \subset I_{Q_{12}}(e_{Q_{12}}(\sigma_1))$.

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We prove (2). By [2, Lemma 3.10], the left-hand side is

$$\sum_{P_1 \supset Q \supsetneq Q_{11}} I_{\langle Q, Q_{12}\rangle}(e_{\langle Q,Q_{12}\rangle}(\sigma_1)),$$

where $\langle Q, Q_{12}\rangle$ is the subgroup generated by Q and Q_{12}. We prove

$$\{\langle Q, Q_{12}\rangle \mid P_1 \supset Q \supsetneq Q_{11}\} = \{Q \mid Q \supsetneq Q_{12}\}.$$

If Q satisfies $P_1 \supset Q \supsetneq Q_{11}$, then there exists $\beta \in \Delta_Q \setminus \Delta_{Q_{11}}$. We have $\beta \in \Delta_Q \subset \Delta_{P_1} = \Delta \setminus \{\alpha\}$. Therefore, we have $\beta \neq \alpha$. Hence, $\beta \notin \Delta_{Q_{11}} \cup \{\alpha\} = \Delta_{Q_{12}}$. On the other hand, $\beta \in \Delta_Q \subset \Delta_{\langle Q,Q_{12}\rangle}$. Namely, we have $\beta \in \Delta_{\langle Q,Q_{12}\rangle} \setminus \Delta_{Q_{12}}$. Obviously, $\langle Q, Q_{12}\rangle \supset Q_{12}$. Therefore, we get $\langle Q, Q_{12}\rangle \supsetneq Q_{12}$.

On the other hand, assume that $Q \supsetneq Q_{12}$. Then $\alpha \in \Delta_Q$ since $\alpha \in \Delta_{Q_{12}}$. Let Q' be the parabolic subgroup corresponding to $\Delta_Q \setminus \{\alpha\}$. Then we have $\Delta_{Q'} \subset \Delta \setminus \{\alpha\} = \Delta_{P_1}$ and $\Delta_{Q'} = \Delta_Q \setminus \{\alpha\} \supsetneq \Delta_{Q_{12}} \setminus \{\alpha\} = \Delta_{Q_{11}}$. Hence, $P_1 \supset Q' \supsetneq Q_{11}$. We have $\Delta_{\langle Q',Q_{12}\rangle} = \Delta_{Q'} \cup \Delta_{Q_{12}} = (\Delta_Q \setminus \{\alpha\}) \cup \Delta_{Q_{11}} \cup \{\alpha\} = \Delta_Q \cup \Delta_{Q_{11}} \cup \{\alpha\}$. This is Δ_Q since $\Delta_Q \supset \Delta_{Q_{12}} = \Delta_{Q_{11}} \cup \{\alpha\}$. Hence, $Q = \langle Q', Q_{12}\rangle$. We get the existence of the exact sequence (3.1).

Assume that $\alpha \in \Delta_{Q_2}$. Then $\alpha \in \Delta_{Q_2}$ and $\alpha \notin \Delta_{Q_2 \cap P_1}$. Hence, $\Delta_{Q_2} \neq \Delta_{Q_2 \cap P_1} \cup \Delta_P$. Therefore, $R_{P_1}(\mathrm{St}_{Q_2}(\sigma_2)) = 0$ by [4, Proposition 5.11]. We have an exact sequence

$$\begin{aligned}&\mathrm{Ext}^i_{\mathcal{H}}(I_{P_1}(\mathrm{St}^{P_1}_{Q_{11}}(\sigma_1)), \mathrm{St}_{Q_2}(\sigma_2)) \to \mathrm{Ext}^i_{\mathcal{H}}(\mathrm{St}_{Q_{12}}(\sigma_1), \mathrm{St}_{Q_2}(\sigma_2))\\ &\to \mathrm{Ext}^{i+1}_{\mathcal{H}}(\mathrm{St}_{Q_{11}}(\sigma_1), \mathrm{St}_{Q_2}(\sigma_2)) \to \mathrm{Ext}^{i+1}_{\mathcal{H}}(I_{P_1}(\mathrm{St}^{P_1}_{Q_{11}}(\sigma_1)), \mathrm{St}_{Q_2}(\sigma_2)).\end{aligned}$$

Since $R_{P_1}(\mathrm{St}_{Q_2}(\sigma_2)) = 0$, for any j, we have

$$\mathrm{Ext}^j_{\mathcal{H}}(I_{P_1}(\mathrm{St}_{Q_{12}}(\sigma_1)), \mathrm{St}_{Q_2}(\sigma_2)) = \mathrm{Ext}^j_{\mathcal{H}}(\mathrm{St}_{Q_{12}}(\sigma_1), R_{P_1}(\mathrm{St}_{Q_2}(\sigma_2))) = 0.$$

Therefore, we get

$$\mathrm{Ext}^i_{\mathcal{H}}(\mathrm{St}_{Q_{12}}(\sigma_1), \mathrm{St}_{Q_2}(\sigma_2)) \simeq \mathrm{Ext}^{i+1}_{\mathcal{H}}(\mathrm{St}_{Q_{11}}(\sigma_1), \mathrm{St}_{Q_2}(\sigma_2)).$$

Next assume that $\alpha \notin \Delta_{Q_2}$. Let Q^c_{11} (resp. Q^c_{12}, Q^c_2) be the parabolic subgroup corresponding to $(\Delta \setminus \Delta_{Q_{11}}) \cup \Delta_P$ (resp. $(\Delta \setminus \Delta_{Q_{12}}) \cup \Delta_P, (\Delta \setminus \Delta_{Q_2}) \cup \Delta_P$). Let $\iota = \iota_G \colon \mathcal{H} \to \mathcal{H}$ be the involution defined by $\iota(T_w) = (-1)^{\ell(w)} T^*_w$, and set $\pi^\iota = \pi \circ \iota$ for an $\mathcal{H}$-module π. Then we have

$$\begin{aligned}\mathrm{Ext}^i_{\mathcal{H}}(\mathrm{St}_{Q_{11}}(\sigma_1), \mathrm{St}_{Q_2}(\sigma_2)) &\simeq \mathrm{Ext}^i_{\mathcal{H}}((\mathrm{St}_{Q_{11}}(\sigma_1))^\iota, (\mathrm{St}_{Q_2}(\sigma_2))^\iota)\\ &\simeq \mathrm{Ext}^i_{\mathcal{H}}(\mathrm{St}_{Q^c_{11}}(\sigma^{\iota_P}_{1,\ell-\ell_P}), \mathrm{St}_{Q^c_2}(\sigma^{\iota_P}_{2,\ell-\ell_P}))\end{aligned}$$

by [2, Theorem 3.6]. Now we have $\alpha \in \Delta_{Q^c_2}$. Applying the lemma (where $Q_{11} = Q^c_{12}$ and $Q_{12} = Q^c_{11}$), we have

$$\begin{aligned}\mathrm{Ext}^i_{\mathcal{H}}(\mathrm{St}_{Q^c_{11}}(\sigma^{\iota_P}_{1,\ell-\ell_P}), \mathrm{St}_{Q^c_2}(\sigma^{\iota_P}_{2,\ell-\ell_P})) &\simeq \mathrm{Ext}^{i-1}_{\mathcal{H}}(\mathrm{St}_{Q^c_{12}}(\sigma^{\iota_P}_{1,\ell-\ell_P}), \mathrm{St}_{Q^c_2}(\sigma^{\iota_P}_{2,\ell-\ell_P}))\\ &\simeq \mathrm{Ext}^{i-1}_{\mathcal{H}}((\mathrm{St}_{Q_{12}}(\sigma_1))^\iota, (\mathrm{St}_{Q_2}(\sigma_2))^\iota)\\ &\simeq \mathrm{Ext}^{i-1}_{\mathcal{H}}(\mathrm{St}_{Q_{12}}(\sigma_1), \mathrm{St}_{Q_2}(\sigma_2)).\end{aligned}$$

We get the lemma. □

For sets X, Y, let $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$ be the symmetric difference.

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Theorem 3.8. *We have*

$$\mathrm{Ext}^i_{\mathcal{H}}(\mathrm{St}_{Q_1}(\sigma_1),\mathrm{St}_{Q_2}(\sigma_2)) \simeq \mathrm{Ext}^{i-\#(\Delta_{Q_1}\triangle\Delta_{Q_2})}_{\mathcal{H}}(e_G(\sigma_1),e_G(\sigma_2)).$$

Proof. By applying Lemma 3.7 several times, we have

$$\mathrm{Ext}^i_{\mathcal{H}}(\mathrm{St}_{Q_1}(\sigma_1),\mathrm{St}_{Q_2}(\sigma_2)) \simeq \mathrm{Ext}^{i-r_1}_{\mathcal{H}}(e_G(\sigma_1),\mathrm{St}_{Q_2}(\sigma_2)),$$

where $r_1 = \#\{\alpha \in \Delta\setminus\Delta_{Q_1} \mid \alpha\in\Delta_{Q_2}\} - \#\{\alpha\in\Delta\setminus\Delta_{Q_1} \mid \alpha\notin\Delta_{Q_2}\}$. Set $Q_2' = n_{w_Gw_{Q_2}}Q_2^{\mathrm{op}}n_{w_Gw_{Q_2}}^{-1}$. Then by Lemma 3.5, we get

$$\begin{aligned}\mathrm{Ext}^{i-r_1}_{\mathcal{H}}(e_G(\sigma_1),\mathrm{St}_{Q_2}(\sigma_2)) &\simeq \mathrm{Ext}^{i-r_1}_{\mathcal{H}}((\mathrm{St}_{Q_2}(\sigma_2))^*,e_G(\sigma_1)^*)\\ &\simeq \mathrm{Ext}^{i-r_1}_{\mathcal{H}}(\mathrm{St}_{Q_2'}(\sigma_2^*),e_G(\sigma_1^*)).\end{aligned}$$

Again using Lemma 3.7, we have

$$\mathrm{Ext}^{i-r_1}_{\mathcal{H}}(\mathrm{St}_{Q_2'}(\sigma_2^*),e_G(\sigma_1^*)) \simeq \mathrm{Ext}^{i-r_1-r_2}_{\mathcal{H}}(e_G(\sigma_2^*),e_G(\sigma_1^*)),$$

where $r_2 = \#(\Delta\setminus\Delta_{Q_2'}) = \#(\Delta\setminus\Delta_{Q_2})$. Applying Lemma 3.5 again, we get

$$\mathrm{Ext}^{i-r_1-r_2}_{\mathcal{H}}(e_G(\sigma_2^*),e_G(\sigma_1^*)) \simeq \mathrm{Ext}^{i-r_1-r_2}_{\mathcal{H}}(e_G(\sigma_1),e_G(\sigma_2)).$$

Since $r_1+r_2 = \#(\Delta_{Q_1}\triangle\Delta_{Q_2})$, we get the lemma. □

Recall that the trivial module $\mathbf{1}$ is defined by $\mathbf{1}(T_w) = q_w$. We denote the restriction of $\mathbf{1}$ to $\mathcal{H}_{\mathrm{aff}}$ by $\mathbf{1}_{\mathcal{H}_{\mathrm{aff}}}$.

Corollary 3.9. *We have* $\mathrm{Ext}^i_{\mathcal{H}_{\mathrm{aff}}}(\mathbf{1}_{\mathcal{H}_{\mathrm{aff}}},\mathbf{1}_{\mathcal{H}_{\mathrm{aff}}}) = 0$ *for* $i>0$.

Proof. Let $\overline{\mathcal{H}}_{\mathrm{aff}}$ be the quotient of $\mathcal{H}_{\mathrm{aff}}$ by the ideal generated by $\{T_t - 1 \mid t\in Z_\kappa\cap W_{\mathrm{aff}}(1)\}$. Then this is the Hecke algebra attached to the Coxeter system $(W_{\mathrm{aff}},S_{\mathrm{aff}})$. Let π_2 (resp. $\overline{\pi}_1$) be an an $\mathcal{H}_{\mathrm{aff}}$-module (resp. $\overline{\mathcal{H}}_{\mathrm{aff}}$-module). Then we have $\mathrm{Hom}_{\mathcal{H}_{\mathrm{aff}}}(\overline{\pi}_1,\pi_2) = \mathrm{Hom}_{\overline{\mathcal{H}}_{\mathrm{aff}}}(\overline{\pi}_1,\pi_2^{Z_\kappa})$. In particular, $\pi_2\mapsto\pi_2^{Z_\kappa}$ sends injective $\mathcal{H}$-modules to injective $\overline{\mathcal{H}}_{\mathrm{aff}}$-modules. Since the functor $\pi_2\mapsto\pi_2^{Z_\kappa}$ is exact, we have $\mathrm{Ext}^i_{\mathcal{H}_{\mathrm{aff}}}(\overline{\pi}_1,\pi_2)\simeq\mathrm{Ext}^i_{\overline{\mathcal{H}}_{\mathrm{aff}}}(\overline{\pi}_1,\pi_2^{Z_\kappa})$. Therefore, we have $\mathrm{Ext}^i_{\mathcal{H}_{\mathrm{aff}}}(\mathbf{1}_{\mathcal{H}_{\mathrm{aff}}},\mathbf{1}_{\mathcal{H}_{\mathrm{aff}}})\simeq\mathrm{Ext}^i_{\overline{\mathcal{H}}_{\mathrm{aff}}}(\mathbf{1}_{\overline{\mathcal{H}}_{\mathrm{aff}}},\mathbf{1}_{\overline{\mathcal{H}}_{\mathrm{aff}}})$. Consider the root system which defines $(W_{\mathrm{aff}},S_{\mathrm{aff}})$, and let H be the split simply-connected semisimple group with this root system. Then the affine Hecke algebra attached to H is $\overline{\mathcal{H}}_{\mathrm{aff}}$. Let $\mathcal{H}'$ be the pro-p-Iwahori Hecke algebra for H. Then $H' = H$ by [5, II.3.Proposition]; hence, $\mathcal{H}'_{\mathrm{aff}} = \mathcal{H}'$. Therefore, the above argument implies that $\mathrm{Ext}^i_{\mathcal{H}'}(\mathbf{1}_{\mathcal{H}'},\mathbf{1}_{\mathcal{H}'})\simeq\mathrm{Ext}^i_{\overline{\mathcal{H}}_{\mathrm{aff}}}(\mathbf{1}_{\overline{\mathcal{H}}_{\mathrm{aff}}},\mathbf{1}_{\overline{\mathcal{H}}_{\mathrm{aff}}})$.

Therefore, it is sufficient to prove $\mathrm{Ext}^i_{\mathcal{H}}(\mathbf{1}_G,\mathbf{1}_G) = 0$ for $i>0$ assuming G is a split, simply connected semisimple group. By [15, Proposition 6.20], the projective dimension of $\mathbf{1}_G$ is equal to the semisimple rank of G, namely $\#\Delta$. Therefore, $\mathrm{Ext}^{i+\#\Delta}_{\mathcal{H}}(\mathbf{1}_G,\mathrm{St}_B(\mathbf{1}_B)) = 0$ for $i>0$. The left-hand side is $\mathrm{Ext}^i_{\mathcal{H}}(\mathbf{1}_G,\mathbf{1}_G)$ by Theorem 3.8. □

3.4. Extension between extensions

Let P be a parabolic subgroup and σ an $\mathcal{H}_P$-module which has the extension $e_G(\sigma)$ to $\mathcal{H}$. In particular, Δ_P and $\Delta\setminus\Delta_P$ are orthogonal to each other. Let P_2 be a parabolic subgroup corresponding to $\Delta\setminus\Delta_P$. Let $J\subset\mathcal{H}$ be an ideal generated by $\{T_w^*-1\mid w\in W_{\mathrm{aff},P_2}(1)\}$. Then $e_G(\sigma)(J)=0$. Hence, for any module π of $\mathcal{H}$, we have

$$\mathrm{Hom}_{\mathcal{H}}(e(\sigma),\pi)=\mathrm{Hom}_{\mathcal{H}/J}(e(\sigma),\{v\in\pi\mid vJ=0\}).$$

Note that $T_w^{P_2}\mapsto T_w$ defines the injection $\mathcal{H}_{\mathrm{aff},P_2}\to\mathcal{H}$ since the restriction of ℓ on $W_{\mathrm{aff},P_2}(1)$ is ℓ_{P_2}. Since any generator of J is in $\mathcal{H}_{\mathrm{aff},P_2}$, we have $\{v\in\pi\mid vJ=0\}=\{v\in\pi\mid v(J\cap\mathcal{H}_{\mathrm{aff},P_2})=0\}$. Since the trivial representation $\mathbf{1}_{\mathcal{H}_{\mathrm{aff},P_2}}$ of $\mathcal{H}_{\mathrm{aff},P_2}$ is isomorphic to $\mathcal{H}_{\mathrm{aff},P_2}/(J\cap\mathcal{H}_{\mathrm{aff},P_2})$, we get

$$\{v\in\pi\mid v(J\cap\mathcal{H}_{\mathrm{aff},P_2})=0\}=\mathrm{Hom}_{\mathcal{H}_{\mathrm{aff},P_2}}(\mathbf{1}_{\mathcal{H}_{\mathrm{aff},P_2}},\pi).$$

Hence, we get

$$\mathrm{Hom}_{\mathcal{H}}(e_G(\sigma),\pi)=\mathrm{Hom}_{\mathcal{H}/J}(e_G(\sigma),\mathrm{Hom}_{\mathcal{H}_{\mathrm{aff},P_2}}(\mathbf{1}_{\mathcal{H}_{\mathrm{aff},P_2}},\pi)).$$

This isomorphism can be generalized as

$$\mathrm{Hom}_{\mathcal{H}}(\pi_1,\pi)=\mathrm{Hom}_{\mathcal{H}/J}(\pi_1,\mathrm{Hom}_{\mathcal{H}_{\mathrm{aff},P_2}}(\mathbf{1}_{\mathcal{H}_{\mathrm{aff},P_2}},\pi))$$

for any $\mathcal{H}/J$-module π_1. In particular, $\pi\mapsto\mathrm{Hom}_{\mathcal{H}_{\mathrm{aff},P_2}}(\mathbf{1}_{\mathcal{H}_{\mathrm{aff},P_2}},\pi)$ from the category of $\mathcal{H}$-modules to the category of $\mathcal{H}/J$-modules preserves injective modules. Hence, we have a spectral sequence

$$\mathrm{Ext}^i_{\mathcal{H}/J}(e_G(\sigma),\mathrm{Ext}^j_{\mathcal{H}_{\mathrm{aff},P_2}}(\mathbf{1}_{\mathcal{H}_{\mathrm{aff},P_2}},\pi))\Rightarrow\mathrm{Ext}^{i+j}_{\mathcal{H}}(e(\sigma),\pi).$$

Now let σ_1,σ_2 be $\mathcal{H}_P$-modules such that both have the extensions $e_G(\sigma_1),e_G(\sigma_2)$ to $\mathcal{H}$. Since $e_G(\sigma_2)|_{\mathcal{H}_{\mathrm{aff},P_2}}$ is a direct sum of the trivial representations, we have

$$\mathrm{Ext}^j_{\mathcal{H}_{\mathrm{aff},P_2}}(\mathbf{1}_{\mathcal{H}_{\mathrm{aff},P_2}},e(\sigma_2))=0$$

for $j>0$ by Corollary 3.9. Hence,

$$\mathrm{Ext}^i_{\mathcal{H}}(e_G(\sigma_1),e_G(\sigma_2))\simeq\mathrm{Ext}^i_{\mathcal{H}/J}(e_G(\sigma_1),e_G(\sigma_2)).$$

Lemma 3.10 ([6, Proposition 3.5]). *Let I be the ideal of $\mathcal{H}_P$ generated by $\{T_\lambda^P-1\mid\lambda\in\Lambda(1)\cap W_{\mathrm{aff},P_2}(1)\}$. Then we have $\mathcal{H}/J\simeq\mathcal{H}_P/I$.*

Therefore, we get

$$\mathrm{Ext}^i_{\mathcal{H}}(e_G(\sigma_1),e_G(\sigma_2))\simeq\mathrm{Ext}^i_{\mathcal{H}_P/I}(\sigma_1,\sigma_2).$$

Proposition 3.11. *Set $W'_{\mathrm{aff}}=W_{\mathrm{aff},P}$, $S'_{\mathrm{aff}}=S_{\mathrm{aff},P}$, $W'=W_P/(\Lambda\cap W_{\mathrm{aff},P_2})$, $\Omega'=\Omega_P/(\Lambda\cap W_{\mathrm{aff},P_2})$, $W'(1)=W_P(1)/(\Lambda(1)\cap W_{\mathrm{aff},P_2}(1))$, $Z'_\kappa=Z_\kappa/(Z_\kappa\cap W_{\mathrm{aff},P_2}(1))$. Then $(W'_{\mathrm{aff}},S'_{\mathrm{aff}},\Omega',W',W'(1),Z'_\kappa)$ satisfies the condition of subsection 2.1, and the attached algebra is $\mathcal{H}/I$. Moreover, Ω' is commutative.*

Proof. Since $\Delta=\Delta_P\cup\Delta_{P_2}$ is the orthogonal decomposition, we have $W_{\mathrm{aff}}=W_{\mathrm{aff},P}\times W_{\mathrm{aff},P_2}$ and $S_{\mathrm{aff}}=S_{\mathrm{aff},P}\cup S_{\mathrm{aff},P_2}$. The pair $(W'_{\mathrm{aff}},S'_{\mathrm{aff}})=(W_{\mathrm{aff},P},S_{\mathrm{aff},P})$ is a Coxeter system, and Ω_P acts on it. Since W_{aff,P_2} commutes with $W_{\mathrm{aff},P}$, this gives the action of Ω' on $(W'_{\mathrm{aff}},S'_{\mathrm{aff}})$. We have $W'_{\mathrm{aff}}\subset W_P$, and since $W_{\mathrm{aff},P}\cap W_{\mathrm{aff},P_2}$ is trivial, we have the embedding $W'_{\mathrm{aff}}\subset W'$. We also have $\Omega_P\subset W'$. Since $W_P=W_{\mathrm{aff},P}\Omega_P$, we have $W' = W'_{\mathrm{aff}}\Omega'$. Since $W_{\mathrm{aff},P} \cap \Omega_P = \{1\}$, we have $W'_{\mathrm{aff}} \cap \Omega' = \{1\}$ in W'. Hence, $W' = W'_{\mathrm{aff}} \rtimes \Omega'$. Since Z_κ is finite and commutative, Z'_κ is also a finite commutative group. The existence of the exact sequence

$$1 \to Z'_\kappa \to W'(1) \to W' \to 1$$

is obvious. Note that the length function $\ell' \colon W'(1) \to \mathbb{Z}_{\geq 0}$ is given by $\ell_P \colon W'_P(1) \to \mathbb{Z}_{\geq 0}$ since $S'_{\mathrm{aff}} = S_{\mathrm{aff},P}$ and Ω' is the image of Ω_P.

We put $q'_s = q_s$ for $s \in S_{\mathrm{aff},P}$. (Note that $q_s = q_{s,P}$ since $\Delta = \Delta_P \cup \Delta_{P_2}$ is an orthogonal decomposition.) For $s \in \mathrm{Ref}(W'(1))$, take its lift $\widetilde{s} \in \mathrm{Ref}(W_P(1))$ and let c'_s be the image of $c_{\widetilde{s}}$ in $C[Z'_\kappa]$. We prove that this is well-defined. Let $\widetilde{s}'$ be another lift, and take $\lambda \in \Lambda(1) \cap W_{\mathrm{aff},P_2}(1)$ such that $\widetilde{s}' = \widetilde{s}\lambda$. The image of $\widetilde{s}$ in W is in $\mathrm{Ref}(W_P) \subset W_{P,\mathrm{aff}}$ since $S_{P,\mathrm{aff}} \subset W_{P,\mathrm{aff}}$ and $W_{P,\mathrm{aff}}$ is normal. (Recall that a reflection is an element which is conjugate to a simple reflection.) Let $\overline{\lambda}$ be the image of λ in Λ. Since $\widetilde{s},\widetilde{s}' \in W_{P,\mathrm{aff}}(1)$, we have $\overline{\lambda} \in \Lambda \cap W_{\mathrm{aff},P_2} \cap W_{\mathrm{aff},P} = \{1\}$. Hence, $\lambda \in Z_\kappa$. Since $\lambda \in W_{\mathrm{aff},P_2}(1)$, we have $\lambda \in Z_\kappa \cap W_{\mathrm{aff},P_2}(1)$. Hence, the image of $c_{\widetilde{s}'} = c_{\widetilde{s}\lambda} = c_{\widetilde{s}}\lambda$ is the same as that of $c_{\widetilde{s}}$ in $C[Z'_\kappa]$.

We get the parameter (q',c') and let $\mathcal{H}' = \bigoplus_{w \in W'(1)} T'_w$ be the attached algebra. Consider the linear map $\Phi \colon \mathcal{H}_P \to \mathcal{H}'$ defined by $T^P_w \mapsto T'_{\overline{w}}$, where $w \in W_P(1)$ and $\overline{w} \in W'(1)$ is the image of w.

First, we prove that the map Φ preserves the relations. Let $s \in W_P(1)$ be a lift of an affine simple reflection in $S_{\mathrm{aff},P}$. Then we have $(T^P_s)^2 = q_s T^P_{s^2} + c_s T^P_s$. Let $\overline{s}$ be the image of s in $W'(1)$. Then we have $(T'_{\overline{s}})^2 = q'_{\overline{s}} T'_{\overline{s}^2} + c'_{\overline{s}} T'_{\overline{s}}$. The definition of (q',c') says $q'_{\overline{s}} = q_s$ and $\Phi(c_s) = c'_{\overline{s}}$. Hence, Φ preserves the quadratic relations. The compatibility between ℓ_P and ℓ' implies that Φ preserves the braid relations.

Obviously, Φ is surjective. We prove that $\operatorname{Ker}\Phi = I$. Clearly, we have $I \subset \operatorname{Ker}\Phi$. Let $\sum_{w \in W_P(1)} c_w T_w \in \operatorname{Ker}\Phi$, where $c_w \in C$. Fix a section x of $W_P(1) \to W'(1)$. Then we have $\sum_{w \in W_P(1)} c_w T^P_w = \sum_{w \in W'(1)} \sum_{\lambda \in \Lambda(1) \cap W_{\mathrm{aff},P_2}(1)} c_{x(w)\lambda} T^P_{x(w)\lambda}$. Hence,

$$0 = \Phi\left(\sum_{w \in W_P(1)} c_w T^P_w\right) = \sum_{w \in W'(1)} \left(\sum_{\lambda \in \Lambda(1) \cap W_{\mathrm{aff},P_2}(1)} c_{x(w)\lambda}\right) T'_w.$$

Therefore, for each $w \in W'(1)$, we have $\sum_{\lambda \in \Lambda(1) \cap W_{\mathrm{aff},P_2}(1)} c_{x(w)\lambda} = 0$. Hence,

$$\begin{aligned}
&\sum_{w \in W_P(1)} c_w T^P_w \\
&= \sum_{w \in W'(1)} \sum_{\lambda \in \Lambda(1) \cap W_{\mathrm{aff},P_2}(1)} c_{x(w)\lambda} T^P_{x(w)\lambda} \\
&= \sum_{w \in W'(1)} \left(\sum_{\lambda \in \Lambda(1) \cap W_{\mathrm{aff},P_2}(1)} c_{x(w)\lambda} T^P_{x(w)\lambda} - \sum_{\lambda \in \Lambda(1) \cap W_{\mathrm{aff},P_2}(1)} c_{x(w)\lambda} T^P_{x(w)}\right) \\
&= \sum_{w \in W'(1)} \left(\sum_{\lambda \in \Lambda(1) \cap W_{\mathrm{aff},P_2}(1)} c_{x(w)\lambda} T^P_{x(w)} (T^P_\lambda - 1)\right) \in I.
\end{aligned}$$

Finally, Ω' is commutative since Ω_P is commutative. □

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Remark 3.12. The data do not come from a reductive group in general.

3.5. Example

Let $G = \mathrm{PGL}_2$. We have $\Lambda(1) \simeq F^\times/(1+(\varpi)) \simeq \mathbb{Z}\times\kappa^\times$. Consider $\widetilde{G} = \mathrm{SL}_2$. Then G' is the image of $\widetilde{G} \to G$ [5, II.4 Proposition]. By this description, we have $\Lambda(1)\cap W_{\mathrm{aff}}(1) = \{\lambda^2 \mid \lambda \in \Lambda(1)\}$. Therefore, with the notation in Proposition 3.11, we have $W'_{\mathrm{aff}} = \{1\}$, $S'_{\mathrm{aff}} = \emptyset$, $W'(1) = \Lambda(1)/\{\lambda^2 \mid \lambda\in\Lambda(1)\}$, $Z_\kappa = Z_\kappa/\{t^2 \mid t\in Z_\kappa\}$. We have $\mathcal{H}_B/I = C[W'(1)]$.

Consider the trivial module $\mathbf{1}_G$. Then we have $\mathbf{1}_G = e_G(\mathbf{1}_B)$, and we have

$$\mathrm{Ext}^i_{\mathcal{H}}(\mathbf{1}_G,\mathbf{1}_G) \simeq \mathrm{Ext}^i_{\mathcal{H}_B/I}(\mathbf{1}_B,\mathbf{1}_B) \simeq \mathrm{Ext}^i_{C[W'(1)]}(\mathbf{1}_B,\mathbf{1}_B) = H^i(W'(1),C).$$

Here, C is the trivial $W'(1)$-module. Since the group $W'(1)$ is a 2-group, this cohomology is zero if the characteristic of C is not 2. However, if the characteristic of C is 2, since $W'(1) \simeq \mathbb{Z}/2\mathbb{Z}$ $(p=2)$ or $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ $(p\ne 2)$, $H^i(W'(1),C)\ne 0$ if i is even. Therefore, we have infinitely many i with $\mathrm{Ext}^i_{\mathcal{H}}(\mathbf{1}_G,\mathbf{1}_G)\ne 0$. This recovers Koziol's example [12, Example 6.2].

3.6. Summary

Now we get a reduction. The Ext^1 between simple modules is equal to Ext^{1-r} between supersingular simple modules for some $r\ge 0$ or zero. In particular, if $r\ge 2$, then Ext^1 between simple modules is zero. If $r=1$, then $\mathrm{Ext}^{1-r} = \mathrm{Hom}$, so it is zero or one-dimensional. If $r=0$, we have to calculate Ext^1 between supersingular simple modules. Therefore, the only remaining task is to calculate Ext^1 between supersingular simple modules.

4. Ext^1 between supersingular modules

In this section, we fix data $(W_{\mathrm{aff}},S_{\mathrm{aff}},\Omega,W,W(1),Z_\kappa)$ and let $\mathcal{H}$ be the algebra attached to this data. We do not assume that these data come from a group. We also assume:

- our parameter q_s is zero.
- $\#Z_\kappa$ is prime to p.

As in subsection 2.9, let $W^{\mathrm{aff}}(1)$ be the inverse image of W_{aff} in $W(1)$, and put $\mathcal{H}^{\mathrm{aff}} = \bigoplus_{w\in W^{\mathrm{aff}}(1)} CT_w$.

For a character χ of Z_κ and $w\in W$, we define $(w\chi)(t) = \chi(\widetilde{w}^{-1}t\widetilde{w})$ where $\widetilde{w}\in W(1)$ is a lift of W. Since Z_κ is commutative, this does not depend on a lift $\widetilde{w}$ and defines a character $w\chi$ of Z_κ. For a character Ξ of $\mathcal{H}^{\mathrm{aff}}$ and $\omega\in\Omega(1)$, we write $\Xi\omega$ for the character $T_w \mapsto \Xi(T_{\omega w\omega^{-1}})$ for $w\in W^{\mathrm{aff}}(1)$. Since $\Xi\omega$ only depends on the image $\overline{\omega}$ of ω in Ω, we also write $\Xi\overline{\omega}$.

Note that, since $s\cdot c_{\widetilde{s}} = c_{\widetilde{s}}$ for $s\in S_{\mathrm{aff}}$ with a lift $\widetilde{s}$ by the conditions of the parameter c, we have $(s\chi)(c_{\widetilde{s}}) = \chi(c_{\widetilde{s}})$.

4.1. Ext^1 for $\mathcal{H}^{\mathrm{aff}}$

Let χ,χ' be characters of Z_κ and $J\subset S_{\mathrm{aff},\chi}, J'\subset S_{\mathrm{aff},\chi'}$ subsets. Then we have characters $\Xi = \Xi_{J,\chi}$, $\Xi' = \Xi_{J',\chi'}$ of $\mathcal{H}^{\mathrm{aff}}$. We calculate $\mathrm{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\Xi,\Xi')$.

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To express the space of extensions, we need some notation. For each $s\in S_{\mathrm{aff}}$, let C_s be the set of functions a on $\{\widetilde{s}\in W(1)\mid \widetilde{s}\mapsto s\in W\}$ such that $a(t\widetilde{s})=\chi'(t)a(\widetilde{s})$, $a(\widetilde{s}t)=a(\widetilde{s})\chi(t)$ for any $t\in Z_\kappa$. Then $C_s\neq 0$ if and only if $\chi'=s\chi$ and if $\chi'=s\chi$ then $\dim_C C_s=1$.

Now we define some subsets of S_{aff}. First, consider the sets

$$\begin{aligned}
A_1(\Xi,\Xi')&=\{s\in S_{\mathrm{aff}}\mid \Xi(T_{\widetilde{s}})=\Xi'(T_{\widetilde{s}})=0\},\\
A_2(\Xi,\Xi')&=\{s\in S_{\mathrm{aff}}\mid \Xi(T_{\widetilde{s}})\neq 0,\ \Xi'(T_{\widetilde{s}})=0\},\\
A_3(\Xi,\Xi')&=\{s\in S_{\mathrm{aff}}\mid \Xi(T_{\widetilde{s}})=0,\ \Xi'(T_{\widetilde{s}})\neq 0\},\\
A_4(\Xi,\Xi')&=\{s\in S_{\mathrm{aff}}\mid \Xi(T_{\widetilde{s}})\neq 0,\ \Xi'(T_{\widetilde{s}})\neq 0\},
\end{aligned}$$

where $\widetilde{s}$ is a lift of s. We define

$$\begin{aligned}
&S_2(\Xi,\Xi')=A_2(\Xi,\Xi')\cup A_3(\Xi,\Xi'),\\
&S_1(\Xi,\Xi')=\{s\in A_1(\Xi,\Xi')\setminus S_{\mathrm{aff},\chi}\mid s\chi=\chi',\ (ss_1)^2\neq 1 \text{ for any } s_1\in S_2(\Xi,\Xi')\}.
\end{aligned}$$

If $s\in S_{\mathrm{aff},\chi}$ and $a\in C_s$, then $a(\widetilde{s})\chi(c_{\widetilde{s}})^{-1}\in C$ does not depend on a lift $\widetilde{s}$ of s. We denote it by $a\chi(c_s)^{-1}$. We also have that if $s\in S_{\mathrm{aff},\chi'}$, then $a(\widetilde{s})\chi'(c_{\widetilde{s}})^{-1}$ does not depend on a lift $\widetilde{s}$. We denote it by $a\chi'(c_s)^{-1}$. If $a\neq 0$, then $\chi'=s\chi$. Hence, if $s\in S_{\mathrm{aff},\chi}$, then $s\in S_{\mathrm{aff},\chi'}$ and $a\chi(c_s)^{-1}=a\chi'(c_s)^{-1}$.

For the Hecke algebra attached to a finite Coxeter system, the following proposition is [8, Theorem 5.1], and we use a similar proof.

Proposition 4.1. *Consider the subspace $E_2(\Xi,\Xi')$ of $\bigoplus_{s\in S_2(\Xi,\Xi')}C_s$ consisting (a_s) such that*

- *If $s_1,s_2\in A_2(\Xi,\Xi')$, then $a_{s_1}\chi(c_{s_1})^{-1}=a_{s_2}\chi(c_{s_2})^{-1}$.*
- *If $s_1,s_2\in A_3(\Xi,\Xi')$, then $a_{s_1}\chi'(c_{s_1})^{-1}=a_{s_2}\chi'(c_{s_2})^{-1}$.*
- *If $s_1\in A_2(\Xi,\Xi')$, $s_2\in A_3(\Xi,\Xi')$ and $(s_1s_2)^2=1$, then $a_{s_1}\chi(c_{s_1})^{-1}+a_{s_2}\chi'(c_{s_2})^{-1}=0$,*

and put $E_1(\Xi,\Xi')=\bigoplus_{s\in S_1(\Xi,\Xi')}C_s$, $E(\Xi,\Xi')=E_1(\Xi,\Xi')\oplus E_2(\Xi,\Xi')$. For $(a_s)\in E(\Xi,\Xi')$, consider the linear map $\mathcal{H}\to M_2(C)$ defined by

$$T_{\widetilde{s}}\mapsto\begin{pmatrix}\Xi(T_{\widetilde{s}}) & 0\\ a_s(\widetilde{s}) & \Xi'(T_{\widetilde{s}})\end{pmatrix},$$

where $a_s=0$ if $s\notin S_1(\Xi,\Xi')\cup S_2(\Xi,\Xi')$. Then this gives an extension of Ξ by Ξ', and it gives a surjective map $E(\Xi,\Xi')\to \operatorname{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\Xi,\Xi')$. The kernel is

$$\left\{(a_s)\in E(\Xi,\Xi')\;\middle|\;\begin{array}{l}a_{s_1}\chi(c_{s_1})^{-1}+a_{s_2}\chi'(c_{s_2})^{-1}=0\\ \qquad\qquad (s_1\in A_2(\Xi,\Xi'),s_2\in A_3(\Xi,\Xi'))\\ a_s=0\ (s\in S_1(\Xi,\Xi'))\end{array}\right\}. \tag{4.1}$$

Remark 4.2. Let V_2 be a subspace of $\bigoplus_{s\in A_2(\Xi,\Xi')}C_s$ consisting (a_s) such that $a_{s_1}\chi(c_{s_1})^{-1}=a_{s_2}\chi(c_{s_2})^{-1}$ for any $s_1,s_2\in A_2(\Xi,\Xi')$. Then $\dim V_2\leq 1$ and $V_2\neq 0$ if and only if $C_s\neq 0$ for any $s\in A_2(\Xi,\Xi')$, namely $s\chi=\chi'$ for any $s\in A_2(\Xi,\Xi')$. Define V_3 by the similar way. Then $\dim V_3\leq 1$ and $V_3\neq 0$ if and only if $s\chi=\chi'$ for any $s\in A_3(\Xi,\Xi')$. If there is no $s_1\in A_2(\Xi,\Xi')$ and $s_2\in A_3(\Xi,\Xi')$ such that $(s_1s_2)^2=1$, then $E_2(\Xi,\Xi')=V_2\oplus V_3$. Otherwise, $\dim E_2(\Xi,\Xi')=\max\{0,\dim V_2+\dim V_3-1\}$.

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Proof. Let M be an extension of Ξ by Ξ'. Since $\#Z_\kappa$ is prime to p, the representation of Z_κ over C is completely reducible. Hence, we can take a basis e_1,e_2 such that $T_te_1 = \chi(t)e_1$ and $T_te_2 = \chi'(t)e_2$. With this basis, the action of $T_{\widetilde{s}}$ where $\widetilde{s} \in S_{\mathrm{aff}}(1)$ with the image $s \in S_{\mathrm{aff}}$ is described as

$$T_{\widetilde{s}} = \begin{pmatrix} \Xi(T_{\widetilde{s}}) & 0 \\ a_s(\widetilde{s}) & \Xi'(T_{\widetilde{s}}) \end{pmatrix}.$$

for some $a_s(\widetilde{s}) \in C$. The action of T_t where $t \in Z_\kappa$ is given by

$$\begin{pmatrix} \chi(t) & 0 \\ 0 & \chi'(t). \end{pmatrix}$$

Since $T_tT_{\widetilde{s}} = T_{t\widetilde{s}}$, we have

$$\begin{pmatrix} \chi(t) & 0 \\ 0 & \chi'(t). \end{pmatrix} \begin{pmatrix} \Xi(T_{\widetilde{s}}) & 0 \\ a_s(\widetilde{s}) & \Xi'(T_{\widetilde{s}}) \end{pmatrix} = \begin{pmatrix} \Xi(T_{t\widetilde{s}}) & 0 \\ a_s(t\widetilde{s}) & \Xi'(T_{t\widetilde{s}}) \end{pmatrix}.$$

Hence, $a_s(t\widetilde{s}) = \chi'(t)a_s(\widetilde{s})$. Similarly, we have $a_s(\widetilde{s}t) = a_s(\widetilde{s})\chi(t)$. Hence, $a_s \in C_s$.

Now we check the conditions that the map defines an action of $\mathcal{H}^{\mathrm{aff}}$. Since we have

$$\begin{pmatrix} \Xi(T_{\widetilde{s}}) & 0 \\ a_s(\widetilde{s}) & \Xi'(T_{\widetilde{s}}) \end{pmatrix}^2 = \begin{pmatrix} \Xi(T_{\widetilde{s}})^2 & 0 \\ a_s(\widetilde{s})(\Xi(T_{\widetilde{s}}) + \Xi'(T_{\widetilde{s}})) & \Xi'(T_{\widetilde{s}})^2 \end{pmatrix},$$

this satisfies the quadratic relation $T_{\widetilde{s}}^2 = T_{\widetilde{s}}c_{\widetilde{s}}$ if and only if

$$a_s(\widetilde{s})(\Xi(T_{\widetilde{s}}) + \Xi'(T_{\widetilde{s}})) = a_s(\widetilde{s})\chi(c_{\widetilde{s}}). \tag{4.2}$$

If $s \in A_1(\Xi,\Xi')$, then $a_s(\widetilde{s}) = 0$ or $\chi(c_{\widetilde{s}}) = 0$, namely $a_s = 0$ or $s \notin S_{\mathrm{aff},\chi}$.

If $s \in A_2(\Xi,\Xi')$, then $a_s = 0$ or $\Xi(T_{\widetilde{s}}) = \chi(c_{\widetilde{s}})$. Since $\Xi(T_{\widetilde{s}}) \neq 0$, we always have $\Xi(T_{\widetilde{s}}) = \chi(c_{\widetilde{s}})$. Hence, equation (4.2) is always satisfied.

If $s \in A_3(\Xi,\Xi')$, then $a_s = 0$ or $\Xi'(T_{\widetilde{s}}) = \chi(c_{\widetilde{s}})$. Note that if $a_s \neq 0$, then $s\chi = \chi'$; hence, $S_{\mathrm{aff},\chi} = S_{\mathrm{aff},\chi'}$ and $\chi(c_{\widetilde{s}}) = \chi'(c_{\widetilde{s}})$. Therefore, under $a_s \neq 0$, we have $\Xi'(T_{\widetilde{s}}) = \chi(c_{\widetilde{s}})$ if and only if $\Xi'(T_{\widetilde{s}}) = \chi'(c_{\widetilde{s}})$. This always hold since $\Xi'(T_{\widetilde{s}}) \neq 0$. Hence, equation (4.2) is always satisfied.

If $s \in A_4(\Xi,\Xi')$, then we have $\Xi(T_{\widetilde{s}}) = \chi(c_{\widetilde{s}})$. Hence, we have $a_s(\widetilde{s})\Xi'(T_{\widetilde{s}}) = 0$. Therefore, we have $a_s = 0$ since $\Xi'(T_{\widetilde{s}}) \neq 0$.

Consequently, the quadratic relation holds if and only if $a_s = 0$ or $s \in (A_1(\Xi,\Xi') \setminus S_{\mathrm{aff},\chi}) \cup A_2(\Xi,\Xi') \cup A_3(\Xi,\Xi')$. The action of $T_{\widetilde{s}}$ is given by one of the following matrix:

$$\begin{pmatrix} 0 & 0 \\ a_s(\widetilde{s}) & 0 \end{pmatrix}, \begin{pmatrix} \chi(c_{\widetilde{s}}) & 0 \\ a_s(\widetilde{s}) & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ a_s(\widetilde{s}) & \chi'(c_{\widetilde{s}}) \end{pmatrix}, \begin{pmatrix} \chi(c_{\widetilde{s}}) & 0 \\ 0 & \chi'(c_{\widetilde{s}}) \end{pmatrix}. \tag{4.3}$$

Here, each $\chi(c_{\widetilde{s}})$ and $\chi'(c_{\widetilde{s}})$ is not zero, and in the first matrix, we assume that $s \notin S_{\mathrm{aff},\chi}$ if $a_s \neq 0$.

Now we check the braid relations. Let $s_1,s_2 \in S_{\mathrm{aff}}$ and $\widetilde{s}_1,\widetilde{s}_2$ their lifts. We consider a braid relation $s_1s_2\cdots = s_2s_1\cdots$. It is easy to see that the action satisfies the braid relation for some lifts $\widetilde{s}_1,\widetilde{s}_2$ if and only if it is satisfied for any lifts $\widetilde{s}_1,\widetilde{s}_2$. Take $\widetilde{s}_1,\widetilde{s}_2$ such that $\widetilde{s}_1\widetilde{s}_2\cdots = \widetilde{s}_2\widetilde{s}_1\cdots$. It is easy to see that if $s_1 \in A_4(\Xi,\Xi')$ or $s_2 \in A_4(\Xi,\Xi')$, then the braid relations hold automatically. So we assume that $s_1,s_2 \in A_1(\Xi,\Xi') \cup A_2(\Xi,\Xi') \cup A_3(\Xi,\Xi')$.

Assume that $s_1 \in A_1(\Xi,\Xi')$. We have

$$\begin{pmatrix} 0 & 0 \\ a_{s_1}(\widetilde{s}_1) & 0 \end{pmatrix}\begin{pmatrix} \Xi(T_{\widetilde{s}_2}) & 0 \\ a_{s_2}(\widetilde{s}_2) & \Xi'(T_{\widetilde{s}_2}) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a_{s_1}(\widetilde{s}_1)\Xi(T_{\widetilde{s}_2}) & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 \\ a_{s_1}(\widetilde{s}_1) & 0 \end{pmatrix}\begin{pmatrix} \Xi(T_{\widetilde{s}_2}) & 0 \\ a_{s_2}(\widetilde{s}_2) & \Xi'(T_{\widetilde{s}_2}) \end{pmatrix}\begin{pmatrix} 0 & 0 \\ a_{s_1}(\widetilde{s}_1) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} \Xi(T_{\widetilde{s}_2}) & 0 \\ a_{s_2}(\widetilde{s}_2) & \Xi'(T_{\widetilde{s}_2}) \end{pmatrix}\begin{pmatrix} 0 & 0 \\ a_{s_1}(\widetilde{s}_1) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \Xi'(T_{\widetilde{s}_2})a_{s_1}(\widetilde{s}_1) & 0 \end{pmatrix},$$

$$\begin{pmatrix} \Xi(T_{\widetilde{s}_2}) & 0 \\ a_{s_2}(\widetilde{s}_2) & \Xi'(T_{\widetilde{s}_2}) \end{pmatrix}\begin{pmatrix} 0 & 0 \\ a_{s_1}(\widetilde{s}_1) & 0 \end{pmatrix}\begin{pmatrix} \Xi(T_{\widetilde{s}_2}) & 0 \\ a_{s_2}(\widetilde{s}_2) & \Xi'(T_{\widetilde{s}_2}) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \Xi'(T_{\widetilde{s}_2})\Xi(T_{\widetilde{s}_2})a_{s_1}(\widetilde{s}_1) & 0 \end{pmatrix},$$

$$\begin{pmatrix} \Xi(T_{\widetilde{s}_2}) & 0 \\ a_{s_2}(\widetilde{s}_2) & \Xi'(T_{\widetilde{s}_2}) \end{pmatrix}\begin{pmatrix} 0 & 0 \\ a_{s_1}(\widetilde{s}_1) & 0 \end{pmatrix}\begin{pmatrix} \Xi(T_{\widetilde{s}_2}) & 0 \\ a_{s_2}(\widetilde{s}_2) & \Xi'(T_{\widetilde{s}_2}) \end{pmatrix}\begin{pmatrix} 0 & 0 \\ a_{s_1}(\widetilde{s}_1) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, the braid relation is satisfied if and only if

- $a_{s_1} = 0$
- or $\Xi(T_{\widetilde{s}_2}) = \Xi'(T_{\widetilde{s}_2})$ and the order of s_1s_2 is 2
- or $\Xi(T_{\widetilde{s}_2})\Xi'(T_{\widetilde{s}_2}) = 0$ and the order of s_1s_2 is 3
- or the order of s_1s_2 is greater than 3.

If $s_2 \in A_1(\Xi,\Xi')$, then the condition always holds. If $s_2 \in A_2(\Xi,\Xi') \cup A_3(\Xi,\Xi')$, then the condition holds if and only if $a_{s_1} = 0$ or the order of s_1s_2 is not 2, namely $(s_1s_2)^2 \neq 1$.

Replacing s_1 with s_2, if $s_2 \in A_1(\Xi,\Xi')$, we have the similar condition.

Assume that $s_1,s_2 \in A_2(\Xi,\Xi')$. We have

$$\begin{pmatrix} \chi(c_{\widetilde{s}_1}) & 0 \\ a_{s_1}(\widetilde{s}_1) & 0 \end{pmatrix}\begin{pmatrix} \chi(c_{\widetilde{s}_2}) & 0 \\ a_{s_2}(\widetilde{s}_2) & 0 \end{pmatrix} = \begin{pmatrix} \chi(c_{\widetilde{s}_1})\chi(c_{\widetilde{s}_2}) & 0 \\ a_{s_1}(\widetilde{s}_1)\chi(c_{\widetilde{s}_2}) & 0 \end{pmatrix} = \begin{pmatrix} \chi(c_{\widetilde{s}_1}) & 0 \\ a_{s_1}(\widetilde{s}_1) & 0 \end{pmatrix}\chi(c_{\widetilde{s}_2}).$$

By this calculation, the braid relation is satisfied if and only if $a_{s_1}(\widetilde{s}_1)\chi(c_{\widetilde{s}_2})\cdots = a_{s_2}(\widetilde{s}_2)\chi(c_{\widetilde{s}_1})\cdots$. By [20, Proposition 4.13 (6)], we have $c_{\widetilde{s}_1}c_{\widetilde{s}_2}\cdots = c_{\widetilde{s}_2}c_{\widetilde{s}_1}\cdots$. Hence, the braid relation is satisfied if and only if $a_{s_1}(\widetilde{s}_1)\chi(c_{\widetilde{s}_1})^{-1} = a_{s_2}(\widetilde{s}_2)\chi(c_{\widetilde{s}_2})^{-1}$, namely $a_{s_1}\chi(c_{s_1})^{-1} = a_{s_2}\chi(c_{s_2})^{-1}$. By a similar calculation, if $s_1,s_2 \in A_3(\Xi,\Xi')$, then the braid relation is satisfied if and only if $a_{s_1}\chi'(c_{s_1})^{-1} = a_{s_2}\chi'(c_{s_2})^{-1}$.

Finally, we assume that $s_1 \in A_2(\Xi,\Xi')$ and $s_2 \in A_3(\Xi,\Xi')$. We have

$$\begin{pmatrix} \chi(c_{\widetilde{s}_1}) & 0 \\ a_{s_1}(\widetilde{s}_1) & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ a_{s_2}(\widetilde{s}_2) & \chi'(c_{\widetilde{s}_2}) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 \\ a_{s_2}(\widetilde{s}_2) & \chi'(c_{\widetilde{s}_2}) \end{pmatrix}\begin{pmatrix} \chi(c_{\widetilde{s}_1}) & 0 \\ a_{s_1}(\widetilde{s}_1) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a_{s_2}(\widetilde{s}_2)\chi(c_{\widetilde{s}_1}) + a_{s_1}(\widetilde{s}_1)\chi'(c_{\widetilde{s}_2}) & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 \\ a_{s_2}(\widetilde{s}_2) & \chi'(c_{\widetilde{s}_2}) \end{pmatrix}\begin{pmatrix} \chi(c_{\widetilde{s}_1}) & 0 \\ a_{s_1}(\widetilde{s}_1) & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ a_{s_2}(\widetilde{s}_2) & \chi'(c_{\widetilde{s}_2}) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, the braid relation is satisfied if and only if

- $a_{s_2}(\widetilde{s}_2)\chi(c_{\widetilde{s}_1}) + a_{s_1}(\widetilde{s}_1)\chi'(c_{\widetilde{s}_2}) = 0$ and the order of s_1s_2 is 2.
- the order of s_1s_2 is greater than 2.

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We notice that $a_{s_2}(\widetilde{s}_2)\chi(c_{\widetilde{s}_1}) + a_{s_1}(\widetilde{s}_1)\chi'(c_{\widetilde{s}_2}) = 0$ if and only if $a_{s_1}\chi(c_{s_1})^{-1} + a_{s_2}\chi'(c_{s_2})^{-1} = 0$.

We get the following table which shows the condition for the braid relation:

<table>
<tr><th>s_2 \ s_1</th><th>A_1</th><th>A_2</th><th>A_3</th><th>A_4</th></tr>
<tr><td>A_1</td><td>always</td><td colspan="2">$a_{s_2} = 0$ or $(s_1s_2)^2 \neq 1$</td><td>always</td></tr>
<tr><td>A_2</td><td rowspan="2">$a_{s_1} = 0$ or $(s_1s_2)^2 \neq 1$</td><td>$a_{s_1}\chi(c_{s_1})^{-1} = a_{s_2}\chi(c_{s_2})^{-1}$</td><td>$a_{s_1}\chi'(c_{s_1})^{-1} + a_{s_2}\chi(c_{s_2})^{-1} = 0$ or $(s_1s_2)^2 \neq 1$</td><td>always</td></tr>
<tr><td>A_3</td><td>$a_{s_1}\chi(c_{s_1})^{-1} + a_{s_2}\chi'(c_{s_2})^{-1} = 0$ or $(s_1s_2)^2 \neq 1$</td><td>$a_{s_1}\chi'(c_{s_1})^{-1} = a_{s_2}\chi'(c_{s_2})^{-1}$</td><td>always</td></tr>
<tr><td>A_4</td><td>always</td><td>always</td><td>always</td><td>always</td></tr>
</table>

Now we assume that $(a_s) \in \bigoplus_{s\in S_{\mathrm{aff}}} C_s$ defines an action of $\mathcal{H}$. First, recall that $C_s \neq 0$ if and only if $s\chi = \chi'$. Hence, $a_s \neq 0$ implies $s\chi = \chi'$. Since the quadratic relations hold, if $a_s \neq 0$, then $s \in (A_1(\Xi,\Xi') \setminus S_{\mathrm{aff},\chi}) \cup S_2(\Xi,\Xi')$. If $s \in A_1(\Xi,\Xi') \setminus S_{\mathrm{aff},\chi}$ and $(ss_1)^2 = 1$ for some $s_1 \in S_2(\Xi,\Xi')$, then the table says that $a_s = 0$. Therefore, if $a_s \neq 0$, then $s \in S_1(\Xi,\Xi') \cup S_2(\Xi,\Xi')$. Hence, again by the table, (a_s) belongs to $E(\Xi,\Xi')$.

Conversely, if $(a_s) \in E(\Xi,\Xi')$, then $a_s \neq 0$ implies $s \in S_1(\Xi,\Xi') \cup S_2(\Xi,\Xi') \subset (A_1(\Xi,\Xi') \setminus S_{\mathrm{aff},\chi}) \cup S_2(\Xi,\Xi')$. Hence, each $T_{\widetilde{s}}$ satisfies the quadratic relation. Let $s_1, s_2 \in S_{\mathrm{aff}}$. If $s_1 \in A_1(\Xi,\Xi')$ and $s_2 \in A_2(\Xi,\Xi') \cup A_3(\Xi,\Xi')$, then the definition of $S_1(\Xi,\Xi')$ says that $(s_1s_2)^2 \neq 1$ or $a_{s_1} = 0$. Then by the table, the braid relation for s_1, s_2 holds. For other cases, the condition on $E_2(\Xi,\Xi')$ and the table imply that the braid relation holds too.

Therefore, the map $E(\Xi,\Xi') \to \mathrm{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\Xi,\Xi')$ is well-defined and surjective.

Assume that the extension given by (a_s) splits, namely each matrix in equation (4.3) is simultaneous diagonalizable. If $s \in A_1(\Xi,\Xi')$, then the matrix corresponding to s is diagonalizable if and only if $a_s = 0$. Let $s_1, s_2 \in A_2(\Xi,\Xi')$. Then by $a_{s_1}\chi(c_{s_1})^{-1} = a_{s_2}\chi(c_{s_2})^{-1}$, the matrices corresponding to s_1, s_2 commute with each other. Hence, these matrices are simultaneous diagonalizable. Similarly, matrices corresponding to $A_3(\Xi,\Xi')$ are simultaneous diagonalizable.

If $s_1 \in A_2(\Xi,\Xi')$ and $s_2 \in A_3(\Xi,\Xi')$, then the corresponding matrices are

$$\begin{pmatrix} \chi(c_{\widetilde{s}_1}) & 0 \\ a_s(\widetilde{s}_1) & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ a_s(\widetilde{s}_2) & \chi'(c_{\widetilde{s}_2}) \end{pmatrix},$$

and these commute with each other if and only if $a_{s_1}\chi(c_{s_1})^{-1} + a_{s_2}\chi(c_{s_2})^{-1} = 0$. Hence, the kernel is equation (4.1). □

Remark 4.3. Let $\omega \in \Omega(1)_{\Xi} \cap \Omega(1)_{\Xi'}$. Then ω acts on $\mathrm{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\Xi,\Xi')$, and by this action, $\mathrm{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\Xi,\Xi')$ is a right $\Omega(1)_{\Xi} \cap \Omega(1)_{\Xi'}$-module. We also have the action of ω on $E(\Xi,\Xi')$ as follows: Let $s \in S_{\mathrm{aff}}$, and put $s_1 = \omega^{-1}s\omega \in S_{\mathrm{aff}}$. Then $a \mapsto (\widetilde{s}_1 \mapsto a(\omega\widetilde{s}_1\omega^{-1}))$ gives an

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isomorphism $C_s \simeq C_{s_1}$. We denote this map by $a \mapsto a \cdot \omega$. Then the action is given by $(a_s) \mapsto (a_{\omega^{-1}s\omega} \cdot \omega)$. This action commutes with the action of ω on $\mathrm{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\Xi, \Xi')$.

4.2. Semidirect product

The argument in this subsection is general. Let A be a C-algebra and Γ a group acting on A. We assume that a finite commutative normal subgroup $\Gamma' \subset \Gamma$ and an embedding $C[\Gamma'] \hookrightarrow A$ are given. Here, we assume that for $\gamma' \in \Gamma'$, the action of γ' on A as an element in Γ is given by $a \mapsto \gamma' a (\gamma')^{-1}$. We put $B = C[\Gamma] \otimes_{C[\Gamma']} A$ and define a multiplication by $(\gamma_1 \otimes a_1)(\gamma_2 \otimes a_2) = \gamma_1\gamma_2 \otimes (\gamma_2^{-1} \cdot a_1)a_2$ for $a_1, a_2 \in A$ and $\gamma_1, \gamma_2 \in \Gamma$. Of course, the example in our mind is $A = \mathcal{H}^{\mathrm{aff}}$, $\Gamma = \Omega(1)$ and $\Gamma' = Z_\kappa$. We have $B = \mathcal{H}$.

Let M_1, M_2 be right B-modules. Then $\mathrm{Hom}_A(M_1, M_2)$ has the structure of a Γ-module defined by $(f\gamma)(m) = f(m\gamma^{-1})\gamma$. This action factors through $\Gamma \to \Gamma/\Gamma'$, and we have $\mathrm{Hom}_B(M_1, M_2) = \mathrm{Hom}_A(M_1, M_2)^{\Gamma/\Gamma'}$. Let N be a Γ/Γ'-module and $\varphi \in \mathrm{Hom}_{\Gamma/\Gamma'}(N, \mathrm{Hom}_A(M_1, M_2))$. Set $f \colon N \otimes M_1 \to M_2$ by $f(n \otimes m) = \varphi(n)(m)$ for $n \in N$ and $m \in M_1$. Then for $\gamma \in \Gamma$, we have $f(n\gamma \otimes m\gamma) = \varphi(n\gamma)(m\gamma) = \varphi(n)(m)\gamma = f(n \otimes m)\gamma$. Namely, f is Γ-equivariant. We define an action of $a \in A$ on $N \otimes M_1$ by $(n \otimes m)a = n \otimes ma$. Then it coincides with the action of Γ on $C[\Gamma']$, and it gives an action of B. This correspondence gives an isomorphism

$$\mathrm{Hom}_{\Gamma/\Gamma'}(N, \mathrm{Hom}_A(M_1, M_2)) \simeq \mathrm{Hom}_B(N \otimes M_1, M_2).$$

In particular, if M_2 is an injective B-module, then $\mathrm{Hom}_A(M_1, M_2)$ is an injective Γ/Γ'-module. Therefore, from $\mathrm{Hom}_B(M_1, M_2) = \mathrm{Hom}_A(M_1, M_2)^{\Gamma/\Gamma'}$, we get a spectral sequence

$$E_2^{ij} = H^i(\Gamma/\Gamma', \mathrm{Ext}^j_A(M_1, M_2)) \Rightarrow \mathrm{Ext}^{i+j}_B(M_1, M_2).$$

In particular, we have an exact sequence

$$0 \to H^1(\Gamma/\Gamma', \mathrm{Hom}_A(M_1, M_2)) \to \mathrm{Ext}^1_B(M_1, M_2) \to \mathrm{Ext}^1_A(M_1, M_2)^{\Gamma/\Gamma'} \tag{4.4}$$

Moreover, we assume the following situation. Let Γ_1 be a finite index subgroup of Γ which contains Γ', and put $B_1 = A \otimes_{C[\Gamma']} C[\Gamma_1]$. Then this is a subalgebra of B, and B is a free left B_1-module with a basis given by a complete representative of $\Gamma_1 \backslash \Gamma$. Assume that M_1 has a form $L_1 \otimes_{B_1} B$ for some B_1-module L_1. We have $M_1 = \bigoplus_{\gamma \in \Gamma_1 \backslash \Gamma} L_1 \otimes \gamma$. Since B is flat over B_1, we have

$$\mathrm{Ext}^1_B(M_1, M_2) \simeq \mathrm{Ext}^1_{B_1}(L_1, M_2).$$

We have a B_1-module embedding $L_1 \hookrightarrow M_1$. This is in particular an A-homomorphism, and we get

$$\mathrm{Ext}^i_A(M_1, M_2) \to \mathrm{Ext}^1_A(L_1, M_2).$$

Since $L_1 \hookrightarrow M_1$ is a B_1-homomorphism, this is a Γ_1-homomorphism. Hence, this induces

$$\mathrm{Ext}^i_A(M_1, M_2) \to \mathrm{Ind}^{\Gamma}_{\Gamma_1}(\mathrm{Ext}^i_A(L_1, M_2)).$$

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The decomposition $M_1 = \bigoplus_{\gamma \in \Gamma_1 \backslash \Gamma} L_1 \otimes \gamma$ respects the A-action. Hence,

$$\operatorname{Ext}^i_A(M_1, M_2) = \bigoplus_{\gamma \in \Gamma_1 \backslash \Gamma} \operatorname{Ext}^i_A(L_1 \otimes \gamma, M_2) = \bigoplus_{\gamma \in \Gamma_1 \backslash \Gamma} \operatorname{Ext}^i_A(L_1, M_2)\gamma.$$

Therefore, the above homomorphism is an isomorphism

$$\operatorname{Ext}^i_A(M_1, M_2) \simeq \operatorname{Ind}^{\Gamma}_{\Gamma_1}(\operatorname{Ext}^i_A(L_1, M_2)).$$

This implies

$$H^1(\Gamma/\Gamma', \operatorname{Hom}_A(M_1, M_2)) \simeq H^1(\Gamma_1/\Gamma', \operatorname{Hom}_A(L_1, M_2)),$$
$$\operatorname{Ext}^1_A(M_1, M_2)^{\Gamma/\Gamma'} \simeq \operatorname{Ext}^1_A(L_1, M_2)^{\Gamma_1/\Gamma'}$$

and a commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & H^1(\Gamma/\Gamma', \operatorname{Hom}_A(M_1, M_2)) & \longrightarrow & \operatorname{Ext}^1_B(M_1, M_2) & \longrightarrow & \operatorname{Ext}^1_A(M_1, M_2)^{\Gamma/\Gamma'} \\ & \Big| \wr & & \Big| \wr & & \Big| \wr \\ 0 \longrightarrow & H^1(\Gamma_1/\Gamma', \operatorname{Hom}_A(L_1, M_2)) & \longrightarrow & \operatorname{Ext}^1_{B_1}(L_1, M_2) & \longrightarrow & \operatorname{Ext}^1_A(L_1, M_2)^{\Gamma_1/\Gamma'}. \end{array}$$

We also assume that there exists a finite index subgroup Γ_2 of Γ which contains Γ' and $M_2 = L_2 \otimes_{B_2} B$, where $B_2 = A \otimes_{C[\Gamma']} C[\Gamma_2]$. Let $\{\gamma_1, \dots, \gamma_r\}$ be a set of complete representatives of $\Gamma_2 \backslash \Gamma / \Gamma_1$. Then the decomposition $M_2 = \bigoplus_{\gamma \in \Gamma_2 \backslash \Gamma} L_2 \otimes \gamma = \bigoplus_i \bigoplus_{\gamma \in (\Gamma_1 \cap \gamma_i^{-1} \Gamma_2 \gamma_i) \backslash \Gamma_1} L_2 \otimes \gamma_i \gamma$ gives

$$M_2|_{B_1} = \bigoplus_i L_2\gamma_i \otimes_{B_1 \cap \gamma_i^{-1} B_2 \gamma_i} B_1,$$

where $L_2\gamma_i$ is a $\gamma_i^{-1} B_2 \gamma_i$-module defined by: $L_2\gamma_i = L_2$ as a vector space and the action is given by $l(\gamma_i^{-1} b \gamma_i) = l \cdot b$ for $l \in L_2$ and $b \in B_2$, here $\cdot$ is the original action of $b \in B_2$ on L_2. From this isomorphism, we get

$$H^i(\Gamma_1/\Gamma', \operatorname{Ext}^j_A(L_1, M_2)) \simeq \bigoplus_i H^i((\Gamma_1 \cap \gamma_i^{-1} \Gamma_2 \gamma_i)/\Gamma', \operatorname{Ext}^j_A(L_1, L_2\gamma_i))$$

and

$$\operatorname{Ext}^i_{B_1}(L_1, M_2) = \bigoplus_i \operatorname{Ext}^i_{B_1 \cap \gamma_i^{-1} B_2 \gamma_i}(L_1, L_2\gamma_i)$$

which is compatible with the exact sequence in equation (4.4).

Set $A = \mathcal{H}^{\mathrm{aff}}$, $\Gamma = \Omega(1)$, $\Gamma' = Z_\kappa$, $\Gamma_1 = \Omega(1)_\Xi$ and $\Gamma_2 = \Omega(1)_{\Xi'}$. Then we get the following lemma. Recall that Ω is assumed to be commutative. For a character Ξ of $\mathcal{H}$, we defined $\Omega(1)_\Xi$ as the stabilizer of Ξ in $\Omega(1)$ and $\mathcal{H}_\Xi = \mathcal{H}^{\mathrm{aff}} C[\Omega(1)_\Xi]$ in subsection 2.9.

Lemma 4.4. *Let χ, χ' be characters of Z_κ and $J \subset S_{\mathrm{aff},\chi}, J' \subset S_{\mathrm{aff},\chi'}$. Put $\Xi = \Xi_{\chi,J}$, $\Xi' = \Xi_{\chi',J'}$, and let V, V' be irreducible $C[\Omega(1)_\Xi]$, $C[\Omega(1)_{\Xi'}]$-modules, respectively. Let $\{\omega_1, \dots, \omega_r\}$ be a set of complete representatives of $\Omega_\Xi \backslash \Omega / \Omega_{\Xi'} = \Omega / \Omega_\Xi \Omega_{\Xi'}$, and define Ξ'_i by $\Xi'_i(X) = \Xi'(\omega_i X \omega_i^{-1})$. Consider the representation of $C[\Omega(1)_{\Xi'_i}] = \omega_i^{-1} C[\Omega(1)_{\Xi'}] \omega_i$*

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twisting V' by ω_i, and we denote it by V'_i. Put $\Omega_{\Xi,\Xi'} = \Omega_\Xi \cap \Omega_{\Xi'}$ and $\mathcal{H}_{\Xi,\Xi'} = \mathcal{H}_\Xi \cap \mathcal{H}_{\Xi'}$. Then we have a commutative diagram with exact rows:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
H^1(\Omega, \operatorname{Hom}_{\mathcal{H}^{\mathrm{aff}}}(\pi_{J,\chi,V},\pi_{J',\chi',V'})) & \xrightarrow{\sim} & \bigoplus_i H^1(\Omega_{\Xi,\Xi'_i}, \operatorname{Hom}_{\mathcal{H}^{\mathrm{aff}}}(\Xi\otimes V, \Xi'_i \otimes V'_i)) \\
\downarrow & & \downarrow \\
\operatorname{Ext}^1_{\mathcal{H}}(\pi_{\chi,J,V},\pi_{\chi',J',V'}) & \xrightarrow{\sim} & \bigoplus_i \operatorname{Ext}^1_{\mathcal{H}_{\Xi,\Xi'_i}}(\Xi\otimes V,\Xi'_i\otimes V'_i) \\
\downarrow & & \downarrow \\
\operatorname{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\pi_{\chi,J,V},\pi_{\chi',J',V'})^{\Omega} & \xrightarrow{\sim} & \bigoplus_i \operatorname{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\Xi\otimes V,\Xi'_i\otimes V'_i)^{\Omega_{\Xi,\Xi'_i}}
\end{array}$$

The following theorem will be proved in subsection 4.4.

Theorem 4.5. *The map $\operatorname{Ext}^1_{\mathcal{H}_{\Xi,\Xi'}}(\Xi\otimes V,\Xi'\otimes V') \to \operatorname{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\Xi\otimes V,\Xi'\otimes V')^{\Omega_{\Xi,\Xi'}}$ is surjective.*

Combining with Lemma 4.4, we get the surjectivity of $\operatorname{Ext}^1_{\mathcal{H}}(\pi_{\chi,J,V},\pi_{\chi',J',V'}) \to \operatorname{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\pi_{\chi,J,V},\pi_{\chi',J',V'})^{\Omega}$ stated in the introduction of this paper.

4.3. $\operatorname{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\Xi\otimes V,\Xi'\otimes V')^{\Omega_{\Xi,\Xi'}}$

To prove Theorem 4.5, we analyze $\operatorname{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\Xi\otimes V,\Xi'\otimes V')^{\Omega_{\Xi,\Xi'}}$. First, we have

$$\operatorname{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\Xi\otimes V,\Xi'\otimes V') \simeq \operatorname{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\Xi,\Xi')\otimes \operatorname{Hom}_C(V,V')$$

and the surjective homomorphism $E(\Xi,\Xi') \to \operatorname{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\Xi,\Xi')$. We have the decomposition $E(\Xi,\Xi') = E_1(\Xi,\Xi') \oplus E_2(\Xi,\Xi')$. Let $E'_1(\Xi,\Xi')$ (resp. $E'_2(\Xi,\Xi')$) be the image of $E_1(\Xi,\Xi')$ (resp. $E_2(\Xi,\Xi')$). By the description of the kernel (4.1), we have:

- $\operatorname{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\Xi,\Xi') = E'_1(\Xi,\Xi') \oplus E'_2(\Xi,\Xi')$.
- $E_1(\Xi,\Xi') \xrightarrow{\sim} E'_1(\Xi,\Xi')$.
- the dimension of the kernel of $E_2(\Xi,\Xi') \to E'_2(\Xi,\Xi')$ is at most 1.

Define E_i $(i=2,3)$ by $E_i = E_2(\Xi,\Xi') \cap \bigoplus_{s\in A_i(\Xi,\Xi')} C_s$. Then $\dim E_2, \dim E_3 \le 1$ and $E_i \ne 0$ if and only if for any $s \in A_i(\Xi,\Xi')$ we have $s\chi = \chi'$.

Assume that $s\chi = \chi'$ for any $s \in A_2(\Xi,\Xi')$. Fix $s_0 \in A_2(\Xi,\Xi')$. Then $a = (a_s) \mapsto a_{s_0}\chi(c_{s_0})^{-1}$ gives an isomorphism $E_2 \simeq C$. Let $\omega \in \Omega_{\Xi,\Xi'}(1)$. Then

$$(a\omega)_{s_0}\chi(c_{s_0})^{-1} = a_{\omega s \omega^{-1}}(\omega \widetilde{s}_0 \omega^{-1})\chi(c_{\widetilde{s}_0})^{-1}.$$

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Since ω stabilizes χ, we have $\chi(c_{\widetilde{s}_0}) = (\omega^{-1}\chi)(c_{\widetilde{s}_0}) = \chi(\omega \cdot c_{\widetilde{s}_0}) = \chi(c_{\omega\widetilde{s}_0\omega^{-1}})$. Therefore, we have

$$\begin{aligned}(a\omega)_{s_0}\chi(c_{s_0})^{-1} &= a_{\omega s\omega^{-1}}(\omega\widetilde{s}_0\omega^{-1})\chi(c_{\omega\widetilde{s}_0\omega^{-1}})^{-1}\\ &= a_{\omega s\omega^{-1}}\chi(c_{\omega s\omega^{-1}})^{-1}\\ &= a_s\chi(c_s)^{-1}.\end{aligned}$$

Here, the last equality follows from the definition of $E_2'(\Xi,\Xi')$. Namely, $\Omega_{\Xi,\Xi'}$ acts trivially on E_2. By the same argument, $\Omega_{\Xi,\Xi'}$ also acts trivially on E_3. Therefore, it also acts trivially on $E_2(\Xi,\Xi')$, hence on $E_2'(\Xi,\Xi')$. Hence,

$$(E_2'(\Xi,\Xi') \otimes \operatorname{Hom}_C(V,V'))^{\Omega_{\Xi,\Xi'}} = E_2'(\Xi,\Xi') \otimes \operatorname{Hom}_{\Omega_{\Xi,\Xi'}}(V,V').$$

4.4. Proof of Theorem 4.5

Now we prove Theorem 4.5. Take $e \in \operatorname{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\Xi\otimes V, \Xi'\otimes V')^{\Omega_{\Xi,\Xi'}}$, and first, assume that $e \in E_1'(\Xi,\Xi')$. Therefore, e gives $f_s \in C_s \otimes \operatorname{Hom}_C(V,V')$. The space $C_s \otimes \operatorname{Hom}_C(V,V')$ is the space of functions f_s on $\{\widetilde{s} \mid \widetilde{s} \text{ is a lift of } s\}$ with values in $\operatorname{Hom}_C(V,V')$ such that $f(t_1\widetilde{s}t_2) = \chi'(t_1)f(\widetilde{s})\chi(t_2)$ for $t_1,t_2 \in Z_\kappa$. Using this f_s, we define an $\mathcal{H}$-module structure on $V \oplus V'$ by

$$T_{\widetilde{s}} \mapsto \begin{pmatrix}\Xi(T_{\widetilde{s}}) & 0\\ f_s(\widetilde{s}) & \Xi'(T_{\widetilde{s}})\end{pmatrix}, \quad T_\omega \mapsto \begin{pmatrix}V(\omega) & 0\\ 0 & V'(\omega)\end{pmatrix},$$

where $\widetilde{s} \in S_{\mathrm{aff}}$, s its image in S_{aff} and $f_s = 0$ if $s \notin S_1(\Xi,\Xi')$. Since e is $\Omega_{\Xi,\Xi'}$-invariant and $E_1(\Xi,\Xi') \to E_1'(\Xi,\Xi')$ is injective, we have $V'(\omega)f_s(\widetilde{s})V(\omega^{-1}) = f_{\omega s\omega^{-1}}(\omega\widetilde{s}\omega^{-1})$. Hence,

$$\begin{aligned}&\begin{pmatrix}V(\omega) & 0\\ 0 & V'(\omega)\end{pmatrix}\begin{pmatrix}\Xi(T_{\widetilde{s}}) & 0\\ f_s(\widetilde{s}) & \Xi'(T_{\widetilde{s}})\end{pmatrix}\begin{pmatrix}V(\omega) & 0\\ 0 & V'(\omega)\end{pmatrix}^{-1}\\ &= \begin{pmatrix}\Xi(T_{\omega\widetilde{s}^{-1}}) & 0\\ f_s(\omega\widetilde{s}\omega^{-1}) & \Xi'(T_{\omega\widetilde{s}\omega^{-1}})\end{pmatrix}.\end{aligned}$$

Namely, the above action gives an action of $\mathcal{H}_{\mathrm{aff}}C[\Omega_{\Xi,\Xi'}]$. Hence, this gives an extension class in $\operatorname{Ext}^1_{\mathcal{H}_{\Xi,\Xi'}}(\Xi\otimes V, \Xi'\otimes V')$, and its image in $\operatorname{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\Xi\otimes V, \Xi'\otimes V')$ corresponds to e.

Next, we assume that e comes from E_2-part. Then we may assume that there exist $\varphi \in \operatorname{Hom}_{\Omega(1)_{\Xi,\Xi'}}(V,V')$ and $e_0 \in E_2'(\Xi,\Xi')$ such that e is given by $e_0 \otimes \varphi$. Take a lift (a_s) of e_0 in $E_2(\Xi,\Xi')$, and consider the action of $\mathcal{H}^{\mathrm{aff}}C[\Omega(1)_{\Xi,\Xi'}]$ on $V \oplus V'$ defined by

$$T_{\widetilde{s}} \mapsto \begin{pmatrix}\Xi(T_{\widetilde{s}}) & 0\\ a_s(\widetilde{s})\varphi & \Xi'(T_{\widetilde{s}})\end{pmatrix}, \quad T_\omega \mapsto \begin{pmatrix}V(\omega) & 0\\ 0 & V'(\omega)\end{pmatrix},$$

where $\widetilde{s} \in S_{\mathrm{aff}}(1)$, s its image in S_{aff} and $a_s = 0$ if $s \notin S_2(\Xi,\Xi')$. Recall that $\Omega(1)_{\Xi,\Xi'}$ acts trivially on (a_s). Since φ is $\Omega(1)_{\Xi,\Xi'}$-equivariant, the calculation as above shows that this gives an action of $\mathcal{H}_{\Xi,\Xi'}$.

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4.5. Calculation of the extensions

We have

$$\mathrm{Ext}^1_{\mathcal{H}}(\pi_{\chi,J,V},\pi_{\chi',J',V'}) \simeq \bigoplus_i \mathrm{Ext}^1_{\mathcal{H}_{\Xi,\Xi'_i}}(\Xi\otimes V,\Xi'_i\otimes V'_i).$$

Hence, it is sufficient to calculate $\mathrm{Ext}^1_{\mathcal{H}_{\Xi,\Xi'_i}}(\Xi\otimes V,\Xi'_i\otimes V'_i)$. Now replacing (Ξ'_i,V_i) with (Ξ',V'), we explain how to calculate $\mathrm{Ext}^1_{\mathcal{H}_{\Xi,\Xi'}}(\Xi\otimes V,\Xi'\otimes V')$.

Theorem 4.5 implies

$$\begin{aligned}&\dim \mathrm{Ext}^1_{\mathcal{H}_{\Xi,\Xi'}}(\Xi\otimes V,\Xi'\otimes V')\\ &= \dim H^1(\Omega_{\Xi,\Xi'},\mathrm{Hom}_{\mathcal{H}^{\mathrm{aff}}}(\Xi\otimes V,\Xi'\otimes V')) + \dim \mathrm{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\Xi\otimes V,\Xi'\otimes V')^{\Omega_{\Xi,\Xi'}}.\end{aligned}$$

Since $\mathcal{H}^{\mathrm{aff}}$ acts trivially on V and V's, we have

$$\mathrm{Hom}_{\mathcal{H}^{\mathrm{aff}}}(\Xi\otimes V,\Xi'\otimes V') = \mathrm{Hom}_{\mathcal{H}^{\mathrm{aff}}}(\Xi,\Xi')\otimes \mathrm{Hom}_C(V,V'),$$

and it is zero if $\Xi\neq\Xi'$. If $\Xi=\Xi'$, then

$$\mathrm{Hom}_{\mathcal{H}^{\mathrm{aff}}}(\Xi,\Xi')\otimes \mathrm{Hom}_C(V,V') = \mathrm{Hom}_C(V,V'),$$

and hence,

$$H^1(\Omega_{\Xi,\Xi'},\mathrm{Hom}_{\mathcal{H}^{\mathrm{aff}}}(\Xi\otimes V,\Xi'\otimes V'_i)) \simeq H^1(\Omega_{\Xi,\Xi'},\mathrm{Hom}_C(V,V')).$$

This is a group cohomology of an abelian group.

We also have

$$\mathrm{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\Xi\otimes V,\Xi'\otimes V') = \mathrm{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\Xi,\Xi')\otimes \mathrm{Hom}_C(V,V'),$$

and

$$\mathrm{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\Xi,\Xi') = E'_1(\Xi,\Xi')\oplus E'_2(\Xi,\Xi').$$

As we saw in subsection 4.3, $\Omega_{\Xi,\Xi'}$ acts trivially on $E'_2(\Xi,\Xi')$. Hence,

$$(E'_2(\Xi,\Xi')\otimes \mathrm{Hom}_C(V,V'))^{\Omega_{\Xi,\Xi'}} = E'_2(\Xi,\Xi')\otimes \mathrm{Hom}_{\Omega_{\Xi,\Xi'}(1)}(V,V'),$$

and it is not difficult to calculate this.

Finally, we consider $(E'_1(\Xi,\Xi')\otimes \mathrm{Hom}_C(V,V'))^{\Omega_{\Xi,\Xi'}}$. By Proposition 4.1, we have $E'_1(\Xi,\Xi')\simeq E_1(\Xi,\Xi') = \bigoplus_{s\in S_1(\Xi,\Xi')} C_s$. Fix $s_0\in S_1(\Xi,\Xi')$, and let $\Omega(1)_{\Xi,\Xi',s_0}$ be the stabilizer of s_0 in $\Omega(1)_{\Xi,\Xi'}$. Then C_{s_0} is an $\Omega(1)_{\Xi,\Xi',s_0}$-representation. Consider an $\Omega(1)_{\Xi,\Xi'}$-orbit $\mathcal{S}\subset S_1(\Xi,\Xi')$. The subspace $\bigoplus_{s\in\mathcal{S}} C_s$ is $\Omega(1)_{\Xi,\Xi'}$-stable, and we have an isomorphism

$$\bigoplus_{s\in\mathcal{S}} C_s \simeq \mathrm{Ind}^{\Omega(1)}_{\Omega(1)_{\Xi,\Xi',s_0}} C_{s_0}$$

defined by

$$(a_s)\mapsto (\omega\mapsto(\widetilde{s}_0\mapsto a_{\omega^{-1}s\omega})(\omega^{-1}\widetilde{s}_0\omega)).$$

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Let $\{s_1,\dots,s_r\}$ be a complete representative of the $\Omega(1)_{\Xi,\Xi'}$-orbits in $S_1(\Xi,\Xi')$. Then we have

$$E_1'(\Xi,\Xi') \simeq \bigoplus_i \operatorname{Ind}_{\Omega(1)_{\Xi,\Xi',s_i}}^{\Omega(1)_{\Xi,\Xi'}} C_{s_i}.$$

Hence,

$$(E_1(\Xi,\Xi')\otimes \operatorname{Hom}_C(V,V'))^{\Omega_{\Xi,\Xi'}} = \bigoplus_i (C_{s_i}\otimes \operatorname{Hom}_C(V,V'))^{\Omega_{\Xi,\Xi',s_i}}.$$

4.6. Example: $G = \mathrm{GL}_n$

Assume that the data come from GL_n. Then the data are as follows (see [17]).

We have $W_0 = S_n$, $W = S_n \ltimes (F^\times/\mathcal{O}^\times)^n \simeq S_n \ltimes \mathbb{Z}^n$, $W(1) = S_n \ltimes (F^\times/(1+(\varpi)))^n = S_n \ltimes (\mathbb{Z}\times\kappa^\times)^n$ and $W^{\mathrm{aff}} = S_n \ltimes \{(x_i)\in\mathbb{Z}^n \mid \sum x_i = 0\}$. Set

$$\omega = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{pmatrix}(0,\dots,0,1) \in S_n \ltimes \mathbb{Z}^n \subset W(1),$$

and denote its image in W by the same letter ω. Then Ω is generated by ω and $\Omega(1) = \langle\omega\rangle(\kappa^\times)^n$. We have $\omega^n = (1,\dots,1)$, and it belongs to the center of $W(1)$. The element $c_{s_i}\in C[Z_\kappa]$ is given by $c_{s_i} = \sum_{t\in\kappa^\times} T_{\nu_i(t)\nu_{i+1}(t)^{-1}}$, where $\nu_i\colon \kappa^\times \to (\kappa^\times)^n$ is an embedding to i-th entry and $\nu_{n+1} = \nu_1$.

Let $\pi_{\chi,J,V}$ and $\pi_{\chi',J',V'}$ be simple supersingular modules, and we assume that the dimension of the modules are both n.

Remark 4.6. An importance of n-dimensional simple supersingular modules is revealed by a work of Grosse-Klönne [9]. He constructed a correspondence between supersingular n-dimensional modules of $\mathcal{H}$ and irreducible modulo p n-dimensional representations of $\mathrm{Gal}(\overline{F}/F)$.

We have $\dim\pi_{\chi,J,V} = (\dim V)[\Omega:\Omega_\Xi]$. Since $\Omega(1)$ is (hence, $\Omega(1)_\Xi$ is) commutative, we have $\dim V = 1$. Therefore, our assumption implies $[\Omega:\Omega_\Xi] = n$. Since ω^n is in the center, $\langle\omega^n\rangle\subset\Omega_\Xi$. Hence, $\Omega_\Xi = \langle\omega^n\rangle$ and $\Omega_\Xi = \langle\omega^n\rangle(\kappa^\times)^n$. Set $\lambda = V(\omega^n)$. Since $V|_{(\kappa^\times)^n} = \chi$, V is determined by χ and λ. We also put $\lambda' = V'(\omega^n)$.

We define $\chi_j\colon\kappa^\times\to C^\times$ by $\chi(t_1,\dots,t_n) = \chi_1(t_1)\dots\chi_n(t_n)$, and we extend it for any $j\in\mathbb{Z}$ by $\chi_{j\pm n} = \chi_j$. Then

$$(s_i\chi)_j = \begin{cases} \chi_j & (j\ne i,i+1),\\ \chi_{i+1} & (j=i),\\ \chi_i & (j=i+1). \end{cases}$$

The description of c_{s_i} shows $\chi(c_{s_i}) = 0$ if and only if $\chi_i = \chi_{i+1}$ if and only if $s_i\chi = \chi$. Therefore, $S_{\mathrm{aff},\chi} = \{s_i\in S_{\mathrm{aff}} \mid \chi_i = \chi_{i+1}\}$.

We consider $\operatorname{Ext}^1_{\mathcal{H}_{\Xi,\Xi'}}(\Xi\otimes V,\Xi'\otimes V')$. By Theorem 4.5, we have the exact sequence

$$0\to H^1(\Omega_{\Xi,\Xi'},\operatorname{Hom}_{\mathcal{H}^{\mathrm{aff}}}(\Xi,\Xi')\otimes\operatorname{Hom}_C(V,V'))\to \operatorname{Ext}^1_{\mathcal{H}_{\Xi,\Xi'}}(\Xi\otimes V,\Xi'\otimes V')$$
$$\to \operatorname{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\Xi\otimes V,\Xi'\otimes V')^{\Omega_{\Xi,\Xi'}}\to 0.$$

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The space $\mathrm{Hom}_{\mathcal{H}^{\mathrm{aff}}}(\Xi,\Xi')$ is C if $\Xi=\Xi'$ and 0 otherwise. We have $\Omega_{\Xi,\Xi'}=\langle\omega^n\rangle\simeq\mathbb{Z}$ and ω^n acts on $\mathrm{Hom}_C(V,V')$ by $\lambda^{-1}\lambda'$. Therefore, $H^1(\Omega_{\Xi,\Xi'},\mathrm{Hom}_C(V,V'))=C$ if $\lambda=\lambda'$ and 0 otherwise. Namely, we get

$$\dim H^1(\Omega_{\Xi,\Xi'},\mathrm{Hom}_{\mathcal{H}^{\mathrm{aff}}}(\Xi,\Xi')\otimes\mathrm{Hom}_C(V,V'))=\begin{cases}1 & \Xi=\Xi',V=V',\\ 0 & \text{otherwise.}\end{cases}\tag{4.5}$$

Note that $\Omega_{\Xi,\Xi'}$ acts on S_{aff} trivially since $\Omega_{\Xi,\Xi'}$ is in the center of W. Hence, the stabilizer of each $s\in S_{\mathrm{aff}}$ in $\Omega_{\Xi,\Xi'}$ is $\Omega_{\Xi,\Xi'}$ itself, and each orbit is a singleton. Therefore, by the previous subsection, we have

$$\begin{aligned}&\mathrm{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\Xi\otimes V,\Xi'\otimes V')^{\Omega_{\Xi,\Xi'}}\\ &=\bigoplus_{s\in S_1(\Xi,\Xi')}(C_s\otimes\mathrm{Hom}_C(V,V'))^{\Omega_{\Xi,\Xi'}}\oplus E_2'(\Xi,\Xi')\otimes\mathrm{Hom}_{\Omega_{\Xi,\Xi'}}(V,V').\end{aligned}$$

Since $\omega^n\in\Omega_{\Xi,\Xi',s}=\Omega_{\Xi,\Xi'}$ is in the center of $W(1)$, it acts trivially on C_s. Hence, $(C_s\otimes\mathrm{Hom}_C(V,V'))^{\Omega_{\Xi,\Xi',s}}=\mathrm{Hom}_{\Omega_{\Xi,\Xi'}}(V,V')$, and it is not zero if and only if $\lambda=\lambda'$. Hence,

$$\mathrm{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\Xi\otimes V,\Xi'\otimes V')^{\Omega_{\Xi,\Xi'}}\simeq\begin{cases}C^{S_1(\Xi,\Xi')}\oplus E_2'(\Xi,\Xi') & \lambda=\lambda',\\ 0 & \text{otherwise.}\end{cases}$$

A complete representative of $\Omega/\Omega_\Xi\Omega_{\Xi'}$ is given by $\{1,\omega,\dots,\omega^{n-1}\}$. Put $\Xi_i'=\Xi'\omega^i$. This is parametrized by $(\chi\omega^i,J_i=\omega^iJ\omega^{-i})$. We have $(\chi\omega^i)_j=\chi_{j+i}$ and $\omega^iJ\omega^{-i}=\{s_{j+i}\mid s_j\in J\}$. We put $V_i'=V'\omega^i$. Then $V_i'(\omega)=V'(\omega)$ and $V_i'|_{Z_\kappa}=\chi\omega^i$.

The cohomology group $H^1(\Omega_{\Xi,\Xi_i'},\mathrm{Hom}_{\mathcal{H}^{\mathrm{aff}}}(\Xi,\Xi_i')\otimes\mathrm{Hom}_C(V,V_i'))$ is zero if and only if $(\Xi,V)\neq(\Xi_i',V_i')$ by equation (4.5). There exists at most one i such that $(\Xi,\Xi_i')\neq(V,V_i')$, and such i exists if and only if (Ξ,V) is Ω-conjugate to (Ξ',V'). Hence,

$$\begin{aligned}&\dim\bigoplus_{i=0}^{n-1}H^1(\Omega_{\Xi,\Xi_i'},\mathrm{Hom}_{\mathcal{H}^{\mathrm{aff}}}(\Xi,\Xi_i')\otimes\mathrm{Hom}_C(V,V_i'))\\ &=\begin{cases}1 & (\Xi,V)\text{ is }\Omega\text{-conjugate to }(\Xi',V'),\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

We also have

$$\begin{aligned}&\dim\bigoplus_{i=0}^{n-1}\mathrm{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\Xi\otimes V,\Xi'\otimes V')^{\Omega_{\Xi,\Xi'}}\\ &\simeq\begin{cases}\sum_{i=0}^{n-1}(\#S_1(\Xi,\Xi_i')+\dim E_2'(\Xi,\Xi_i')) & \lambda=\lambda',\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

and each term can be calculated by the description in Proposition 4.1.

4.7. GL_2

Now we assume $n=2$, and we compute $\mathrm{Ext}^1_{\mathcal{H}}(\pi_{\chi,J,V},\pi_{\chi',J',V'})$. We continue to use the notation in the previous subsection. Then ω switches s_0 and s_1.

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Lemma 4.7. *The nonvanishing of* $\mathrm{Ext}^1_{\mathcal{H}}(\pi_{\chi,J,V},\pi_{\chi',J',V'})$ *implies that* (χ,J,V) *is conjugate to* (χ',J',V') *by* Ω.

Proof. As we have seen in the above, nonvanishing of Ext^1 implies $V(\omega)=V'(\omega)=(V'\omega)(\omega)$. Hence, it is sufficient to prove that (χ,J') is conjugate to (χ',J').

If $H^1(\Omega_{\Xi,\Xi'},\mathrm{Hom}_{\mathcal{H}^{\mathrm{aff}}}(\Xi\otimes\Xi')\otimes\mathrm{Hom}_C(V,V'))\neq 0$, we have $\Xi=\Xi'$ and $V=V'$. Hence, we have the lemma.

If $\mathrm{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\Xi\otimes V,\Xi'\otimes V')^{\Omega_{\Xi,\Xi'}}\neq 0$, then $C_s\neq 0$, hence $\chi'=s\chi$ for some $s\in S_{\mathrm{aff}}$. Since we assume $G=\mathrm{GL}_2$, $s_0\chi=s_1\chi=\chi\omega$. Since $\pi_{\chi,J,V}$ and $\pi_{\chi',J',V'}$ are both supersingular, the possibility of (J,J') is $(\emptyset,\emptyset)$, $(\{s_0\},\{s_1\})$, $(\{s_0\},\{s_1\})$, $(\{s_0\},\{s_0\})$, $(\{s_1\},\{s_1\})$ and except the last two cases, we have $J=\omega J'\omega^{-1}$. If $J=J'=\{s_0\}$, then $s_0\in S_{\mathrm{aff},\chi}$; hence, $s_0\chi=\chi$. Since $s_0\chi=s_1\chi$, we have $S_{\mathrm{aff},\chi}=S_{\mathrm{aff}}$. Hence, $S_1(\Xi,\Xi')=\emptyset$. We also have $A_2(\Xi,\Xi')=A_3(\Xi,\Xi')=\emptyset$. Therefore, we get $\mathrm{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\Xi\otimes V,\Xi'\otimes V')^{\Omega_{\Xi,\Xi'}}=0$. By the same way, if $J=J'=\{s_1\}$, then $\mathrm{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\Xi\otimes V,\Xi'\otimes V')^{\Omega_{\Xi,\Xi'}}=0$. □

Since $\pi_{\chi,J,V}$ only depends on the Ω-orbit of (χ,J,V), we may assume $(\chi,J,V)=(\chi',J',V')$. In this case, $H^1(\Omega_{\Xi,\Xi'},\mathrm{Hom}_{\mathcal{X}^{\mathrm{aff}}}(\Xi\otimes V,\Xi_i\otimes V_i))$ is one-dimensional if $i=0$ and zero if $i=1$.

(1) The case of $\chi_1=\chi_2$. Then we have $S_{\mathrm{aff},\chi}=S_{\mathrm{aff}}$. By the proof of Lemma 4.7, we have $\mathrm{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\Xi\otimes V,\Xi\otimes V)=0$. We have $S_1(\Xi,\Xi_1)=\emptyset$, $A_2(\Xi,\Xi_1)=J_1=\omega J\omega^{-1}$ and $A_3(\Xi,\Xi_1)=J_0=J$. Hence, the description in Proposition 4.1 shows that $\dim E_2'(\Xi,\Xi')=1$, and hence, $\dim\mathrm{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\Xi\otimes V,\Xi\otimes V)=1$.

(2) The case of $\chi_1\neq\chi_2$. Then we have $S_{\mathrm{aff},\chi}=\emptyset$. Since $\chi\neq s\chi=s\chi_0$ for $s=s_0,s_1$, $C_s=0$. Therefore, $\mathrm{Ext}^1_{\mathcal{H}^{\mathrm{aff}}}(\Xi\otimes V,\Xi_0\otimes V_0)=0$. Since $S_{\mathrm{aff},\chi}=\emptyset$, $\Xi(T_s)=\Xi'(T_s)=0$ for any $s\in S_{\mathrm{aff}}$. We have $A_2(\Xi,\Xi_1)=A_3(\Xi,\Xi_1)=\emptyset$ and $S_1(\Xi,\Xi_1)=S_{\mathrm{aff}}$. Therefore, $E_1'(\Xi,\Xi_1)=0$ and $\dim E_2'(\Xi,\Xi_1)=\#S_1(\Xi,\Xi_1)=\#S_{\mathrm{aff}}=2$.

Hence, we have

$$\dim\mathrm{Ext}^1_{\mathcal{H}}(\pi_{\chi,J,V},\pi_{\chi',J',V'})=\begin{cases}0 & (\pi_{\chi,J,V}\not\simeq\pi_{\chi',J',V'}),\\ 2 & (\pi_{\chi,J,V}\simeq\pi_{\chi',J',V'},\ \chi_1=\chi_2),\\ 3 & (\pi_{\chi,J,V}\simeq\pi_{\chi',J',V'},\ \chi_1\neq\chi_2).\end{cases}$$

This recovers [7, Corollary 6.7]. (Note that in [7], they calculate the extensions with fixed central character. Since we do not fix the central character here, the dimension calculated here is one greater than the dimension they calculated.)

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