Periodic data rigidity of Anosov automorphisms with Jordan blocks

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Abstract. Anosov automorphisms with Jordan blocks are not periodic data rigid. We introduce a refinement of the periodic data and show that this refined periodic data characterizes C^{1+} conjugacy for Anosov automorphisms on $\mathbb{T}^4$ with a Jordan block.

1. *Introduction*

An Anosov diffeomorphism of a Riemannian manifold M is a diffeomorphism $F \colon M \to M$ such that TM continuously splits into two DF-invariant bundles E^u and E^s such that vectors in E^u are uniformly expanded by DF and vectors in E^s are uniformly contracted by DF. If $L \in \mathrm{SL}(n, \mathbb{Z})$ does not have any eigenvalues of unit modulus, then the map induced by L on $\mathbb{T}^n$ is an Anosov diffeomorphism, which is called an *Anosov automorphism* due to its algebraic construction. Anosov diffeomorphisms of tori exhibit strong topological rigidity. If $F \colon \mathbb{T}^n \to \mathbb{T}^n$ is Anosov and in the homotopy class of $L \in \mathrm{SL}(n, \mathbb{Z})$, then by work of Franks [Fra69] and Manning [Man74], there exists a homeomorphism h such that $h^{-1}Fh = L$. The map h is called a *conjugacy* and is Hölder continuous. In this paper, we study ridigity of Anosov automorphisms defined by a matrix $L \in \mathrm{SL}(4, \mathbb{Z})$ that have a Jordan block.

This paper is the first to show rigidity of an Anosov diffeomorphism with a Jordan block. There are two main contributions of this paper. The first is the identification of the correct refinement of the periodic data for this setting and showing that these data exist and are well defined. The second main contribution is a new approach to studying the regularity of conjugacies that does not rely on either conformality or an abundance of dynamically invariant foliations. In fact, in this paper, we contend with Anosov automorphisms that,

restricted to their unstable manifold, preserve only a single one-dimensional foliation and whose differential has polynomial growth of conformal distortion.

In this paper, we study a rigidity property of conjugacies between Anosov diffeomorphisms. Specifically, we will investigate conditions that imply that a conjugacy h is C^1. One well-known obstruction to the existence of a C^1 conjugacy is the periodic data, which we now describe. Suppose that F and G are two diffeomorphisms that are C^1 conjugate by a conjugacy h. If p is a periodic point of period n, then $h(p)$ is a periodic point of G of period n. By the chain rule, we must have that

$$D_p F^n = D_{h(p)} h \, D_{h(p)} G^n (D_{h(p)} h)^{-1}.$$

Thus, the return maps of DF and DG at corresponding periodic points are conjugate. Given two diffeomorphisms F and G with a conjugacy h between them, we say that F and G have the same *periodic data with respect to h* if for each periodic point p of period n, we have that $D_p F^n$ and $D_{h(p)} G^n$ are conjugate as linear maps. In some situations, if F and G have the same periodic data, then h is C^1.

We say that a diffeomorphism is C^{k+} when it is C^k and its kth derivative is α-Hölder continuous for some $\alpha > 0$. We write $\mathrm{Diff}^{k+}(M)$ for the group of diffeomorphisms of M that are C^{k+}. This leads us to the definition of rigidity in which we are interested here.

Definition 1.1. We say that an Anosov diffeomorphism $F \colon M \to M$ is *periodic data rigid* if for any Anosov diffeomorphism $G \in \mathrm{Diff}^{2+}(M)$ in the homotopy class of F, if F and G have the same periodic data with respect to a conjugacy h, then h is C^{1+}.

The main examples of diffeomorphisms F exhibiting this type of rigidity are Anosov automorphisms.

Periodic data rigidity is well studied. Early work was done by De la Llave, Marco, and Moriyon. See, for example, [**dlL87**, **dlL92**, **MM87**]. More recently, a paper by Gogolev, Kalinin, and Sadovskaya [**GKS11**] showed local periodic data rigidity of an Anosov automorphism of $\mathbb{T}^n$, under the assumption that no three eigenvalues of L have the same modulus and that L and L^4 are both irreducible. This result implies, as a consequence, earlier results obtained in [**Gog08**]. See also the recent preprint [**DG24**], which obtains non-perturbative versions of these results. Closely related to periodic data rigidity is Lyapunov spectrum rigidity, see [**GKS20**, **SY19**] for generalizations of these results to this setting. The paper [**SY19**] proves local Lyapunov spectrum rigidity for perturbations of Anosov automorphisms with Lyapunov exponents of multiplicity one as well as some further generalizations to the partially hyperbolic setting. These results rely on a careful study of entropy along invariant foliations. The paper [**GKS20**] uses a related technique to generalize the results of [**GKS11**] from periodic data rigidity to Lyapunov spectrum rigidity.

The previously mentioned results apply to Anosov automorphisms of tori. The author recently also obtained periodic data rigidity results for Anosov automorphisms of nilmanifolds [**DeW21**]. Recently, Gogolev and Rodriguez Hertz [**GRH22**] have proved additional periodic data rigidity results relying on a novel condition called *very non-algebraicity* using a technique of matching functions that they introduced. They also extended these results to codimension one Anosov diffeomorphisms in [**GRH21**].

In this paper, we consider Anosov automorphisms of $\mathbb{T}^4$ with a Jordan block, such as the following:

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}. \tag{1}$$

Such automorphisms with Jordan blocks are known to not be periodic data rigid due to examples of de la Llave in [**dlL02**, §6], and do not seem to have been studied after the construction of those counterexamples. In fact, any Anosov automorphism of $\mathbb{T}^4$ that has a Jordan block in its periodic data is not periodic data rigid. In this paper, we nonetheless recover a periodic data rigidity result for such automorphisms by introducing a refinement of the periodic data, which we call the Jordan periodic data. As we will explain after the theorem when we define Jordan periodic data, this extra data will allow us to detect precisely when the derivative cocycle DF is cohomologous to constant.

THEOREM 1.2. *If $L \in \mathrm{SL}(4, \mathbb{Z})$ defines an Anosov automorphism of $\mathbb{T}^4$ with a Jordan block and F is any C^{2+} Anosov diffeomorphism of $\mathbb{T}^4$ in the homotopy class of L with the same periodic data as L, then the two have well-defined Jordan periodic data and are C^{1+} conjugate if and only if their full Jordan periodic data coincide. Further, the condition of having the same full Jordan data as L is equivalent to DF being continuously cohomologous to the constant cocycle given by L.*

In Proposition A.1, we give an explicit description of to which Anosov automorphisms this theorem applies. In fact, the work of Kalinin, Sadovskaya, and Wang [**KSW23**] gives the following as an immediate corollary.

COROLLARY 1.3. *Suppose that L is as in Theorem 1.2. Then there exists $k \in \mathbb{N}$ such that if F is as in Theorem 1.2 and F is a C^k small perturbation of L, then F is C^∞ conjugate to L.*

This follows from the main theorem [**KSW23**, Theorem 1.3], which is a 'bootstrapping' result that says C^1 conjugacies between a weakly irreducible Anosov automorphism and its perturbation are C^∞. The automorphisms we consider are weakly irreducible due to [**KSW23**, Lemma 3.1] and Proposition A.1.

In the course of the proof of Theorem 1.2, we investigate how one may think about periodic data in this setting, which leads us to the Jordan periodic data, which we describe below.

1.1. *Jordan periodic data.* Fix some $\lambda > 0$. If we have a cocycle taking values in the set of matrices

$$U_\lambda = \left\{ \begin{bmatrix} \lambda & a \\ 0 & \lambda \end{bmatrix} \mid a \in \mathbb{R} \right\},$$

then knowing that two such cocycles have the same periodic data may not be enough to determine that they are cohomologous as $\mathrm{GL}(2, \mathbb{R})$ cocycles. In fact, if we have two continuous cocycles $\mathcal{A}$ and $\mathcal{B}$ over a map $\sigma \colon \Sigma \to \Sigma$ taking values in U_λ, and all of the periodic data of $\mathcal{A}$ and $\mathcal{B}$ has a Jordan block, then their periodic data are identical because

all such Jordan blocks are conjugate. If we write these cocycles as

$$\begin{bmatrix} \lambda & \alpha_A \\ 0 & \lambda \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \lambda & \alpha_B \\ 0 & \lambda \end{bmatrix},$$

then these two cocycles are continuously cohomologous as $\mathrm{GL}(2, \mathbb{R})$ cocycles precisely when there exists a constant C and a continuous map $\psi \colon \Sigma \to \mathbb{R}$ such that

$$\alpha_A = C\alpha_B + \psi \circ \sigma - \psi. \tag{2}$$

We then define $[\alpha_A]$ to be the set of all functions α_B such that there is a solution C, ψ to equation (2). We call $[\alpha_A]$ a cohomology class. One can check that $[\alpha_A]$ is determined by the periodic data of α_A, i.e. $[\alpha_A]$ is determined by the sums of α_A along all periodic orbits. We refer to $[\alpha_A]$ as the *Jordan periodic data* associated to $\mathcal{A}$. For the formal definition of Jordan periodic data, see Definition 3.2 as well as the surrounding discussion, which explains the relationship between this cohomology class and periodic data.

Not all cocycles are immediately recognizable as taking values in U_λ. We say that a cocycle is U_λ-*framed* if it may be continuously conjugated into U_λ. If a cocycle is U_λ-framed and has a Jordan block, then its Jordan periodic data may be defined, and indeed is well defined independent of the conjugacy into U_λ.

Our main result has two parts. The first part shows that if an Anosov diffeomorphism has the same periodic data as an Anosov automorphism with a Jordan block, then it has well-defined Jordan periodic data.

PROPOSITION 1.4. *Let L be an Anosov automorphism of $\mathbb{T}^4$ with a Jordan block and eigenvalues λ, λ^{-1}. Suppose that F is a C^2 Anosov diffeomorphism with the same periodic data as L. Then the stable and unstable bundles of F are trivial, and U_λ and $U_{\lambda^{-1}}$-framed, respectively.*

In the case that the stable and unstable bundles admit such framings, one obtains $[\alpha_u^F]$ and $[\alpha_s^F]$ as before; we refer to these as the unstable and stable Jordan periodic data, respectively. We refer to the pair $[\alpha_u^F]$ and $[\alpha_s^F]$ as the *full* Jordan periodic data.

The second part of the main result shows that if an Anosov diffeomorphism has the same Jordan periodic data as an Anosov automorphism, then the two are C^{1+}-conjugate.

PROPOSITION 1.5. *Suppose that L is an Anosov automorphism of $\mathbb{T}^4$ with a Jordan block and that F is a C^{2+} Anosov diffeomorphism such that:*
(1) *F has the same periodic data as L with respect to a conjugacy h;*
(2) *the stable and unstable bundles of F are trivial and admit $U_{\lambda^{-1}}$ and U_λ-framings, respectively;*
(3) *F has the same full Jordan periodic data as L.*
Then h is a C^{1+} conjugacy between F and L.

These immediately combine to prove Theorem 1.2.

There are several directions for further investigation beyond the results of this paper. The first is to consider what happens for higher dimensional Jordan blocks. In this paper, we have only addressed the case of a two-dimensional Jordan block. Another direction

would also be to consider Anosov automorphisms with Jordan blocks that have more than two Lyapunov exponents. A third direction would be to move past rigidity of Anosov automorphisms to the case where the derivative cocycle is not cohomologous to constant. For example, the author does not know the answer to the following question, which would give a nonlinear generalization of Theorem 1.2.

Question 1.6. Suppose that F and G are Anosov diffeomorphisms with the same periodic data as a linear Anosov diffeomorphism of $\mathbb{T}^4$ that has a Jordan block. If the full Jordan data of F and G coincide, then are F and G C^1 conjugate?

1.2. *Outline of proof.* The proof follows the following outline.
(1) We introduce the 'Jordan' periodic data.
(2) We do some preliminary work to produce a DF-invariant flag $\mathcal{E}$ on which the Jordan periodic data are well defined.
(3) We show the existence of 'slow' foliations within each unstable leaf tangent to $\mathcal{E}$ and their unique integrability.
(4) We show that the conjugacy h intertwines these slow foliations and is uniformly C^{1+} along them.
(5) We show that the 'slow' foliation is C^{1+} and that its holonomies are uniformly regular over long distances.
(6) Using the de la Llave argument [dlL02], we construct a C^1 model of the conjugacy, h_0, such that $F^n \circ h_0 \circ L^{-n} \to h$ uniformly. By studying the derivatives $F^n h_0 L^{-n}$, we obtain that h is Lipschitz.
(7) Once we know that h is suitably Lipschitz, we can differentiate it and use cocycle rigidity to conclude that its derivative is continuous and hence h is differentiable.

2. *Preliminaries*
For definitions concerning foliations, we refer the reader to [PSW97], which contains a thorough discussion of the topic. We use the same terminology as that paper, which is standard. We now set some notation that will be of particular use in our argument. For a foliation $\mathcal{F}$, we write $\mathcal{F}(x)$ for the leaf containing the point x. As an extension of this notation, if S is a set, then we write $\mathcal{F}(S)$ for $\bigcup_{x \in S} \mathcal{F}(x)$. If $\mathcal{F}$ is a foliation with C^1 leaves that foliates a Riemannian manifold M and if $x \in \mathcal{F}(y)$, then we denote by $d_{\mathcal{F}}(x, y)$ the distance between x and y as measured along the immersed submanifold $\mathcal{F}$, where $\mathcal{F}$ is endowed with the pullback Riemannian metric obtained from its inclusion into M. If $h \colon M \to M$ is a map, then we say that h intertwines the foliations $\mathcal{F}$ and $\mathcal{G}$ if $h(\mathcal{F}(x)) = \mathcal{G}(h(x))$ for all points $x \in M$.

We say that a map f is C^{1+} if there exists $0 < \alpha < 1$ such that the derivative of f is C^{α}. When we say that a map of a non-compact space is uniformly C^{α}, we mean that there exist $\alpha, \delta > 0$ such that, restricted to balls of radius δ, the map is C^{α} Hölder with uniform constant.

In the following argument, we will often work with uniform transversals to a C^1 foliation $\mathcal{F} \subset M$. What we mean by this encompasses two things. First, that there is a uniform upper bound on the C^1 norm of the transversals when we regard them as maps

of a disk into M. Second, that there is a uniform lower bound on the angle between the tangent space to γ and $T\mathcal{F}$. In other words, when we work with uniform transversals, we are assuming a uniform upperbound on the C^1 norm of these transversals as well as a uniform lower bound on the angle that they make with the foliation to which they are transverse.

If W is a set such that each leaf of $\mathcal{F}$ intersects W at most once, then we write Π_W for the projection onto W along the leaves of $\mathcal{F}$. Specifically, if z is a point such that $z \in \mathcal{F}(x)$ for some $x \in W$, then $\Pi_W(z) = x$. We say that sets T and W are holonomy related if $W \subset \mathcal{F}(T)$ and *vice versa*. We will typically use this notation for projections onto a curve contained in an unstable leaf that is transverse to a subfoliation of the the unstable foliation.

For an Anosov diffeomorphism F, we denote by $\mathcal{W}^{u,F}$ the unstable foliation of F. This foliation has uniformly C^{2+} leaves when F is C^{2+}. We denote by $E^{u,F}$ the unstable bundle of F; similarly we define the stable foliation $\mathcal{W}^{s,F}$ and the stable bundle $E^{s,F}$.

If E is an n-dimensional vector bundle over a manifold M, then by a *framing* of E, we mean a choice of basis $[e_1, \ldots, e_n]$ of E_x for each x that varies continuously with x. For an Anosov automorphism L on $\mathbb{T}^4$ with a Jordan block, we have a translation invariant framing $[e_1^L, e_2^L]$ that presents the differential of $DL|_{E^{u,L}}$ as a Jordan block:

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}.$$

3. *Jordan periodic data*

Suppose that $A \colon \Sigma \to \mathrm{GL}(2, \mathbb{R})$ is a continuous function defining a cocycle over a transitive hyperbolic system $\sigma \colon \Sigma \to \Sigma$, such as a transitive Anosov diffeomorphism. The natural notion of equivalence for two cocycles is the notion of *conjugacy*. We say that two continuous cocycles $A_1, A_2 \colon \Sigma \to \mathrm{GL}(2, \mathbb{R})$ are continuously conjugate if there exists a continuous function $B \colon \Sigma \to \mathrm{GL}(2, \mathbb{R})$ such that for all $x \in \Sigma$,

$$A_2(x) = (B(\sigma(x)))^{-1} A_1(x) B(x).$$

We analogously speak of two cocycles being Hölder conjugate if we may take B to be Hölder. We speak of other regularities analogously. Further, we say that a cocycle A is conjugate into a subgroup $H \subseteq \mathrm{GL}(2, \mathbb{R})$ if there exists $B \colon \Sigma \to \mathrm{GL}(2, \mathbb{R})$ such that for every $x \in \Sigma$,

$$(B(\sigma(x)))^{-1} A_1(x) B(x) \in H.$$

Suppose that A is continuously conjugate to a cocycle taking values in the group $U_\lambda \subset \mathrm{GL}(2, \mathbb{R})$ given by

$$U_\lambda = \left\{ \begin{bmatrix} \lambda & a \\ 0 & \lambda \end{bmatrix} \mid a \in \mathbb{R} \right\}.$$

We say that such a cocycle is reducible to a cocycle taking values in U_λ, or is U_λ-framed. By assumption, the cocycles of interest in this paper have non-trivial Jordan blocks in their periodic data. If a cocycle has a Jordan block in its periodic data for every periodic point, then we say that this cocycle has *Jordan-full* periodic data. Note that this happens when a cocycle has the same periodic data as a constant cocycle defined by a Jordan block. We

now consider the cohomology of such cocycles that are Hölder conjugate to U_λ and have Jordan-full periodic data. We may write such a cocycle as

$$\begin{bmatrix} \lambda & \alpha_A \\ 0 & \lambda \end{bmatrix} \tag{3}$$

for $\alpha_A \colon \Sigma \to \mathbb{R}$ a Hölder continuous function. The function α_A characterizes A up to Hölder $GL(2, \mathbb{R})$-conjugacy: two U_λ-valued cocycles A and B are Hölder conjugate if and only if α_A and α_B lie in the same 'projective' cohomology class. The following follows from a much more detailed result of Sadovskaya [**Sad13**, Proposition 5.1], which gives a detailed description of $GL(2, \mathbb{R})$ cocycles.

LEMMA 3.1. *Suppose that* (Σ, σ) *is a transitive hyperbolic system and that* $A, B \colon \Sigma \to GL(2, \mathbb{R})$ *are two Hölder continuous functions defining cocycles reducible to* U_λ *that are Jordan-full. Then, A and B are Hölder conjugate if and only if there exists a constant* $C \neq 0$ *and a Hölder continuous function* $\phi \colon \Sigma \to \mathbb{R}$ *such that*

$$\alpha_A = C\alpha_B + \phi \circ \sigma - \phi.$$

In fact, if A and B take values in U_λ, then any conjugacy between A and B takes values in upper triangular matrices.

By the usual abelian Livsic theorem, $\alpha_A = C\alpha_B + \phi \circ \sigma - \phi$ if and only if there exists C such that for each periodic point p of period k, $\sum_{i=1}^{k} \alpha_A(\sigma^i p) = C \sum_{i=1}^{k} \alpha_B(\sigma^i p)$. If there is such a C, we say that the periodic data of A and B are in the same *projective class*. Given this preliminary, the following definition is well defined.

Definition 3.2. Suppose that $A \colon \Sigma \to GL(2, \mathbb{R})$ is a function defining a cocycle over a transitive hyperbolic system that is reducible to U_λ. Let $Per(\Sigma)$ be the set of periodic points of Σ. As before, associated to A is the function $\alpha_A \colon \Sigma \to \mathbb{R}$. We define the *Jordan periodic data* of A to be the projective class of the function $Per(\Sigma) \to \mathbb{R}$ that sends a periodic point p of period k to the sum of α_A along its orbit:

$$p \mapsto \sum_{i=1}^{k} \alpha_A(\sigma^i(p)).$$

Lemma 3.1 shows that the Jordan periodic data characterize up to conjugacy the cocycles for which they are defined.

The discussion above defines the Jordan periodic data for cocycles over a single fixed system. If we have two cocycles over different conjugate systems, then we may pull back the cocycle by the conjugacy. We then say that two cocycles have the same Jordan periodic data with respect to a conjugacy h if the pulled back cocycle has the same Jordan periodic data as the original cocycle.

If F is an Anosov diffeomorphism such that $E^{u,F}$ and $E^{s,F}$ are trivial bundles and admit U_λ and $U_{\lambda^{-1}}$-framings, then we write $[\alpha_u^F]$ for the Jordan data of $DF|_{E^{u,F}}$ and $[\alpha_s^F]$ for the Jordan data of the stable bundle. We call the pair the *full* Jordan data of F.

4. *Preliminary reductions*

In this section, we give some preliminary reductions that bring us closer to producing the U_λ-framings we need to define the periodic data.

LEMMA 4.1. *Suppose L is as in Proposition A.1 and that F is a C^2 Anosov diffeomorphism with the same periodic data as L. Then there exists a non-trivial Hölder continuous DF-invariant flag*

$$0 \subset \mathcal{E} \subset E^{u,F}$$

and Hölder continuous Riemannian metrics on $\mathcal{E}$ and $E^{u,F}/\mathcal{E}$ such that with respect to these metrics, the differential of F has pointwise norms exactly λ. The same holds for $E^{s,F}$.

Proof. Because F has the same periodic data as L, the top and bottom Lyapunov exponents of the measures supported on periodic orbits are equal. Thus, we may apply the continuous amenable reduction [KS13, Theorem 3.9] to conclude that either $DF|_{E^{u,F}}$ is conformal with respect to some Hölder continuous metric, or that there exists a Hölder continuous DF invariant flag

$$0 \subset \mathcal{E} \subset E^{u,F}. \tag{4}$$

As $DF|_{E^{u,F}}$ has Jordan blocks in its periodic data, it cannot be conformal, and thus we obtain the non-trivial flag in equation (4).

We now appeal to our knowledge of the periodic data for the claim about the metric. Note that because $\mathcal{E}$ is one-dimensional, the map $N: \mathbb{T}^4 \to \mathbb{R}^\times$ given by $x \mapsto \|D_x F|_\mathcal{E}\|$ defines a cocycle over F. Further, note that the assumption on periodic data implies that all the periodic data of N are a power of λ. Thus, by the abelian Livsic theorem [KH97, Theorem 19.2.1], N is Hölder cohomologous to the constant cocycle λ, that is, $N = \lambda(\phi \circ F)\phi^{-1}$ for some Hölder $\phi: \mathbb{T}^4 \to \mathbb{R}^\times$. If we replace the reference metric $\|\cdot\|$ on $\mathcal{E}$ with the metric $\phi\|\cdot\|$, then for this new metric, we have that the norm of $DF|_\mathcal{E}$ is exactly λ at every point.

Precisely the same considerations show that $E^{u,F}/\mathcal{E}$ admits a norm with the same property.   □

Remark 4.2. The reason that the above lemma is phrased in terms of a norm instead of in terms of a cocycle conjugacy is that there is nothing telling us that the bundle $\mathcal{E}$ is orientable.

Remark 4.3. After this point, we will always equip $E^{u,F}$ with a metric obtained from Lemma 4.1. The lemma gives metrics on $\mathcal{E}$ and $E^{u,F}/\mathcal{E}$. Here is how we can obtain a metric on $E^{u,F}$. To begin, fix a complement $\mathcal{E}^c$ to $\mathcal{E}$ inside of $E^{u,F}$; this can be done by taking the orthogonal complement to $\mathcal{E}$ with respect to any fixed reference metric, for instance. Then we have a projection $\pi: E^{u,F} \to E^{u,F}/\mathcal{E}$. We may pull back the metric on $E^{u,F}/\mathcal{E}$ to $\mathcal{E}^c$ via π. We then obtain a metric on $E^{u,F}$ by declaring the metrics on $\mathcal{E}^c$ and $\mathcal{E}$ to be orthogonal. We will use this choice of metric below.

We now remark on some useful properties of the new metric we just constructed. The most useful property is that if we measure distance along unstable leaves by using this metric, then curves tangent to $\mathcal{E}$ will be uniformly stretched by a factor of exactly λ under

the dynamics. Throughout the rest of the proof, we will measure distance along unstable leaves with respect to this new metric. The old and new metrics on $E^{u,F}$ each make the leaves of the unstable foliation into metric spaces. A useful observation is that the (metric space) metrics on the leaves of the unstable foliation are bi-Lipshitz equivalent whether we choose to measure distance with the old or the new metric.

We will continue using this metric until we can obtain a better one, which happens once we know that $E^{u,F}$ is in fact U_λ-framed. See Remark 6.7.

5. *The 'slow' foliation*

Note that if we identify a $\mathcal{W}^{u,L}$ leaf with $\mathbb{R}^2$ by use of the framing $[e_1^L, e_2^L]$, then $\mathcal{W}^{u,L}$ has an invariant foliation tangent to the e_1^L line field. We call this foliation $\mathcal{S}^L$. We think of this foliation as 'slow' because vectors tangent to it grow at only rate $O(\lambda^n)$ when we iterate L. The following proposition constructs an analogous 'slow' foliation $\mathcal{S}^F$ that subfoliates the unstable foliation $\mathcal{W}^{u,F}$.

In this proof, we will use the notion of a quasi-isometry. Consider a surjection of metric spaces $f : (M_1, d_1) \to (M_2, d_2)$. Then f is an (A, B)-quasi-isometry if there exist constants $A \geq 1$ and $B \geq 0$ such that for any $x, y \in M_1$,

$$\frac{1}{A} d_1(x, y) - B \leq d_2(f(x), f(y)) \leq A d_1(x, y) + B.$$

PROPOSITION 5.1. *Suppose that F is a C^{2+} Anosov diffeomorphism that has the same periodic data as $L \in \mathrm{SL}(4, \mathbb{Z})$, which is hyperbolic and has a Jordan block, and that h is a conjugacy between F and L. Restricted to a $\mathcal{W}^{u,F}$ leaf, the bundle spanned by e_1^F is uniquely integrable and tangent to a foliation $\mathcal{S}^F$, which has uniformly C^{1+} leaves. Further, h intertwines $\mathcal{S}^F$ with the foliation $\mathcal{S}^L$.*

Proof. Throughout this proof, we endow M with the metrics obtained from Lemma 4.1 as described in Remark 4.3. By [**DeW21**, Cor ollary 2.7], there exist uniform constants A, B such that for all $x \in \mathbb{T}^4$, $h|_{\mathcal{W}^{u,L}(x)} : \mathcal{W}^{u,L}(x) \to \mathcal{W}^{u,F}(h(x))$ is an (A, B)-quasi-isometry.

Note that

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^n = \lambda^n \begin{bmatrix} 1 & n/\lambda \\ 0 & 1 \end{bmatrix}.$$

This implies that if $x \in \mathcal{W}^{u,L}(y)$ and $x \notin \mathcal{S}^L(y)$, then there exists $C > 0$ such that

$$d_{\mathcal{W}^{u,L}}(L^n(x), L^n(y)) \geq C\lambda^n n.$$

As h is a quasi-isometry, this implies that

$$d_{\mathcal{W}^{u,F}}(F^n(h(x)), F^n(h(y))) = d_{\mathcal{W}^{u,F}}(h(L^n(x)), L^n(y)) \geq A^{-1}C\lambda^n n - B. \quad (5)$$

However, if $q \in \mathcal{W}^{u,F}(r)$, and q and r are two points connected by a curve tangent to $\mathcal{E}$ contained within $\mathcal{W}^{u,F}(r)$, then

$$d_{\mathcal{W}^{u,F}}(F^n(q), F^n(r)) \leq \lambda^n \quad (6)$$

because $\|DF|_{\mathcal{E}}\| = \lambda$ when we use the metric from Lemma 4.1. Thus, if q and r are connected by such a curve, then $h^{-1}(q)$ and $h^{-1}(r)$ must lie in the same $\mathcal{S}^L$ leaf inside

of $\mathcal{W}^{u,L}$ because otherwise, equation (6) would contradict equation (5). This implies that the curves inside an unstable leaf tangent to $\mathcal{E}$ uniquely integrate to a foliation. Further, this foliation has uniformly C^{1+} leaves because the $\mathcal{E}$ distribution is uniformly Hölder by Proposition 4.1. □

6. *Differentiability along the slow foliation and the Jordan periodic data*

The approach we follow for showing that h is differentiable along $\mathcal{S}^L$ is not new and follows exactly the same argument as in [**DeW21**, §2.5]. Consequently, we will just give an outline of the idea. To establish differentiability of h along $\mathcal{S}^L$, we will use the following result, which is a C^{1+} version of Saghin and Yang [**SY19**, Theorem G]. All the definitions used in this section and a thorough discussion may be found in [**DeW21**, §2.5].

Definition 6.1. Let $\mathcal{F}$ be an expanding foliation for a C^{1+} diffeomorphism f. An f-invariant measure μ is a *Gibbs expanding state* along $\mathcal{F}$ if for any foliation chart of $\mathcal{F}$, the disintegration of μ along the plaques of the chart is equivalent to the Lebesgue measure on the plaque for μ-almost every plaque.

Using this definition, we may state the following.

LEMMA 6.2. [**DeW21**, Lemma 13] *Let M be a smooth closed manifold and let $f, g \in \mathrm{Diff}^{1+}(M)$. Let $\mathcal{F}$ be a one-dimensional expanding foliation for f, and let $\mathcal{G}$ be an expanding foliation for g such that $\mathcal{F}$ and $\mathcal{G}$ have uniformly C^{1+} leaves. Let μ be a Gibbs expanding state of f along $\mathcal{F}$. Suppose that f and g are topologically conjugate by a homeomorphism h and that h intertwines $\mathcal{F}$ and $\mathcal{G}$. Then the following two conditions are equivalent:*

(1) $\nu := h_*(\mu)$ *is a Gibbs expanding state of g along the foliation $\mathcal{G}$;*
(2)   *h restricted to each $\mathcal{F}$ leaf within the support of μ is uniformly C^{1+}.*

The main thing that needs to be verified to apply the previous lemma is that the $\mathcal{S}^F$ foliation is absolutely continuous. This is immediate from the following lemma because $\|DF|_{\mathcal{E}}\| = \lambda$ is constant. This lemma follows from work of Ledrappier and Young [**LY85**]. An equivalent statement that applies to our setting appears in [**SY19**, Theorem 2.5]

LEMMA 6.3. *Let F be a C^{1+} diffeomorphism and let μ be an F-invariant measure. Suppose that $\mathcal{F}$ is an expanding foliation for F. Suppose that ξ is an increasing measurable partition subordinate to $\mathcal{F}$ and μ. Then the conditional measures of μ are absolutely continuous on the leaves of $\mathcal{F}$ if and only if*

$$H_\mu(F^{-1}\xi \mid \xi) = \int \log \|DF \mid_{T\mathcal{F}}\| \, d\mu,$$

where $H_\mu(F^{-1}\xi \mid \xi)$ is the conditional entropy of $F^{-1}\xi$ given ξ.

The previous two lemmas then combine to show the following.

PROPOSITION 6.4. *The conjugacy h in Proposition 5.1 is uniformly C^{1+} on leaves of the $\mathcal{S}^L$ foliation.*

Using this differentiability, we can show the following proposition.

PROPOSITION 6.5. *The bundle $\mathcal{E}$ in Lemma 4.1 is trivial.*

Proof. Recall the linear frame $[e_1^L, e_2^L]$ of $E^{u,L}$. Because h intertwines $\mathcal{S}^L$ and $\mathcal{S}^F$, and is C^{1+} along these foliations, we see that $Dh(e_1^L)$ is a continuous framing of $\mathcal{E}$; hence, $\mathcal{E}$ is trivial. □

We can now prove Proposition 1.4, which says that if F has the same periodic data as L, which has Jordan blocks, then the stable and unstable bundles of F are U_λ and $U_{\lambda^{-1}}$-framed, respectively.

Proof of Proposition 1.4. By Proposition 6.5, the bundle $\mathcal{E}$ from Lemma 4.1 has a framing $\widetilde{e}_1^F$. The periodic data for $DF|_{\mathcal{E}}$ is exactly multiplication by λ. Hence, by application of a Livsic argument, we can rescale the $\widetilde{e}_1^F$ framing by a Hölder function to find a new framing e_1^F with the property that

$$D_x F e_1^F(x) = \lambda e_1^F(F(x)).$$

We now show that E^u and $E^u/\mathcal{E}$ are trivial. By Proposition B.3, the bundle E^u is orientable, and hence, by [**MS74**, Problem 12-A], $w_1(E^u) = 0$. However, E^u is isomorphic to $E^u/\mathcal{E} \oplus \mathcal{E}$, so $w_1(E^u) = w_1(E^u/\mathcal{E}) \oplus w_1(\mathcal{E})$ by additivity of the first Stiefel–Whitney class, see [**MS74**, §4]. Thus, we find that $w_1(E^u/\mathcal{E}) = 0$, and hence this bundle is trivial. This also implies that E^u is trivial as it is the sum of trivial bundles.

Thus, we see that $E^{u,F}/\mathcal{E}$ also admits a non-vanishing section $\overline{e}_2^F$ because it is trivial. By the same Livsic argument as before, we can rescale $\overline{e}_2^F$ so that DF acts on this section as multiplication by λ. Let e_2^F be a Hölder continuous section of E^u projecting to $\overline{e}_2^F$ along $\mathcal{E}$. Then, with respect to the framing $[e_1^F, e_2^F]$, $DF|_{E^u}$ is U_λ-framed. The same considerations apply to the stable bundle. □

Definition 6.6. In the following, we will say that an Anosov diffeomorphism is U_λ-framed if the following hold:

(1) F is in the same homotopy class as an Anosov automorphism $L \in \mathrm{SL}(4, \mathbb{Z})$ with a Jordan block;

(2) the stable and unstable bundles of F are trivial;

(3) $DF|_{E^{u,F}}$ and $DF|_{E^{s,F}}$ admit Hölder continuous U_λ- and $U_{\lambda^{-1}}$-framings, where $\lambda > 1$ is an eigenvalue of L. We refer to the unstable framing as $[e_1^F, e_2^F]$.

Remark 6.7. Once we know that the bundle $E^{u,F}$ is U_λ-framed, for some framing $[e_1^F, e_2^F]$, we may then choose a new metric on $E^{u,F}$ as in Remark 4.3 by declaring e_1^F and e_2^F to be unit length and orthogonal. This new metric will satisfy all of the properties of the metric from that remark, but will have the additional property that it makes $[e_1^F, e_2^F]$ an orthogonal frame. We will use this metric below.

7. Regularity of the 'slow' foliation

To upgrade the regularity of the 'slow' foliation, we will use the normal forms developed by Kalinin and Sadovskaya. The following theorem is a combination of [**Kal24**, Theorem 4.6, Corollary 4.8, and Remark 4.2]. In the following theorem, the required estimate on ϵ is

particularly simple because we have assumed that r is bounded away from 1 (see [**Kal24**, Corollary 4.8]).

THEOREM 7.1. (Non-stationary linearization) *Let F be a C^r, $r \in (3/2, \infty]$ diffeomorphism of a smooth manifold X and let $\mathcal{W}$ be an F-invariant topological foliation of X with uniformly C^r leaves. Let $\chi > 0$ and*

$$0 < \epsilon < \chi/5.$$

Suppose that the linear extension $DF|_{T\mathcal{W}}$ satisfies

$$e^{\chi-\epsilon}\|v\| \le \|D_x F(v)\| \le e^{\chi+\epsilon}\|v\|. \tag{7}$$

Then, there exists a family $\{\mathcal{H}_x\}_{x \in X}$ of C^r diffeomorphisms $\mathcal{H}_x : \mathcal{W}(x) \to T_x\mathcal{W}(x)$ satisfying $\mathcal{H}_x(x) = 0$ and $D_0\mathcal{H}_x = \mathrm{Id}$ such that for each $x \in X$,

$$D_x F|_{T\mathcal{W}(x)} = \mathcal{H}_{F(x)} \circ F \circ \mathcal{H}_x^{-1} : T_x\mathcal{W} \to T_{F(x)}\mathcal{W}(F(x)). \tag{8}$$

The maps $\mathcal{H}_x$ restricted to balls of uniform radius depend continuously on $x \in X$ in the $C^{\lfloor r \rfloor}$ topology and have derivative that is uniformly $r - \lfloor r \rfloor$ Hölder. Further, for each $y \in \mathcal{W}_x$, the map $\mathcal{H}_y \circ \mathcal{H}_x^{-1} : T_x\mathcal{W} \to T_y\mathcal{W}$ is affine.

We now upgrade the regularity of the $\mathcal{S}^F$ foliation. The idea behind the subsequent proofs is to use that the normal forms coordinates are uniformly bilipschitz on small balls. This allows us to compare the rate at which curves shrink in both the normal forms coordinates and in the ambient manifold.

PROPOSITION 7.2. *Suppose that F is a C^{2+} Anosov diffeomorphism that is U_λ-framed as in Definition 6.6 and has the same Jordan data as L. Then, the $\mathcal{S}^F$ foliation is uniformly C^{2+} when restricted to $\mathcal{W}^{u,F}$ leaves.*

Remark 7.3. The following proof uses the normal forms coordinates. One could also prove this by using that the cocycle holonomies restricted to leaves are C^{1+} and show that the e_1^F distribution is invariant under these holonomies by studying the map $F^n H_{F^{-n}x F^{-n}y} F^{-n}$, where H_{xy} denotes the cocycle holonomy between two points x and y in the same unstable leaf.

Proof. To begin, let us explain why normal forms coordinates exist for the $\mathcal{W}^{u,F}$ foliation using Theorem 7.1. The main thing to check is equation (7). Due to our assumption on the periodic data, from [**Kal11**, Theorem 1.3], it follows that for each $\epsilon > 0$, there is $c_\epsilon > 0$ such that for any $v \in E^{u,F}$,

$$c_\epsilon e^{\chi-\epsilon}\|v\| < \|D_x f(v)\| < c_\epsilon^{-1} e^{\chi+\epsilon}\|v\|.$$

We can then eliminate the constant c_ϵ by passing to an adapted metric. Thus, we see that the coordinates from the conclusion of Theorem 7.1 exist. We call these normal forms coordinates.

We now use the normal forms to finish the proof. Note that if $\gamma : [0, 1] \to M$ is a C^1 curve tangent to the e_1^F distribution, then for some $C = \mathrm{len}(\gamma)$, we have

$$d_{\mathcal{W}^{u,F}}(F^{-n}\gamma(0), F^{-n}\gamma(1)) \leq C\lambda^{-n}, \tag{9}$$

because $DF^{-1}(e_1^F) = \lambda^{-1}e_1^F$. Note that for any $x \in M$, the Riemannian distance on $\mathcal{W}_{loc}^{u,F}(x)$ and the Euclidean distance on $\mathcal{W}^{u,L}$ with respect to the normal forms chart on $\mathcal{W}^{u,F}$ are uniformly bi-Lipschitz as the normal forms depend continuously on the basepoint in the C^1 topology.

Note that $T_x\mathcal{W}^{u,F}$ comes equipped with the framing $[e_1^F, e_2^F]$, which gives coordinates on it as a manifold and as a vector space. We will always use this framing to express differentials of maps as matrices. As explained in Remark 6.7, we may choose an adapted metric on $T_x\mathcal{W}^{u,F}$ that makes e_1^F and e_2^F orthonormal.

Note that

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^{-n} = \lambda^{-n} \begin{bmatrix} 1 & -n\lambda^{-1} \\ 0 & 1 \end{bmatrix}.$$

Hence, by equation (8), F^{-n}, when viewed as a map $T_x\mathcal{W}^{u,F} \to T_{F^{-n}(x)}\mathcal{W}^{u,F}$ in normal forms coordinates, is

$$\mathcal{H}_{F^{-n}(x)}F^{-n}\mathcal{H}_x^{-1} = \lambda^{-n} \begin{bmatrix} 1 & -n\lambda^{-1} \\ 0 & 1 \end{bmatrix}. \tag{10}$$

Suppose that $\gamma : [0, 1] \to T_x\mathcal{W}^{u,F}$ is non-constant in the e_2^F direction. Then, it is immediate from equation (10) that there exists C_1 such that

$$d_{T_{F^{-n}(x)}\mathcal{W}^{u,F}}(\mathcal{H}_{F^{-n}(x)}F^{-n}\mathcal{H}_x^{-1}(\gamma(0)), \mathcal{H}_{F^{-n}(x)}F^{-n}\mathcal{H}_x^{-1}(\gamma(1))) \geq C_1 n\lambda^{-n}. \tag{11}$$

Consider now a curve η in $\mathcal{W}^{u,F}$ that is tangent to the e_1^F distribution. Then, as in equation (9), there exists C_2 such that

$$d_{\mathcal{W}^{u,F}(F^{-n}(x))}(F^{-n}(\eta(0)), F^{-n}(\eta(1))) \leq C_2\lambda^{-n}.$$

Note that under backwards iteration, the curve $F^{-n} \circ \gamma$ lies in a uniformly small ball. There exists C_3 such that on such uniformly small balls, $\mathcal{H}_x$ is C_3-bi-Lipschitz. Thus,

$$d_{T_{F^{-n}(x)}\mathcal{W}^{u,F}}(\mathcal{H}_{F^{-n}(x)}F^{-n}(\eta(0)), \mathcal{H}_{F^{-n}(x)}F^{-n}(\eta(1))) \leq C_3 C_2\lambda^{-n}. \tag{12}$$

Thus, $\mathcal{H}_x\eta$ must lie tangent to the linear e_1^F foliation on $T_x\mathcal{W}^{u,F}$ because otherwise, equation (11) would contradict equation (12).

This implies that the linear foliation of $T_x\mathcal{W}^{u,F}$ tangent to e_1^F pushes forward by $\mathcal{H}_x$ to the $\mathcal{S}^F$ foliation. Thus, the $\mathcal{S}^F$ foliation has the same regularity as $\mathcal{H}_x$, as it is the image under $\mathcal{H}_x$ of the e_1^F foliation. As the maps $\mathcal{H}_x$ are uniformly locally C^{2+}, this implies that $\mathcal{S}^F$ is a uniformly C^{2+} foliation subordinate to $\mathcal{W}^{u,F}$. $\square$

We may consider holonomies along the $\mathcal{S}^L$ foliation between distant uniform transversals T and W. As the holonomies of the $\mathcal{S}^L$ foliation are isometries, this implies that the resulting map $T \to W$ is uniformly continuous independent of the distance between T

and W along $\mathcal{S}^L$ leaves. We will now develop a sequence of lemmas and prove that the analogous property holds for $\mathcal{S}^F$.

For $\theta \in [0, \pi]$, we say that a transversal W is θ-transverse to $\mathcal{S}^F$ if at every point, we have that W makes an angle of at least θ with $\mathcal{S}^F$. The following says that transversals to the $\mathcal{S}^F$ foliation travel between uniformly distant leaves in uniform time. To describe the distance between leaves, we use the notion of the Hausdorff distance between two sets in a metric space (M, d). The *Hausdorff distance* is defined as

$$d_H(X, Y) = \max \left\{ \sup_{x \in X} d(x, Y), \sup_{y \in Y} d(y, X) \right\}.$$

If X and Y are two sets which both lie inside a leaf W of another foliation $\mathcal{W}$, then we write

$$d_H^{\mathcal{W}}(X, Y) = \max \left\{ \sup_{x \in X} d_W(x, Y), \sup_{y \in Y} d_W(y, X) \right\},$$

where d_W is the distance along the leaf W with the W term's pullback metric.

We will prove the following by covering the foliation with tiny charts in which the claim is trivial.

LEMMA 7.4. *Suppose that F is U_λ-framed. For any $D > 0$, $\theta \in (0, \pi/2]$, there exist $C > 0$ such that if $x \in \mathcal{W}^{u,F}(y)$ and $d_H^{\mathcal{W}^{u,F}}(\mathcal{S}^F(x), \mathcal{S}^F(y)) < D$, then any θ-uniform transversal from x to $\mathcal{S}^F(y)$ hits $\mathcal{S}^F(y)$ in distance at most C.*

Before we begin the proof, we introduce some definitions. As a reminder, a local product neighborhood for a point in a foliation is a foliation chart on a neighborhood of the point. Let $B_r^n(0)$ denote the ball of r radius in $\mathbb{R}^n$. For what follows, we will work with foliation charts whose domain is of the form $\phi \colon [-a, b] \times B_r^{n-1}(0) \subset \mathbb{R}^n \to M$; the reason for this is so that we can refer to the 'top' and 'bottom' plaques of the chart, by which we mean $\phi(\{-a\} \times B_r^{n-1}(0))$ and $\phi(\{b\} \times B_r^{n-1}(0))$. As the $\mathcal{S}^F$ foliation is transversely orientable, we require that this ordering of the end leaves agrees with the transverse orientation on the foliation.

As the $\mathcal{S}^L$ foliation restricted to $\mathcal{W}^u$ leaves is essentially the same as a foliation of Euclidean $\mathbb{R}^2$ by parallel lines, it makes perfect sense to speak of the vertical 'distance' between two leaves. This distance is precisely equal to the Hausdorff distance between two leaves, though we refer to it as vertical as we are thinking of it in this geometric way.

Definition 7.5. For $\theta \in (0, \pi/2]$, $t_0 > 0$, $\epsilon > 0$, a (θ, t_0, ϵ)-chart for a point x is a local product neighborhood for the $\mathcal{S}^F$ foliation such that x lies in the middle of the bottom plaque of the neighborhood and any unit speed transversal beginning from x that is θ-transverse hits the top leaf of the foliation chart in at most t_0 time. Further, we require that h carries the top and bottom plaques of the chart to leaves of the $\mathcal{S}^L$ foliation at vertical distance at least ϵ from each other.

Proof of Lemma 7.4. To begin, we show that for every $\theta \in (0, \pi]$, there exist $t_\theta, \epsilon > 0$ such that every point $x \in \mathbb{T}^4$ is contained in the bottom leaf of a $(\theta, t_\theta, \epsilon_\theta)$-chart. To prove this, we essentially use two claims, one about the uniform continuity of h and the other about the uniform C^1-ness of $\mathcal{S}^F$.

(1) Note that because h is uniformly continuous, for any $\epsilon > 0$, there exists $\epsilon_L > 0$ such that if P_1 and P_2 are two bounded plaques of the $\mathcal{S}^F$ foliation in the same $\mathcal{W}^{u,F}$ leaf with $d(P_1, P_2) > \epsilon$, then $h(P_1), h(P_2)$ lie in leaves of the $\mathcal{S}^L$ foliation at vertical distance at least $\epsilon_L > 0$ apart.

(2) For a fixed $\theta \in (0, \pi]$, because the $\mathcal{S}^F$ foliation is uniformly C^1, for every sufficiently small $\epsilon > 0$, there exist $t_{\epsilon,\theta} > 0$ and $\delta_{\epsilon,\theta} > 0$ such that every point x is contained in the middle of the bottom plaque of a product neighborhood where a θ-transversal beginning from x hits the top plaque in at most $t_{\epsilon,\theta}$ time and, in addition, $d(P_1, P_2) > \delta_{\epsilon,\theta}$.

By combining these two statements, we obtain that for any $\theta > 0$, there exist $t_\theta, \epsilon_\theta > 0$ and a cover of the $\mathcal{S}^F$ foliation by $(\theta, t_\theta, \epsilon_\theta)$-charts. Further, note that as the $\mathcal{S}^F$ foliation is transversely orientable, we may choose these charts to agree with this transverse orientation.

We now show that uniform transversals uniformly travel 'vertically'. Consider a θ-transverse curve γ to the $\mathcal{S}^F$ foliation passing through a point x. Let B be a $(\theta, t_\theta, \epsilon_\theta)$-neighborhood for x and let P_1, P_2 be the top and bottom plaques of B. Note that the vertical distance between $h(P_1)$ and $h(P_2)$ is at least ϵ_θ. Thus, we see that as γ traverses B, the vertical height of $h(\gamma)$ increases by ϵ_θ. This implies that $h(\gamma(0))$ and $h(\gamma(nt_\theta))$ are vertical distance at least $n\epsilon_\theta$ apart.

If $\mathcal{S}^F(x)$ and $\mathcal{S}^F(y)$ are at Hausdorff distance at most D apart, then because h is a quasi-isometry, there exists $D' > 0$ such that $d_H^{\mathcal{W}^{u,L}}(h(\mathcal{S}^F), h(\mathcal{S}^F)) < D'$. Thus, we see that a unit-speed uniform θ-transversal starting at the $\mathcal{S}^F(x)$ leaf will reach the $\mathcal{S}^F(y)$ leaf in at most $\lceil D'\epsilon_\theta^{-1} t_\theta \rceil$ time. The claim follows. $\qquad\square$

We may now prove the following.

LEMMA 7.6. *Suppose that F is U_λ-framed, then the holonomies of the $\mathcal{S}^F$ foliation are uniformly C^1. That is, if W and T are two uniform C^1 holonomy related transversals, then the map $\Pi_{T,W} \colon T \to W$ has uniformly continuous Jacobian, which is bounded independent of the transversals.*

The approach of the following proof is to use normal forms coordinates to linearize the $\mathcal{S}^F$ foliation and study how transversals to the $\mathcal{S}^F$ foliation are stretched by its holonomies. Because $\mathcal{S}^F$ is intertwined with the $\mathcal{S}^L$ foliation, which has isometric holonomies, the holonomies of $\mathcal{S}^F$ can only stretch curves a limited amount transverse to the $\mathcal{S}^F$ foliation. This observation then implies the result.

Proof. By Lemma 7.4, there exists $C_1 > 0$ such that if T is a transversal to the $\mathcal{S}^F$ foliation of length less than 1, and W is a uniform transversal to the $\mathcal{S}^F$ foliation, then $\Pi_W(T)$ has length less that C_1. This is immediate because $\Pi_W(T)$ is itself a uniform transversal.

Let $\mathcal{H}_x \colon \mathcal{W}^{u,F}(x) \to T_x \mathcal{W}^{u,F}(x)$ be the normal forms coordinates as in Theorem 7.1. On balls of uniform radius $r_0 \gg C_1$, the normal forms coordinates are uniformly D-bi-Lipschitz for some $D \geq 1$. By that theorem, we have in addition that if $x \in \mathcal{W}^{u,F}(y)$, then the map $\mathcal{H}_y \circ \mathcal{H}_x^{-1} \colon T_x \mathcal{W}^{u,F} \to T_y \mathcal{W}^{u,F}$ is affine. In fact, its differential, with

respect to the $[e_1^F, e_2^F]$ framing, is of the form

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}, \tag{13}$$

because the normal forms intertwine the $\mathcal{S}^F$ foliation with the linear foliation of $T_x \mathcal{W}^{u,F}$ tangent to e_1^F as we saw in the proof of Proposition 7.2.

Fix a small number $0 < \epsilon \ll r_0$. Suppose that I is a uniform transversal to the $\mathcal{S}^F$ foliation through the point x that has length at least ϵ, but length less than r_0.

Let $I_x = \mathcal{H}_x \circ I$. Then, I_x is a curve in $T_x \mathcal{W}^{u,F}$. Suppose that dI_x/dt is $\alpha(t)e_1^F + \beta(t)e_2^F$. Because I_x has length at least $D^{-1}\epsilon$ and is uniformly transverse to $\mathcal{S}^F$, there exists a uniform constant $C_2 > 0$ such that

$$\int \beta(t) \, dt \geq C_2 D^{-1} \epsilon. \tag{14}$$

Consider a point $y \in \mathcal{W}^{u,F}$ such that $y \in \mathcal{S}^F(I)$. If we view I_x in the $T_y \mathcal{W}^{u,F}$ normal coordinates, then I_x has differential $D\mathcal{H}_y \mathcal{H}_x^{-1} \circ I_x$, which by equation (13) has differential with e_2^F component equal to

$$c\beta(t)e_2^F.$$

Let J be a curve through 0 in $T_y \mathcal{W}^{u,F}$ tangent to e_2^F. Note that $\mathcal{H}_y^{-1} J$ is a uniform transversal to the $\mathcal{S}^F$ foliation. As a curve in $T_y \mathcal{W}^{u,F}$, $\Pi_J \mathcal{H}_y \mathcal{H}_x^{-1} I_x$ has tangent $c\beta(t)e_2^F$. Thus, by equation (14),

$$\mathrm{len}(\Pi_J \mathcal{H}_y I) \geq cC_2 D^{-1} \epsilon.$$

However, by the uniform D-bi-Lipschitzness of normal coordinates, this implies that

$$\mathrm{len}(\Pi_{\mathcal{H}_y J} I) \geq cC_2 D^{-2} \epsilon.$$

However, by the first paragraph of this proof, the length of this curve is at most C_1. Thus,

$$C_1 \geq cC_2 D^{-2} \epsilon,$$

so

$$c \leq C_1 C_2^{-1} D^2 \epsilon^{-1}.$$

Note that this estimate is independent of x and y. This implies that in normal forms coordinates, the holonomies are uniformly C^1 because the entry c in equation (13) is uniformly bounded. However, this implies that the same result holds for the $\mathcal{S}^F$ holonomies because the normal forms coordinates are uniformly locally C^{2+}. □

The previous discussion shows that the distribution defined by the vector field e_1^F is C^{1+}; however, this is not the same as the vector defining the distribution being C^1. We now upgrade the regularity by using regularity for solutions of Livsic equations following [**NT98**, Theorem 2.2].

LEMMA 7.7. *Suppose that F is U_λ-framed. Then, the vector field e_1^F is uniformly C^{1+} along $\mathcal{W}^{u,F}$.*

Proof. The vector field e_1^F is only Hölder continuous, but we would like a C^{1+} version of it. So, let $\widetilde{e}_1^F$ be a C^{1+} rescaling of e_1^F, which exists because e_1^F is tangent to a C^{1+} distribution by Lemma 7.2.

Let $[e_1^F]$ denote the subspace tangent to e_1^F. Then consider the C^{1+} cocycle $A: \mathbb{T}^4 \to \mathbb{R}$ arising from $DF|_{[e_1^F]}: \mathbb{T}^4 \to \mathbb{R}$ that sends x to the number $\eta(x)$ such that $DF\widetilde{e}_1^F = e^{\eta(x)}\widetilde{e}_1^F$. Note that if p is a periodic point of period p, then $\sum_{i=1}^n A(F^i(p)) = n \ln \lambda$.

We claim that, in fact, A is cohomologous to constant via a transfer function $\phi: \mathbb{T}^4 \to \mathbb{R}$ that is uniformly C^{1+} along $\mathcal{W}^u$ leaves. This follows from the argument in [**NT98**, Theorem 2.2], which we now sketch. If we view the function A as defining a cocycle $\mathcal{A}$ on the space $\mathbb{T}^4 \times \mathbb{R}$, then this cocycle is partially hyperbolic because $\mathbb{R}$ is abelian. Hence, the unstable foliations lift to $\mathbb{T}^4 \times \mathbb{R}$ and these are uniformly C^{1+} foliations $\mathcal{W}^{u,\mathcal{A}}$. If $\mathcal{B}$ is another uniformly $C^{1+}\mathbb{R}$-valued cocycle along $\mathcal{W}^{u,F}$ leaves arising from a function $B: \mathbb{T}^4 \to \mathbb{R}$, then we similarly get uniformly C^{1+} foliations $\mathcal{W}^{u,\mathcal{B}}$. One can then check that the transfer function carries $\mathcal{W}^{u,\mathcal{A}}$ leaves to $\mathcal{W}^{u,\mathcal{B}}$ leaves. As each of these foliations has uniformly C^{1+} leaves, we see by the implicit function theorem that a transfer function must be uniformly C^{1+} as well.

Using the function ϕ to rescale $\widetilde{e}_1^F$, we may find a new vector field $\hat{e}_1^F$ that is uniformly C^{1+} on $\mathcal{W}^{u,F}$ leaves and satisfies $DF\hat{e}_1^F = \lambda \hat{e}_1^F$. It suffices to now check that e_1^F coincides with $\hat{e}_1^F$. Note that $\hat{e}_1^F = \beta(x)e_1^F$ for some continuous function β. Then computing $DF\beta(x)e_1^F$ in two different ways, we find

$$\lambda\beta(x)e_1^F(\sigma(x)) = \lambda\beta(F(x))e_1^F,$$

and hence β is constant on orbits and hence is constant. Thus, in fact $\hat{e}_1^F = ce : 1^F$ and so e_1^F was C^{1+} all along. $\qquad\square$

LEMMA 7.8. *Suppose that F is U_λ-framed and let ϕ^t be the flow along the e_1^F vector field, that is, tangent to slow leaves. Then,*

$$\phi^t = F^n \circ \phi^{\lambda^{-n}t} \circ F^{-n}. \tag{15}$$

Proof. Both sides of the equality are flows of C^1 vector fields, so it suffices to check that they have the same generating field, that is, the right-hand side is generated by e_1^F.

If we write ϕ^t in coordinates as $(x, y) \mapsto (x, y) + te_1^F + O(t^2)$, then we see that the composition takes

$$\begin{aligned}
(x, y) &\mapsto F^{-n}(x, y)\\
&\mapsto F^{-n}(x, y) + t\lambda^{-n}e_1^F + O((\lambda^{-n}t)^{1+\alpha})\\
&\mapsto (x, y) + te_1^F + O(n\lambda^{-n\alpha}t^{1+\alpha}),
\end{aligned}$$

as desired. $\qquad\square$

The following proof is the only place where we use that the Jordan periodic data of F are exactly those of L. It seems that, in general, if G has Jordan periodic data α_G and $\int \alpha_G \, d\,\mathrm{vol} > 0$, then the following proof can still be carried out. It is unknown to the author whether this property necessarily holds for the Jordan data of all Anosov diffeomorphisms with the periodic data of L.

LEMMA 7.9. *Let F be U_λ-framed. Let ϕ^t be the flow along e_1^F and suppose that F has the same Jordan data as L. Then, the flow ϕ^t carries uniform C^1 transversals to $\mathcal{S}^F$ to uniform C^1 transversals to $\mathcal{S}^F$.*

Proof. To do this, we will study the differential of the flow ϕ^t in the $[e_1^F, e_2^F]$ framing. We will do so by first studying the differential of the flow ϕ^t. We write all differentials below with respect to the $[e_1^F, e_2^F]$ framing.

Because e_1^F is a C^{1+} vector field, we have that the flow map $\phi^t(x)$ is C^{1+} in time and initial conditions. (The author is unaware of a reference in the literature for this claim; however, one can show this by reading, for example, [Izz99] and keeping track of the Hölder constant during Picard iteration.) Thus, in a coordinate chart, we may write $D\phi^t = \mathrm{Id} + O(t^\sigma)$ for the differential of the flow of time t for some $0 < \sigma < 1$. We will only be interested in this differential for increasingly short times $\lambda^{-n}t$.

We now find an expression for $D\phi^t$ in terms of the $[e_1^F, e_2^F]$ framings. We can always choose our coordinates to be tangent to the vector fields $[e_1^F, v]$, where v is some smooth transverse field to e_1^F. Then, $v = \alpha e_1^F + \beta e_2^F$ and we can insist that both α and β are C^σ-Hölder and that β is uniformly bounded below by transversality. For ease of notation, let x be the initial point and $y = \phi^{\lambda^{-n}t}$.

To change into the $[e_1^F, e_2^F]$ framing, we must conjugate $\mathrm{Id} + O(\lambda^{-n\sigma}t^\sigma)$ as follows:

$$
\begin{bmatrix} 1 & -\alpha(y)\beta^{-1}(y) \\ 0 & \beta^{-1}(y) \end{bmatrix} [\mathrm{Id} + O(\lambda^{-n\sigma}t^\sigma)] \begin{bmatrix} 1 & \alpha(x) \\ 0 & \beta(x) \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & -\alpha(y)\beta^{-1}(y) \\ 0 & \beta^{-1}(y) \end{bmatrix} \begin{bmatrix} 1 & \alpha(x) \\ 0 & \beta(x) \end{bmatrix} + O(\lambda^{-n\sigma}t^\sigma)
$$

$$
= \begin{bmatrix} 1 & \alpha(x) - \alpha(y)\beta^{-1}(y)\beta(x) \\ 0 & \beta^{-1}(y)\beta(x) \end{bmatrix} + O(\lambda^{-n\sigma}t^\sigma).
$$

Note that $d(x, y) = O(\lambda^{-n}t)$. As β is bounded below, this implies that

$$
\beta^{-1}(y)\beta(x) = 1 + O(\lambda^{-n\sigma}t^\sigma).
$$

Using a Hölder estimate on α and canceling gives that with respect to the $[e_1^F, e_2^F]$ framings,

$$
D\phi^{\lambda^{-n}t} = \mathrm{Id} + O(t^\sigma \lambda^{-n\sigma})
$$

for some $0 < \sigma < 1$.

Hence, with respect to the framings, using the relation $\phi^t = F^n \circ \phi^{\lambda^{-n}t} \circ F^{-n}$, we find

$$
D\phi^t = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix} (\mathrm{Id} + O(t^\sigma \lambda^{-\sigma n})) \begin{bmatrix} \lambda^{-n} & -n\lambda^{-n-1} \\ 0 & \lambda^{-n} \end{bmatrix}.
$$

However, this is converging to Id as $n \to \infty$. Thus, we see that for all times t, $D\phi^t$ is the map that sends $e_1^F \mapsto e_1^F$ and $e_2^F \mapsto e_2^F$ at corresponding base points.

In particular, this implies the result because it shows that curves that are uniformly transverse to the flow direction e_1^F remain uniformly transverse to e_1^F and are not distorted in length as the fields e_1^F and e_2^F are uniformly bounded above and below in length. $\square$

Remark 7.10. It is perhaps not surprising that the e_2^F field is preserved by ϕ^t because the frames $[ce : 1^F, ce : 2^F]$ are the only frames that present the cocycle as a constant Jordan block.

7.1. *The conjugacy is Lipschitz.* Now that we have established stronger properties of the $\mathcal{S}^F$ foliation, we will use these to show that h is Lipschitz.

LEMMA 7.11. *Suppose that F is C^{2+}, U_λ-framed, and has the same Jordan periodic data as L. Then, h is uniformly Lipschitz restricted to each unstable leaf.*

The proof of Lemma 7.11 is based on exhibiting h as the uniform limit of a sequence of uniformly C^1 functions. The main difficulty we encounter is that we cannot establish regularity of h in a single step. Instead, we establish regularity of h incrementally by studying its regularity in some directions before others. As an ansatz, suppose we knew that h was differentiable with

$$D_z h = \begin{bmatrix} a(z) & b(z) \\ 0 & c(z) \end{bmatrix},$$

with respect to the framings $[e_1^L, e_2^L]$ and $[e_1^F, e_2^F]$. It is easy to deduce that $a(z) = C$ for some number C because h intertwines the $\mathcal{S}^L$ and $\mathcal{S}^F$ foliations. It is straightforward to check that $c(z) = C$ as well by using that $b(z)$ is uniformly bounded. Below, we study the 'differential' of h in this manner building up information little by little. We first study the possible diagonal 'entries' of the matrix representing the derivative of h and then study the 'upper right-corner' of Dh.

What makes this approach difficult is that the existence of a particular partial derivative is not a property that is invariant under coordinate change. To see what is meant by this, consider a function $\phi \colon \mathbb{R}^2 \to \mathbb{R}$ that has a partial derivative $\partial_x \phi$ at every point but does not have the partial derivative $\partial_y \phi$ defined at any point. If we precompose ϕ by a diffeomorphism $f \colon \mathbb{R}^2 \to \mathbb{R}^2$, then $\phi \circ f \colon \mathbb{R}^2 \to \mathbb{R}$ may not have any partial derivatives at any point. This may happen, for instance, if f does not preserve the leaves of the foliation of $\mathbb{R}^2$ by horizontal lines. Naturally, $\phi \circ f$ still has derivatives along *some* curves. However, the existence of the derivative is not guaranteed by merely knowing that the curve is tangent to a particular direction. For example, there are smooth curves $\gamma_1, \gamma_2 \colon (-1, 1) \to \mathbb{R}^2$ such that $\dot{\gamma}_1(0) = \dot{\gamma}_2(0)$, where $\phi \circ \gamma_1$ has a derivative at 0 yet $\phi \circ \gamma_2$ is not differentiable at 0. For instance, consider a curve γ_2 that is tangent to a horizontal line but only to first order.

We will deal with the issues alluded to in the above paragraph by using foliations to control the directions in which we differentiate. By proceeding in this manner, we can still recover the information we need by studying a derivative normal to the foliations $\mathcal{S}^L$ and $\mathcal{S}^F$, which we introduce in Definition 7.16.

This section relies on the following lemma of de la Llave. The version we give below is a slight rephrasing of the original adapted to our setting. See [**GKS20**] for a recent example of the use of this lemma in a similar context.

LEMMA 7.12. [**dlL02**, Theorem 2.1] *Let F, G be C^1 Anosov diffeomorphisms of a closed manifold M. Let h be a homeomorphism of M such that:*

(1) $h \circ F = G \circ h$.

Let k be a map—not necessarily invertible or continuous—such that:

(2) $k(\mathcal{W}_x^{u,F}) \subset \mathcal{W}_{h(x)}^{u,G}$; *and*

(3) $\sup_x d_{\mathcal{W}^{u,G}}(k(x), h(x)) < \infty$.

Then,

$$h(x) = \lim_{n \to \infty} G^{-n} \circ k \circ F^n(x),$$

and the limit is reached uniformly with respect to the distances $d_{\mathcal{W}^{u,F}}$ and $d_{\mathcal{W}^{u,G}}$.

As the convergence in Lemma 7.12 is uniform, we can use it to study whether the conjugacy is Lipschitz by exhibiting the conjugacy as a uniform limit of uniformly Lipschitz functions.

In the following, we will write $Dh(e_1^L)$ to mean the derivative of h in the direction of e_1^L along the foliation $\mathcal{S}^L$. By Proposition 6.4, this derivative exists. Note that when we make use of this notation, we are not asserting that h is differentiable.

LEMMA 7.13. *There exists $C_1 \neq 0$ such that $Dhe_1^L = C_1 e_1^F$. Without loss of generality, we may assume that $C_1 = 1$.*

Proof. By Proposition 6.4, we know that this derivative exists and that

$$Dh(e_1^L) = c(x)e_1^F$$

for some Hölder function $c(x)$. Because

$$F \circ h = h \circ L,$$

we can differentiate this equation along $\mathcal{S}^L$ on both sides and use the chain rule to find that

$$\lambda c(x)e_1^F = \lambda Dh(e_1^L) = Dh(e_1^L)\lambda = c(L(x))\lambda e_1^F.$$

Thus, $c(x)$ is constant on orbits. Hence, as $c(x)$ is continuous, it is constant. To ensure that $C_1 = 1$, we replace $[e_1^F, e_2^F]$ with $[C_1 e_1^F, C_1 e_2^F]$. Note that this does not change the Jordan periodic data or the function α representing the data in this framing.   □

We now show our first smoothing lemma.

LEMMA 7.14. *Suppose that F is C^{2+} and U_λ-framed. There exists a function $h_0 \colon M \to M$ such that, writing h_x for the restriction $h_0|_{\mathcal{W}^{u,L}(x)}$, we have that:*

(1)   *h_0 intertwines the $\mathcal{W}^{u,L}$ and $\mathcal{W}^{u,F}$ foliations as well as the $\mathcal{S}^L$ and $\mathcal{S}^F$ foliations;*

(2)   *h_x is C^1 and is uniformly C^0 close to h with respect to $d_{\mathcal{W}^u}$;*

(3)   *with respect to the framings $[e_1^L, e_2^L]$, $[e_1^F, e_2^F]$, the differential of h_0 is*

$$\begin{bmatrix} 1 & b(x) \\ 0 & a(x) \end{bmatrix},$$

where a is a uniformly bounded continuous function on each leaf and the uniformity is independent of the leaf.

Before beginning the proof, we describe how one studies the regularity in a situation like this. If we want to show that a function $k \colon \mathcal{W}^{u,L}(x) \to \mathcal{W}^{u,F}(h(x))$ is uniformly C^1, then it suffices to check that this is true in a particular family of charts adapted to the C^1 norms on these manifolds. While $\mathcal{W}^{u,L}(x)$ admits a chart $\psi^L \colon \mathbb{R}^2 \to \mathcal{W}^{u,L}(x)$ that is an isometry, and hence uniformly C^1, the leaf $\mathcal{W}^{u,F}(h(x))$ does not *a priori* have such a nice parameterization. However, $\mathcal{W}^{u,F}$ does admit a *uniform* family of charts, which we now describe. Let ϕ be the flow of the C^{1+} vector field e_1^F along the leaves of $\mathcal{S}^F$. Note that for any fixed $t_0 > 0$, the maps ϕ^t for $t \in [-t_0, t_0]$ are uniformly C^{1+}. Fixing numbers $N, t_0 > 0$ and a parameter $\theta > 0$, if $\gamma(t) \colon (-\delta, \delta) \to \mathcal{W}^{u,F}(h(x))$ is any unit speed curve of length between N and $2N$ that makes an angle of at least θ with e_1^F distribution, then the map $\psi_\gamma^F \colon (-\delta, \delta) \times (-t_0, t_0) \to \mathcal{W}^{u,F}(h(x))$ defined by

$$(t_1, t_2) \mapsto \phi^{t_2} \gamma(t_1),$$

is a parameterization of a subset of $\mathcal{W}^{u,F}$. In addition, ψ_γ^F is a C^{1+} chart for the $\mathcal{S}^F$ foliation.

In particular, given this uniform family of charts, we may study regularity in the following way. If we want to show that a map $k \colon \mathcal{W}^{u,L}(x) \to \mathcal{W}^{u,F}(h(x))$ is uniformly C^1, it suffices to show that for each uniform pair of charts ψ_γ^F and ψ^L, we have that

$$(\psi_\gamma^F)^{-1} \circ k \circ \psi^L \tag{16}$$

is uniformly C^1 as a map defined on a subset of $\mathbb{R}^2$. In the proof that follows, we will use this family of charts to study the regularity.

As mentioned before, because the leaves of the $\mathcal{W}^{u,L}$ foliation are Euclidean, they admit global uniformly smooth charts. Specifically, on the $\mathcal{W}^{u,L}$ foliation, we exclusively work with the global charts of the form

$$(t_1, t_2) \mapsto x + t_1 e_1^L + t_2 e_2^L.$$

We construct these charts around a specific transversal to the $\mathcal{S}^L$ foliation of the form $T_0 \colon t \mapsto x + t e_2^L$ in a manner analogous to equation (16). We similarly denote such a chart by $\psi_{T_0}^L$. In addition, these transversals $t \mapsto x + t e_2^L$ define a linear foliation, which we denote by $\mathcal{F}^L$.

Before proceeding, we obtain the following lemma that says that usable transversals to the $\mathcal{S}^F$ foliation exist.

LEMMA 7.15. *In each $\mathcal{W}^{u,F}$ leaf, there exists a uniform C^1 transversal to the $\mathcal{S}^F$ foliation. Moreover, this transversal is given by a map $\mathbb{R} \to \mathcal{W}^{u,F}$ with uniformly continuous, bounded derivative. Further, this transversal is uniformly C^0 close to the image of a uniform transversal to the $\mathcal{S}^L$ foliation under h.*

Proof. To begin, observe that there exist uniformly Hölder continuous topological transversals to the $\mathcal{S}^F$ foliation such as $h(T_0)$, where T_0 is a leaf of $\mathcal{F}^L$. Call this transversal T_1. We will mollify this transversal to obtain the lemma.

We first describe how locally we can mollify this transversal to make it uniformly C^1. As the $\mathcal{S}^F$ foliation is a C^{1+}-foliation, we may view T_1 in a small, uniformly C^{1+} foliation

box for the $\mathcal{S}^F$ foliation. Pulling everything back by this chart, we have a foliation box $[-1, 1] \times [-1, 1]$. By choosing the foliation box sufficiently small and correctly placed relative to T_1, we may arrange that within this box, T_1 is represented by the graph of a Hölder continuous map ϕ such that $\phi(0) = 0$ and $\phi(1) \in (-1/2, 1/2)$. We mollify ϕ to obtain a new function $\hat{\phi}$ that is uniformly C^{1+} on the interval $(-3/4, 3/4)$. Note that the derivatives of $\hat{\phi}$ are uniformly bounded because this is always the case for mollifications of uniformly Hölder functions by convolution with a fixed standard mollifier. Thus, we can locally smooth $\hat{\phi}$ with uniform estimates regardless of where we are smoothing. However, these local mollifications might not piece together to form the full mollified transversal while staying C^0 close to the original one. Nevertheless, we can fix this within the chart where we are mollifying: we may use the C^{1+} flow along the e_1^F vector field as in Lemma 7.9 to fix our mollified curve so that it does not drift away from the topological transversal T_1. As we need only flow a finite distance, the tangent of the flowed curve remains uniformly transverse to the $\mathcal{S}^F$ foliation and the curve's derivative stays uniformly bounded. Thus, by inductively proceeding along the transversal T_1, we obtain the result.   □

We now proceed with the proof of the first smoothing lemma.

Proof of Lemma 7.14. We give the construction for a single leaf $\mathcal{W}^{u,L}(x)$. The full result follows by repeating the construction on each leaf.

Fix a uniform C^1 transversal $T_F \colon \mathbb{R} \to \mathcal{W}^{u,F}(h(x))$ to the $\mathcal{S}^F$ foliation as obtained from Lemma 7.15. In particular, we may use that $h(T_0)$ and T_F are uniformly C^0 close as maps $\mathbb{R} \to \mathcal{W}^{u,F}(x)$ and that the transversal has a parameterization with uniformly bounded derivative.

Letting $T_0 \colon \mathbb{R} \to \mathcal{W}^{u,L}$ be a transversal to $\mathcal{S}^L$ tangent to e_2^L, define $P \colon \mathbb{R} \to \mathbb{R}$ by $T_F^{-1} \circ \Pi_{T_F} \circ h \circ T_0$. Note that P is uniformly Hölder because it is the composition of uniformly Hölder maps.

Along the $\mathcal{F}^L$ leaf T_0, we can mollify P as follows to get a function $\widetilde{P}_{T_0}$. Fixing any smooth bump function σ on $\mathbb{R}$, we define

$$\widetilde{P}_{T_0}(t) := \int P(t - x)\sigma(x)\,dx.$$

Note that $\widetilde{P}_{T_0}$ is C^0 close to P and that $T_F \circ \widetilde{P}_{T_0}$ is uniformly C^0 close to $T_F \circ P_{T_F} = \Pi_{T_F} \circ h \circ T_0$. However, as T_F is C^0 close to $h \circ T_0$, this is uniformly C^0 close to $h \circ T_0$ and hence $T_F \circ \widetilde{P}_{T_0}$ is uniformly C^0 close to $h \circ T_0$ as maps $\mathbb{R} \to \mathcal{W}^{u,F}$.

We now define the map h_x. Define $\hat{h}_2$ to be the map $T_F \circ \widetilde{P}_{T_0}$. We use $\hat{h}_2$ defined in this way to ensure that the parameterization of T_F we obtain remains C^0 close to the corresponding parameterization of $h(T_0)$. As before, let ϕ be the flow along the e_1^F vector field tangent to the $\mathcal{S}^F$ foliation. In the coordinates $(t_1, t_2) = x + t_1 e_1^L + t_2 e_2^L$ on $\mathcal{W}^{u,L}$, define

$$h_x \colon (t_1, t_2) \mapsto \phi^{t_1}(\hat{h}_2(t_2)). \tag{17}$$

We now check that h_x has the required regularity properties. From the definition, it is immediate that h_x is C^1. Next, we claim that h_x is C^0 close to h. From earlier, we have

that $h(T_0(t))$ is uniformly close to $T_F \circ \widetilde{P}_{T_0}(t)$. We claim that independent of s,

$$\phi^s(T_F \circ \widetilde{P}_{T_0}(t)) \text{ is uniformly close to } h(s, t) = \phi^s(h(T_0(t))).$$

This follows because the flow ϕ^s is a uniform quasi-isometry independent of the value of s: this follows because h is a quasi-isometry and intertwines the flow ϕ^s with the linear flow along $\mathcal{S}^L$, which is isometric.

We now turn to the differentiability properties of h_x. From the definition of h_x in equation (17), h_x is manifestly C^1 and $Dh_x(e_1^L) = e_1^F$. It just remains to check that when written with respect to the bases $[e_1^L, e_2^L]$ and $[e_1^F, e_2^F]$, the bottom right corner of Dh_x is uniformly bounded.

We begin by checking the corresponding claim for a pair of uniform charts. For a uniform transversal γ to $\mathcal{S}^F$, we have uniform charts ψ_γ^F and ψ^L, and may write h_x as (h_1, h_2) in these charts. We claim that in such uniform charts, the derivative of h_x looks like

$$\begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix}, \tag{18}$$

where a is a uniformly bounded continuous function. To see this, note first that h_2 in these charts is equal to $\gamma^{-1} \circ \Pi_\gamma \circ T_F \circ \widetilde{P}_{T_0}$. By Lemma 7.6, the map Π_γ is uniformly C^1 as a map $T_F \to \gamma$, as these are both uniform transversals. Thus, as the remaining maps in the composition are uniformly C^1, so is the entire composition. Hence, a is uniformly bounded.

Getting uniformity for the specific framing in the lemma requires only slightly more work. The matrix in equation (18) is written with respect to the chart coordinate frames, which are vector fields $[e_1^L, e_2^L]$ and $[e_1^F, \partial_y]$, where ∂_y depends on the chart ψ_γ^F. The field e_2^F in which we are actually interested is a linear combination of e_1^F and the chart coordinate ∂_y, that is, $e_2^F = ce : 1^F + d\partial_y$, where c and d are uniformly Hölder and d is uniformly bounded away from 0 because, in uniform charts, the coordinate directions are uniformly transverse. Thus, written with respect to bases $[e_1^L, e_2^L]$ and $[e_1^F, e_2^F]$, Dh_x has the following form:

$$\begin{bmatrix} 1 & * \\ 0 & ad^{-1} \end{bmatrix}.$$

Thus, the derivative is uniformly bounded as required. □

7.2. *Normal differentiability.* For a function $A \colon \mathbb{R}^2 \to \mathbb{R}^2$, one often studies differentiability of A by studying the partial derivatives of A. However, the usual definition of a partial derivative is far from being formulated in a coordinate independent way: the partial derivatives of a function are its derivatives along the coordinate directions. For a function A that is C^1, then there is no problem with this, because we can recover the derivative of A in any particular direction from the knowledge of the directional derivatives of A in other directions. For a C^1 function A, the derivatives of A are determined by its derivatives along certain curves. If we know the derivative along enough of these curves, then we know the derivative along any curve.

Normal differentiability deals with the situation where we cannot differentiate along curves, so we cannot use curves directly to define a derivative. However, we may be able to obtain a derivative once we quotient by a foliation. Consider a map $\phi \colon \mathbb{R}^2 \to \mathbb{R}^2$ that preserves the foliation by horizontal lines. Then ϕ induces a map $\widetilde{\phi} \colon \mathbb{R} \to \mathbb{R}$, where we identify $\mathbb{R}$ with the space of horizontal lines. It is entirely plausible that this map $\widetilde{\phi}$ is smooth even though the map ϕ restricted to any vertical line is not even differentiable. In this case, we think of ϕ as being differentiable and the derivative of $\widetilde{\phi}$ as being a 'normal' derivative to the foliation by horizontal lines. By quotienting by the foliation, we are able to canonically detect this normal component of ϕ.

We say that a foliation $\mathcal{F}$ has *well-defined* holonomy if for any two transversals I and J such that $I \subset \mathcal{F}(J)$, we have that for each $x \in I$, $|\mathcal{F}(x) \cap J| = 1$. For example, the foliation of $\mathbb{R}^2$ by lines has well-defined holonomy; in particular, note that $\mathcal{S}^L$ and $\mathcal{S}^F$ both have well-defined holonomy when restricted to unstable leaves. A foliation of a closed manifold might not have well-defined holonomy. When restricted to a coordinate neighborhood, all foliations have well-defined holonomy. However, we will not work with such a local notion as it is not necessary in our case.

Definition 7.16. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are two C^1 foliations with well-defined holonomy. Suppose that ϕ is a continuous map intertwining the $\mathcal{F}$ and $\mathcal{G}$ foliations. We say that ϕ is *normally C^1 differentiable* to $\mathcal{F}$ and $\mathcal{G}$ if for any pair T and W of smooth transversals to $\mathcal{F}$ and $\mathcal{G}$, respectively, with $\phi(T) \subset \mathcal{G}(W)$, the map $T \to W$ given by $\Pi_W \circ \phi$ is C^1. (Here, as before, Π_W denotes the projection to W along the leaves of $\mathcal{G}$.)

Note that because $\mathcal{F}$ and $\mathcal{G}$ are C^1 foliations, the holonomy between different transversals is C^1, and hence this notion of differentiability is well defined independent of the choice of transversals.

When we have fixed a pair of transversals T and W, we will speak about differentiability of ϕ along the transversals T and W normal to the foliations $\mathcal{F}$ and $\mathcal{G}$. Further, because the holonomies of $\mathcal{F}$ and $\mathcal{G}$ are absolutely continuous with respect to the Lebesgue measure, it makes sense to speak of normal differentiability almost everywhere along a transversal. We may also speak of normal differentiability at a pair of points x and $y \in \mathcal{G}(\phi(x))$.

We will restrict the use of this definition to the case where the foliations have dimension 1 and subfoliate a space of dimension 2. In fact, if we have differentiability along such foliations and v and w vectors transverse to $\mathcal{F}$ and $\mathcal{G}$ such that the basepoint of w is the image of the basepoint of v, then there is a well-defined number $D_{v,w}\phi$ that we call the normal derivative of ϕ between v and w, which we define in the following manner. Let T and W be two transversals with $dT/dt = v$ and $dW/dt = w$ at corresponding points. We then define $D_{v,w}\phi$ by

$$D(\Pi_W \circ \phi \circ T)(\partial_t) = (D_{v,w}\phi)w.$$

Having made this definition, we now list a number of claims describing the basic and hopefully intuitive properties of this construction for later use.

CLAIM 7.17. *Suppose that $\phi \colon M_1 \to M_2$ intertwines foliations $\mathcal{F}$ and $\mathcal{G}$ with well-defined holonomy. Suppose that $v \in T_x M_1 \setminus T_x \mathcal{F}$ and $w \in T_y M_2 \setminus T_y \mathcal{G}$ such that $D_{v,w}\phi$ exists.*

Then, for any $v' \in T M_1|_{\mathcal{F}(x)} \setminus T\mathcal{F}(x)$ and $w' \in T M_2|_{\mathcal{G}(\phi(x))} \setminus T\mathcal{G}(x)$, the normal derivative $D_{v',w'}\phi$ exists.

Proof. Supposing such a pair exists, we can express the derivative defining $D_{v',w'}\phi$ by composing with smooth maps. By definition, $D_{v,w}\phi$ existing means that if T is a transversal tangent to w at 0, then

$$\Pi_W \circ \phi \circ T : \mathbb{R} \to W$$

is differentiable at 0. However, this implies that if T' is a transversal tangent to v' at $t = 0$ and W' is tangent to w', then

$$\Pi_{W'}\Pi_W \circ \phi \circ T \circ T^{-1} \circ \Pi_T \circ T' = \Pi_{W'} \circ \phi \circ T'$$

is also differentiable at 0 by the chain rule as we have pre- and post-composed with smooth functions. However, $D(\Pi_{W'} \circ \phi \circ T')$ defines $D_{v',w'}\phi$, which we see exists. $\square$

CLAIM 7.18. *Suppose that $\phi \colon M_1 \to M_2$ intertwines foliations $\mathcal{F}$ and $\mathcal{G}$ with well-defined holonomy. Suppose that ψ is another map intertwining these foliations. Then if $\psi \colon M_1 \to M_2$ satisfies $\psi(x) \in \mathcal{F}(\phi(x))$ for all x, ψ and ϕ are both differentiable where either is differentiable and at such points, ϕ and ψ have the same normal derivative.*

Proof. For a pair of transversals W and T, note that $\Pi_W \circ \phi \circ T = \Pi_W \circ \psi \circ T$; the conclusion is immediate. $\square$

CLAIM 7.19. *If ϕ is C^1 and intertwines foliations as before, then the normal derivative agrees with the usual derivative in the appropriate sense. Namely, if $D\phi(v) = \lambda w$, then $D_{v,w}\phi = \lambda$. More broadly, if $T\mathcal{F}$ and $T\mathcal{G}$ are the tangents to $\mathcal{F}$ and $\mathcal{G}$, then*

$$\Pi_w^{T\mathcal{G}} D\phi(v) = (D_{v,w}\phi)w, \tag{19}$$

where $\Pi_w^{T\mathcal{G}}$ denotes the projection onto the subspace spanned by w along $T\mathcal{G}$.

Proof. We begin by checking equation (19). As before, fix C^1 transversals T and W tangent to v and w. The normal derivative $D_{v,w}F$ is equal to the λ satisfying, for C^1 transversals,

$$(D\Pi_W \circ \phi \circ T)(\partial_t) = (D_{v,w}\phi)w.$$

As ϕ is C^1, we can rewrite this as

$$D(\Pi_W \circ \phi \circ T)(\partial_t) = (D\Pi_W)(D\phi \circ T)(\partial_t) = \Pi_w^{T\mathcal{G}}(DF(v)),$$

which gives equation (19) as desired.

The statement in the second sentence of the claim then follows from equation (19). In this case, if $D(\phi(v))$ is a multiple of w, then $\Pi_w^{T\mathcal{G}} D\phi(v) = D\phi(v)$, so $D_{v,w}\phi = \lambda$ by the definition of $D_{v,w}\phi$. $\square$

CLAIM 7.20. *Suppose that ϕ intertwines two foliations $\mathcal{F}$ and $\mathcal{G}$ as before and that $D_{v,w}\phi$ exists. If v', w' are another pair of vectors with the same respective basepoints, then we can compute $D_{v',w'}\phi$ in the following way. Let q be the tangent to $\mathcal{F}$ at v and r be the tangent*

to $\mathcal{G}$ at the basepoint of w. For a pair of vectors a and b in $\mathbb{R}^2$, write Π_b^a for the projection onto b along a. Then, if $\Pi_{w'}^q w = \lambda w'$ and $\Pi_{v'}^r v = \eta v'$, then $D_{v',w'}\phi = \lambda\eta D_{v,w}\phi$.

Proof. As before, we can rewrite the formula that defines the normal derivative $D_{v',w'}\phi$ into one involving the normal derivative $D_{v,w}\phi$. Namely,

$$\Pi_{W'} \circ \phi \circ T = (\Pi_{W'}) \circ \Pi_W \circ \phi \circ T \circ (T^{-1} \circ \Pi_T \circ T').$$

The projections in the statement of the claim are the differentials of the terms in parentheses written in appropriate coordinates. $\square$

7.3. *Normal differentiability of the conjugacy.*

LEMMA 7.21. *For the $\mathcal{S}^L$ and $\mathcal{S}^F$ foliations, for any pair T, W of corresponding transversals, we have that h is normally differentiable at almost every point of T. In fact, for uniform transversals T and W, the induced map $\Pi_W h \colon T \to W$ is uniformly Lipschitz.*

Proof. We will show this only for the case of uniform transversals, as, by the preceding discussion, this implies normal differentiability for all transversals. We will show this by studying the convergence in Lemma 7.12. Let $h_n = F^{-n} \circ h_0 \circ L^n$, where h_0 is as in Lemma 7.14. We calculate the differential $D_z h_n$ with respect to the $[e_1^L, e_2^L]$, $[e_1^F, e_2^F]$ framings,

$$Dh_n = \begin{bmatrix} \lambda^{-n} & -n\lambda^{-n-1} \\ 0 & \lambda^{-n} \end{bmatrix} \begin{bmatrix} 1 & c(L^n z) \\ 0 & a(L^n z) \end{bmatrix} \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix} = \begin{bmatrix} 1 & c_n(z) \\ 0 & a(L^n z) \end{bmatrix},$$

where a is a uniformly bounded continuous function and $c_n(z)$ is some continuous function depending on n. Fix uniform transversals $T \colon (-1, 1) \to \mathcal{W}^{u,L}(x)$ and $W \colon (-1, 1) \to \mathcal{W}^{u,F}(h(x))$ to the $\mathcal{S}^L$ and $\mathcal{S}^F$ foliations, respectively. We can write $\dot{T}$ as $\alpha_L(t)e_1^L + \beta_L(t)e_2^L$ and $\dot{W}$ as $\alpha_F(t)e_1^F + \beta_F(t)e_2^F$. The tangent to $h_n \circ T$ is then

$$\begin{bmatrix} \alpha(t) + \beta(t)c_n(T(t)) \\ a(L^n(T(t)))\beta(t) \end{bmatrix}.$$

Let Π_W be the projection to W along $\mathcal{S}^F$ and let e^W be the tangent field to W. Then, the differential of $\Pi_W \colon \mathcal{F}(W) \to W$, with respect to these framings $[e_1^F, e_2^F]$ and $[e^W]$, is

$$[0, \omega]$$

for some continuous function ω. We claim that ω is uniformly continuous. Note that for a point y, $\omega(y)$ is the derivative of the map $\Pi_w \circ \eta$ for η a uniform transversal tangent to $e_2^F(y)$; by Lemma 7.6, this derivative is uniformly bounded.

Thus, the derivative of $\Pi_W \circ h_n \circ T_L \colon \mathbb{R} \to W$ sends

$$\partial_t \mapsto a(L^n(T(t)))\beta(t)\omega(F^{-n}k(L^n(x)))e_W,$$

which is uniformly bounded. Thus, the sequence of maps $\eta_n = \Pi_W \circ h_n \circ T_L$ is uniformly Lipschitz. As the sequence η_n converges uniformly to the map $\Pi_W \circ h \circ T_L$, we see that this map is uniformly Lipschitz as well, and hence differentiable almost everywhere with uniformly bounded derivative. $\square$

As noted after its definition, the normal derivative is well defined independent of the transversals used to calculate it. We now determine the normal derivative of h.

LEMMA 7.22. *There exists C such that for almost every z, $D_{e_2^L(z), e_2^F(h(z))} h = C$.*

Proof. We have that $F \circ h = h \circ L$. Both of these are normally differentiable almost everywhere to the foliations $\mathcal{S}^F$ and $\mathcal{S}^L$, and thus they have the same normal derivatives. Writing $a(z)$ for the derivative $D_{e_2^L(z), e_2^F(h(z))} h$, by calculating the derivative of $h \circ L$ in two ways, we see that for almost every z,

$$\lambda a(z) = a(L(z)) \lambda.$$

Thus, by ergodicity, as a is almost surely (a.s.) L-invariant, it agrees almost everywhere with a constant C. Thus, there exists C such that $D_{e_2^L, e_2^F(h(z))} h$ is equal to C almost everywhere. □

We now give another smoothing argument to produce an approximation k to h with even better properties. This time, we do not smooth in the normal direction because by Lemma 7.21, h already has some regularity in this direction.

LEMMA 7.23. *There exists $C \neq 0$ and a function $h_0 \colon M \to M$ such that, writing h_x for the restriction $h_0|_{\mathcal{W}^{u,L}(x)}$, we have that:*
1. *h_0 intertwines the $\mathcal{W}^{u,L}$ and $\mathcal{W}^{u,F}$ foliations as well as the $\mathcal{S}^L$ and $\mathcal{S}^F$ foliations;*
2. *h_x is uniformly C^0 close to h;*
3. *h_x is uniformly C^1;*
4. *with respect to the framings $[e_1^L, e_2^L]$, $[e_1^F, e_2^F]$, the differential of h_0 is*

$$Dh_0 = \begin{bmatrix} 1 & b \\ 0 & C \end{bmatrix},$$

where b is a uniformly continuous bounded function on each leaf and the bounded-ness is independent of the leaf.

Proof. We show how to do the smoothing for a particular leaf $\mathcal{W}^{u,L}(x_0)$. The result follows by smoothing on each leaf.

As in the proof of Lemma 7.14, we fix a uniform global transversal T_F to the $\mathcal{S}^F$ foliation and T_0, a parameterization of an $\mathcal{F}^L$ leaf. Then, we define $\omega(t) \in \mathbb{R}$ by

$$\phi^{\omega(t)} h(T_0(t)) \in T_F.$$

Note that ω is bounded and uniformly C^α for some $\alpha > 0$. Thus, as in Lemma 7.15, we may mollify ω to obtain $\widetilde{\omega}$, which is uniformly C^1 and uniformly close to ω. We now define a new version of h, $h_{x_0} \colon \mathcal{W}^{u,L}(x_0) \to \mathcal{W}^{u,L}(h(x_0))$ using the $(t_1, t_2) = x_0 + t_1 e_1^L + t_2 e_2^L$ coordinates on $\mathcal{W}^{u,L}$:

$$h_{x_0}(t_1, t_2) = \phi^{\widetilde{\omega}(t_2) + t_1} \Pi_{T_F} \circ h(T_0(t_2)).$$

We claim that this map satisfies the conclusion of the lemma. The first two claims follow as in the proof of Lemma 7.14. We will show the remaining claims by studying the derivatives of h_{x_0} in uniform charts.

By Lemma 7.9, for all $t \in \mathbb{R}$, $\phi^t(T_F)$ is a uniform transversal to the $\mathcal{S}^F$ foliation. Hence, we may restrict to using a uniform chart on $\mathcal{W}^{u,F}$ formed from such transversals $\phi^t T_F$. This gives us a uniform chart $\psi^F = \psi^F_{\phi^{t_0} T_F} : (t_1, t_2) \mapsto \phi^{t_0 + t_1} T_F(t_2)$. Let us take the chart $\psi^L : (t_1, t_2) \mapsto (x_0 + t_0 e_1^F) + t_1 e_1^F + t_2 e_2^F$ on $\mathcal{W}^L$. Then, in these charts, the map $(\psi^F)^{-1} \circ h_{x_0} \circ \psi^L = (\widetilde{\omega}(t_2) + t_1, t_2)$, which is uniformly C^1.

It now remains to verify the properties of the differential of h_{x_0}. That the differential has block form

$$\begin{bmatrix} 1 & b \\ 0 & c \end{bmatrix}$$

for two continuous function b and c is immediate from its coordinate expression. In particular, we can read off from this that $Dh_{x_0}(e_1^L) = e_1^F$. Also, note that b is uniformly bounded as $\widetilde{\omega}$ is uniformly C^1.

It only remains to check that c is constant. For this, it suffices to show that there exists a fixed $C \neq 0$ such that $Dh_{x_0}(e_2^L) = ce : 2^F + de_1^F$ for some function d. We check this using properties of the normal derivative. Because h_{x_0} is C^1 and intertwines the $\mathcal{S}^L$ and $\mathcal{S}^F$ foliations, by Claim 7.19, we have that at places where h_{x_0} is differentiable, the normal derivative agrees with the actual derivative. This means that if $\Pi^{e_1^F}_{e_2^F}$ denotes the projection of a vector onto e_2^F along e_1^F, then as h_{x_0} is C^1,

$$ce : 2^F = \Pi^{e_1^F}_{e_2^F} Dh_{x_0}(e_2^L) = (D_{e_2^L, e_2^F} h_{x_0}) e_2^F.$$

So by Lemma 7.22, at almost every point, $D_{e_2^L, e_2^F} h_{x_0} = C$. This implies that almost everywhere, $c = C$. However, c is continuous so $c = C$ everywhere. $\qquad\square$

LEMMA 7.24. *In Lemma 7.23, we must have $C = 1$.*

Proof. For the sake of contradiction, suppose that $C \neq 1$. We write $h_n = F^{-n} \circ h_0 \circ L^n$ and apply Lemma 7.12. The differential of h_n at z is equal to

$$\lambda^{-n} \begin{bmatrix} 1 & -1/\lambda \\ 0 & 1 \end{bmatrix}^n \begin{bmatrix} 1 & b(L^n(z)) \\ 0 & C \end{bmatrix} \lambda^n \begin{bmatrix} 1 & 1/\lambda \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n/\lambda - Cn/\lambda + b(L^n(z)) \\ 0 & C \end{bmatrix}.$$

Note that the upper right-hand corner of this matrix is not only unbounded, but is going uniformly to infinity because $b(L^n(z))$ is uniformly bounded. Let $\gamma : [0, 1] \to \mathcal{W}^{u,L}$ be the curve $t \mapsto x + t e_2^L$. Then, by integrating the derivative of h_n, we see that arbitrarily small segments of γ are stretched to uniform length by h_n for sufficiently large n. However, this implies that the sequence h_n can have no uniform modulus of continuity, which contradicts the uniform convergence of h_n to a continuous function h. $\qquad\square$

We can now show that h is Lipschitz on each leaf.

Proof of Lemma 7.11. We can now conclude by applying Lemma 7.12 again. By applying Lemmas 7.23 and 7.24, we obtain a map h_0 that is uniformly C^{1+} on $\mathcal{W}^{u,L}$ leaves and such

that the differential Dh_0 is, with respect to the frames $[e_1^L, e_2^L]$, $[e_1^F, e_2^F]$,

$$\begin{bmatrix} 1 & a(z) \\ 0 & 1 \end{bmatrix},$$

where a is uniformly bounded. As before, let $h_n = F^{-n} \circ h_0 \circ L^n$. Then, we may calculate as before that the differentials of the functions h_n are

$$\begin{bmatrix} 1 & a(L^n(x)) \\ 0 & 1 \end{bmatrix}.$$

However, note that as these differentials are uniformly bounded, the sequence h_n is uniformly Lipschitz. By Lemma 7.12, as h is the uniform limit of the h_n, we obtain that h is uniformly Lipschitz on each leaf. $\qquad\square$

8. *Differentiability of the conjugacy*

We can now prove Proposition 1.5. We will use the following result.

THEOREM 8.1. [**KSW23**, Theorem 2.1] *Let F be a transitive $C^{1+Hölder}$ Anosov diffeomorphism of a compact manifold M, and let A and B be β-Hölder linear cocycles over F. Suppose that A has one Lyapunov exponent and B is fiber bunched.*

Let μ be an ergodic F-invariant measure on M with full support and local product structure. Then, any μ-measurable conjugacy between A and B is β-Hölder continuous, that is, coincides with a β-Hölder continuous conjugacy on a set of full measure.

We now turn to the proof of Proposition 1.5.

Proof of Proposition 1.5. The hypotheses include that F is U_λ-framed. Hence, by Lemma 7.11, h is Lipschitz and hence differentiable almost everywhere along unstable leaves. Its derivative, with respect to the framings $[e_1^L, e_2^L]$ and $[e_1^F, e_2^F]$, satisfies

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} D_x h = D_{L(x)} h \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}. \tag{20}$$

Both $DF|_{E^{u,F}}$ and $DL|_{E^{u,L}}$ have one exponent and are fiber bunched. Moreover, with respect to volume, we have that equation (20) holds almost everywhere. Thus, we see from Theorem 8.1 that Dh agrees volume almost everywhere with a Hölder continuous function. In particular, this implies that h is uniformly C^{1+} along unstable manifolds as h is Lipschitz and hence is the integral of its derivative. The same argument shows that h is C^{1+} along stable manifolds. Thus, by Journé's lemma, h is C^{1+} (see [**Jou88**] or [**DeW21**, Lemma 35]). $\qquad\square$

A. *Appendix. Conjugacy of integer matrices with Jordan blocks*

The purpose of this appendix is to give the characterization of the elements of $SL(4, \mathbb{Z})$ that we consider in this paper. The Anosov automorphisms to which Theorem 1.2 applies may be described quite succinctly.

PROPOSITION A.1. *Suppose that $L \in SL(4, \mathbb{Z})$ is a matrix with a Jordan block and no eigenvalues of modulus* 1. *Then, L is conjugate to a block matrix of the form*

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix},$$

where A, $B \in GL(2, \mathbb{Z})$, $C \neq 0$, and the characteristic polynomials of A and B are equal and have two distinct real roots.

Proof of Proposition A.1. We first show that such a hyperbolic matrix is conjugate to a matrix of the given form. Let $p(\lambda)$ denote the characteristic polynomial of L. Then, because L's eigenvalues each have multiplicity 2 and L is hyperbolic, we see that $p(\lambda) = q(\lambda)^2$ for some irreducible $q \in \mathbb{Z}[\lambda]$. By [New72, Theorem III.12], any integer matrix L is conjugate to a block upper triangular matrix, where the diagonal blocks of the matrix correspond to the irreducible factors of the characteristic polynomial of L. In our case, this implies that L is conjugate to a matrix of the given form. □

Remark A.2. It is possible that the diagonal blocks in Proposition A.1 may not be conjugate through integer matrices. For example, consider the automorphism defined by

$$\begin{bmatrix} 3 & 2 & 1 & 0 \\ 4 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 6 \end{bmatrix}. \tag{A.1}$$

B. *Appendix. Orientability of the unstable bundle*

The purpose of this section is to show that the unstable bundle of an Anosov diffeomorphism on a torus is orientable. This is relatively straightforward to show once one has introduced appropriate definitions. The central idea is just that topological conjugacies preserve the notion of topological orientability of topological foliations. Hence, if we know a foliation is conjugate to an orientable topological foliation, we will use this information to upgrade that topological orientability to an orientation of the tangent to the foliation.

Let $(\Sigma, \mathcal{F})$ be a foliation of an n-manifold by leaves of dimension k that is given by a foliation atlas $\{(U_i, \phi_i)\}$, where each $\phi_i : U_i \subset M \to \mathbb{R}^k \times \mathbb{R}^{n-k}$, so that the transition functions have the form

$$\phi_i \phi_j^{-1}(x, y) = (\alpha_{ij}(x, y)), \gamma_{ij}(y)).$$

Note that even if the maps ϕ_i are C^0, it makes sense to say whether the transition function is orientation preserving. This is because for a fixed y_0, $\alpha_{ij}(x, y_0)$ as a topological map may preserve or reverse the topological orientation at any given point, that is, its map on local homology is ± 1 at every point. If this map is 1 at every point, then α_{ij} is

orientation preserving. If α_{ij} is orientation preserving for each point in its domain, then we say that the transition function $\phi_i \phi_j^{-1}$ is orientation preserving.

Definition B.1. [**HH81**, §2.3] We say that a foliation $(\Sigma, \mathcal{F})$ is orientable if it admits an atlas such that all transition functions $\phi_i \phi_j^{-1}$, as above, are orientation preserving.

When we speak of an orientation of a foliation $\mathcal{F}$ with uniformly C^1 leaves, what we mean is that there exists an orientation on $T\mathcal{F}$ as a continuous bundle. The following proposition says that if a foliation with uniformly C^1 leaf is orientable as a topological foliation, then it is orientable.

PROPOSITION B.2. *Suppose that M is a C^1 manifold and that $\mathcal{F}$ is a foliation of M with uniformly $C^1 k$-dimensional leaves. Then, if $\mathcal{F}$ admits an oriented topological foliation atlas, then $T\mathcal{F}$ is orientable as a continuous bundle over M.*

Proof. We will construct a continuous non-vanishing section of $\Omega^k T\mathcal{F}$, the space of k-forms on $T\mathcal{F}$. Fix a continuous Riemannian metric on TM.

If we have a point $x \in M$, then x is in the domain of some foliation chart $\phi\colon U \to \mathbb{R}^k \times \mathbb{R}^{n-k}$. If we let U_x be the plaque of this chart containing x, then we may find a C^1 disk D_x containing x inside of U_x whose orientation agrees with the chart orientation on the plaque. As $\mathcal{F}$ is uniformly C^1, we may fix a local smooth transversal foliation $\mathcal{T}$ defined in a neighborhood of U_x. This gives us holonomy maps $T_{x,y}\colon D \to D_y$, between D and its image in the plaque U_y via the $\mathcal{T}$ holonomies. Let $\omega \in \Omega^k T D$ be a choice of orientation on D_x agreeing with the chart orientation. Then, $D(T_{x,y})_* \omega$ gives an orientation on D_y. Thus, on a neighborhood $V \subset M$ of x, we may define a continuous unit norm section $\hat{\omega}$ of $\Omega^k T\mathcal{F}$ by setting $\hat{\omega} = \omega/\|\omega\|$.

Note that the oriented atlas of $\mathcal{F}$ gives, at every point $z \in M$, a choice of generator of the local homology $[\mu_z] \in H_n(\mathcal{F}(z), \mathcal{F}(z) \setminus z)$. However, for such a C^1 manifold endowed with a metric, $[\mu_z]$ is associated to a unique element $\hat{\omega}_x \in \Omega_z^{\dim M} TM$ of unit norm. If we let $\{\mu_z\}_{z \in M}$ be the choice of generators of $H_n(\mathcal{F}, \mathcal{F}(z) \setminus z)$ arising from the topological oriented foliation atlas, then observe that $[\mu_z]$ and $\hat{\omega}$ correspond to the same orientation. Thus, we may extend the definition $\hat{\omega}$ globally because any section we construct agrees with $[\mu_z]$. Then, $\hat{\omega}$ is continuous due to its local construction. $\square$

For a discussion of Anosov automorphims of nilmanifolds, see [**DeW21**, §2]. Note that a torus is a nilmanifold, so the following result applies in our setting.

PROPOSITION B.3. *Suppose that $F\colon N/\Gamma \to N/\Gamma$ is an Anosov diffeomorphism of a nilmanifold. Then, the unstable bundle $E^{u,F}$ is orientable.*

Proof. By the work of Franks [**Fra69**] and Manning [**Man74**], there exists an Anosov automorphism $L\colon N/\Gamma \to N/\Gamma$ and a conjugacy h between F and L. From the discussion in [**DeW21**, §2.2], we see that the foliation $\mathcal{W}^{u,L}$ is orientable. Thus, as h intertwines the foliation $\mathcal{W}^{u,F}$ and $\mathcal{W}^{u,L}$, by Proposition B.2, $T\mathcal{W}^{u,F} = E^{u,F}$ is an orientable bundle. $\square$

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