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A COMMENT ON FINITE NILPOTENT GROUPS OF DEFICIENCY ZERO

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A finite group is said to have deficiency zero if it can be presented with an equal number of generators and relations. Finite metacyclic groups of deficiency zero have been classified, see [1] or [6]. Finite non-metacyclic groups of deficiency zero, which we denote by FD_0-groups, are relatively scarce. In [3] I. D. Macdonald introduced a class of nilpotent FD_0-groups all having nilpotent class ≤ 8. The largest nilpotent class known for a Macdonald group is 7 [4]. Only a finite number of nilpotent FD_0-groups, other than the Macdonald groups, seem to be known [5], [7]. In this note we exhibit a class of FD_0-groups which contains nilpotent groups of arbitrarily large nilpotent class.

For any natural number n define the group G_n by

$$G_n = \langle a, b \mid a^{4n-2}ba = b^2ab,\ a^4b^4 = 1\rangle.$$

We shall use the notation C_n for the cyclic group of order n and the notation $[x, y]$ for $x^{-1}y^{-1}xy$. Commutators of higher weight we define by

$$[x_1, x_2, \ldots, x_n] = [[x_1, x_2, \ldots, x_{n-1}], x_n].$$

LEMMA 1. *In G_n the following relations hold:*

(i) $b^{4n-2}ab = a^2ba$

(ii) $a^{32n} = b^{32n} = 1$.

Proof. (i) This follows immediately.

(ii) We have $bab^{-1-4n}a^{-1}b^{-1} = a^2b$ and, raising this to the fourth power,

$$b^{4+16n}(a^2b)^4 = 1 \tag{1}$$

and so

$$[b, (a^2b)^4] = 1. \tag{2}$$

Now

$$a^2b^2a^2 = a^{-1}ba^{-1}b^{-1} \cdot b^{4n-4}. \tag{3}$$

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For, using Lemma 1(i)

$$\begin{aligned} a^2b^2(a^2ba)b^{-1}a &= a^2b^2\cdot b^{4n-2}abb^{-1}a \\ &= a^2b^{4n}a^2 \\ &= b^{4n-4}. \end{aligned}$$

Also

(4) $$b^2a^2b^2 = aba^{-1}b\cdot b^{4n}.$$

Now, using (2), (3), (4) and the relation $a^4b^4=1$, we have

$$\begin{aligned} ((a^2b)^4b^4)^2 &= (a^2b)^3a^2b(a^2b)^4b^8 \\ &= (a^2b)^3a^2(a^2b)^4b^9 \\ &= a^2ba^2b(a^2b^2a^2)ba^2ba^2b^6 \\ &= a^2ba^2ba^{-1}baba^2b^{4n+2} \\ &= a^2ba(b^2a^2b^2)aba^2b^2 \\ &= a^2bab^2a^2b^2ab(a^{-1}ba^{-1}b^{-1}\cdot b^{4n-4})a^{-2} \\ &= a^2ba(b^2a^2b^2\cdot aba^{-1}b\cdot b^{4n-4})a^{-1}b^{-1}a^{-2} \\ &= 1. \end{aligned}$$

But $((a^2b)^4b^4)^2=1$ together with (1) yields $b^{32n}=1$ as required.

THEOREM 1. *The group G_n is nilpotent and has order $2^{10}n$.*

Proof. Let $r=aba^{-1}b^{-1}$, $s=a^2ba^{-1}b^{-1}a^{-1}$, $t=a^4$ and put $H=\langle r, s, t\rangle$. We enumerate the cosets of H in G_n. The enumeration is made easier by introducing the redundant generator $a^3ba^{-1}b^{-1}a^{-2}$ into H. To see that this generator is redundant notice that, using (3) and $a^4b^4=1$, we have

(5) $$aba^{-1}b^{-1} = b^{-2}a^{-2}\cdot a^{4n}.$$

Now use (5) to obtain

$$\begin{aligned} aba^{-1}b^{-1}\cdot a^3ba^{-1}b^{-1}a^{-2} &= b^{-2}aba^{-1}b^{-1}a^{-2}\cdot a^{4n} \\ &= b^{-2}(b^{-2}a^{-2})a^{-2}\cdot a^{8n} \\ &= a^{8n}. \end{aligned}$$

Hence $a^3ba^{-1}b^{-1}a^{-2}=r^{-1}t^{2n}\in H$.

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Using integers to denote coset representatives we have the following table:

1. $a=2$	1. $b=r^{-1}\cdot 4$
2. $a=5$	2. $b=3$
3. $a=s^{-1}\cdot 6$	3. $b=st^{-n-1}\cdot 7$
4. $a=3$	4. $b=r^2t^{-n-1}\cdot 5$
5. $a=7$	5. $b=6$
6. $a=rt^{-2n}\cdot 8$	6. $b=r^{-1}t^n\cdot 1$
7. $a=t\cdot 1$	7. $b=8$
8. $a=r^{-1}st^{2n+1}\cdot 4$	8. $b=s^{-1}t^n\cdot 2$

Note that we have used the fact that t is central, see Lemma 1 (ii).

We next use the Reidemeister–Schreier algorithm to obtain a presentation for H on the generators r, s, t. A considerable simplification is possible using the relations $[r,t]=[s,t]=t^{8n}=1$ which we know hold in H. The following presentation is obtained.

$$H=\langle r,s,t \mid s^2rsr^{-1}=1,\ s^{-1}r^2sr^2=1,\ sr^3s^{-1}r=t^{4n},\ sr^{-1}s^{-1}r=t^{4n},\ r^{-1}s^2r^{-1}sr^2s=1\ [r,t]=[s,t]=t^{8n}=1\rangle$$

This presentation can be simplified using straightforward manipulations to give

$$H=\langle r,s,t \mid r^4=s^4=t^{8n}=1, [r,s]=t^{4n}, [r,t]=[s,t]=1\rangle.$$

Now $H/H'\cong C_4\times C_4\times C_{4_n}$ and $H'\cong C_2$. This proves that the order of G_n is $2^{10}n$ and that the period of a is $32\,n$.

The group G_n is nilpotent of class 6 and soluble of class 3 for any n. The first two factors of the derived series have rank 2 while the third is cyclic of order 2 generated by $(a^2b)^4b^4$.

By Lemma 1 (i) G_n admits an automorphism θ of order 2 induced by the map which interchanges a and b. Form E_n, the split extension of G_n by θ. Let θ be induced by conjugation by an element x so that E_n is generated by x and y where $a=y$, $b=xyx^{-1}$ and $x^2=1$. It is now easy to show that E_n has the presentation

$$\langle x,y \mid x^2=1, (yx)^2=(xy^2)^2(xy)^2y^{4n}, (xy^4)^2=1\rangle.$$

LEMMA 2. *The group E_{2n} is finite and has a 2-generator 2-relation presentation.*

Proof. Since G_{2n} may be presented with the symmetrical presentation

$$\langle a,b \mid b^{4n-2}ab^5=a^{4n-2}ba,\ a^{4n-2}ba^5=b^{4n-2}ab\rangle,$$

the result follows.

The presentation for G_{2n} given in the above proof shows it to be a member of the class studied by Campbell in [2].

The groups E_{2k} are nilpotent FD_0-groups whose class can be made arbitrarily large as the next theorem shows.

THEOREM 2. *For* $n=2^k$, $k\geq 1$, E_n *is a* FD_0*-group which is nilpotent of class* $\geq 4+k$.

Proof. Clearly y^4 belongs to the derived group of E_n. Also

$$[y^4, \underbrace{x, x, \ldots, x}_{t}] = y^{\alpha}$$

where $\alpha = 4\cdot(-2)^t$.

So, choosing $t=2+k$, the result follows.

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