ANZIAM J. 42(2001), 438–444

AN ORDINARY DIFFERENTIAL EQUATION ARISING IN THE RICCI FLOW ON THE PLANE

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(Received 19 August 1996; revised 13 January 1999)

Abstract

We consider an ordinary differential equation arising in the study of the Ricci flow on $\mathbf{R}^2$. The existence and uniqueness of solutions of this equation are derived. We then study the asymptotic behaviour of these solutions at $\pm\infty$.

1. Introduction

In this paper, we study the ordinary differential equation

$$\left(\frac{g'}{g}\right)' + g - \gamma g' = 0, \quad -\infty < y < \infty, \tag{1.1}$$

where γ is a positive constant, $g = g(y)$ and the prime denotes differentiation with respect to y. Equation (1.1) arises in the study of the Ricci flow on $\mathbf{R}^2$.

We recall that any metric on $\mathbf{R}^2$ can be expressed as

$$ds^2 = u\left(dx_1^2 + dx_2^2\right),$$

where x_1, x_2 are standard coordinates in $\mathbf{R}^2$ and u is a function. The Ricci flow on $\mathbf{R}^2$ is described by

$$\frac{\partial}{\partial t} ds^2 = -R\, ds^2,$$

where R represents scalar curvature (*cf.* [6]). The function $u = u(x,t)$ then satisfies the equation

$$u_t = \Delta(\ln u), \quad x \in \mathbf{R}^2,\ t > 0, \tag{1.2}$$

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where Δ is the standard Laplacian in $\mathbf{R}^2$.

The Cauchy problem for (1.2) has recently been the subject of extensive study (see the list of references and the literature cited therein). We note that the Ricci flow on $\mathbf{R}^2$ can be viewed as the limiting equation of the well-known porous medium equation

$$u_t = \Delta\left(\frac{u^m - 1}{m}\right)$$

as $m \to 0$ (*cf.* [9]). Equation (1.2) also arises in the study of thin film dynamics (*cf.* [4]).

It is well-known that some special solutions of (1.2) play important roles in studying the existence and asymptotic behaviour of solutions of the Cauchy problem for (1.2) (see [2, 5, 6, 8, 9]). For example, some special solutions, namely similarity solutions, in the form

$$u(x,t) = t^{-\alpha}\phi(|x|/t^{\sigma}),$$

for some appropriate exponents α and σ, have been found in [5]. Moreover, they are stable in a certain sense (see [5] for details).

Given $\gamma > 0$, we are looking for special solutions of (1.2) for $t \in (0, 1)$ in the form

$$u(x,t) = \frac{1-t}{|x|^2}g(y), \quad y = \ln\frac{|x|}{(1-t)^{\gamma}}, \tag{1.3}$$

where g is smooth, positive, and integrable over $\mathbf{R}$. Then g satisfies (1.1).

It is shown in [6] that given any $\alpha > \beta > 0$, there exists a $\gamma > 0$ such that (1.1) has a solution $g(y)$ with the properties

$$g(y) = \begin{cases} a_1 e^{-\alpha y} + a_2 e^{-2\alpha y} + \cdots & \text{as } y \to \infty, \\ b_1 e^{\beta y} + b_2 e^{2\beta y} + \cdots & \text{as } y \to -\infty. \end{cases}$$

We are interested in the reverse question. That is, given any $\gamma > 0$, can we solve (1.1)? Moreover, what is the asymptotic behaviour of these solutions $g(y)$ of (1.1) as $y \to \pm\infty$? The purpose of this paper is to answer these two questions.

We organize this paper as follows. In Section 2, the existence and uniqueness of solutions of (1.1) are treated. We then study their asymptotic behaviour in Section 3.

2. Existence and uniqueness

In this section, we study the existence and uniqueness of solutions of (1.1). Let $h = \ln g$. Then g satisfies (1.1) if and only if h satisfies

$$h'' + e^h - \gamma e^h h' = 0, \quad -\infty < y < \infty. \tag{2.1}$$

LEMMA 2.1. *For any global solution h of* (2.1), *there exists a unique $y_0 \in \mathbf{R}$ such that $h'(y_0) = 0$. Moreover, $h'(y) > 0$, $\forall y < y_0$ and $h'(y) < 0$, $\forall y > y_0$.*

PROOF. If $h' > 0$ in $\mathbf{R}$, then

$$h'' + e^h > 0 \quad \text{in } \mathbf{R}$$

and so

$$\left[\frac{(h')^2}{2} + e^h\right]' > 0 \quad \text{in } \mathbf{R}.$$

Recall that we are looking for solutions of (1.1) which are integrable over $\mathbf{R}$. Hence $\exp(h(y)) = g(y) \to 0$ as $y \to \infty$ and so the limit

$$\lim_{y\to\infty} h'(y)$$

exists and is positive, a contradiction to the integrability of g over $\mathbf{R}$.

Similarly, h' cannot be negative in $\mathbf{R}$. Therefore there is a $y_0 \in \mathbf{R}$ such that $h'(y_0) = 0$.

We claim that y_0 is the only zero of h'. Indeed, let

$$\rho(y) = \exp\left\{-\gamma \int_{y_0}^{y} e^{h(z)}\, dz\right\}, \quad y \in \mathbf{R}. \tag{2.2}$$

Then (2.1) is equivalent to

$$(\rho h')' + \rho e^h = 0 \tag{2.3}$$

and so

$$\rho(y)h'(y) = -\int_{y_0}^{y} \rho(z)e^{h(z)}\, dz. \tag{2.4}$$

This implies that $h'(y) > 0$, $\forall y < y_0$ and $h'(y) < 0$, $\forall y > y_0$. The lemma follows.

Since (2.1) is autonomous, without loss of generality we may assume that $y_0 = 0$. Therefore any solution h of (2.1) is monotone increasing for $y < 0$ and monotone decreasing for $y > 0$. Also, from (2.2) we have $\rho > 0$ in $\mathbf{R}$ and

$$\rho'(y) = -\gamma\rho(y)e^{h(y)} < 0, \quad \forall y \in \mathbf{R}.$$

In order to derive the existence of a solution of (2.1), we consider the initial value problem for (2.1) with the initial conditions

$$h'(0) = 0, \quad h(0) = a \in \mathbf{R}. \tag{2.5}$$

The local existence and uniqueness of solution of (2.1) and (2.5) follows from the standard theory of ordinary differential equations. Let h be the solution of (2.1) and (2.5) defined in the maximal existence interval (z_0, z_1) with $z_0 < 0 < z_1$.

LEMMA 2.2. *For the maximal existence interval* (z_0, z_1) *with* $z_0 < 0 < z_1$, $z_0 = -\infty$ *and* $z_1 = \infty$.

PROOF. From (2.4), we have

$$h'(y) = -\int_0^y \frac{\rho(z)}{\rho(y)} e^{h(z)}\, dz = \int_0^y \frac{1}{\rho(y)} \frac{\rho'(z)}{\gamma}\, dz$$
$$= \frac{1}{\gamma\rho(y)}[\rho(y) - 1] = \frac{1}{\gamma}\left[1 - \frac{1}{\rho(y)}\right]. \tag{2.6}$$

If $z_1 < \infty$, then either

$$h(y) \to -\infty \quad \text{as } y \to z_1 \tag{2.7}$$

or

$$h'(y) \to -\infty \quad \text{as } y \to z_1. \tag{2.8}$$

Since $h(y) \le a$, $\forall y \in (0, z_1)$ we have

$$\rho(y) \ge \exp\left\{-\gamma z_1 e^a\right\}, \quad \forall y \in (0, z_1).$$

From (2.6) we obtain

$$h'(y) \ge \frac{1}{\gamma}\left[1 - \exp(\gamma z_1 e^a)\right], \quad \forall y \in (0, z_1).$$

Hence (2.8) is impossible. On the other hand,

$$h(y) = a + \int_0^y h'(z)\, dz \ge a + \frac{y}{\gamma}\left[1 - \exp(\gamma z_1 e^a)\right],$$

a contradiction to (2.7). We conclude that $z_1 = \infty$. Similarly, $z_0 = -\infty$.

Hence we have proved the following existence and uniqueness theorem.

THEOREM 2.3. *For any* $a \in \mathbf{R}$, *there is a unique global solution* h *of* (2.1) *with* $h'(0) = 0$ *and* $h(0) = a$. *Conversely, any global solution of* (2.1) *with* $h'(0) = 0$ *must be the solution of the initial value problem* (2.1) *and* (2.5) *with some* $a \in \mathbf{R}$.

3. Asymptotic behaviour

Let g be the smooth positive integrable (over $\mathbf{R}$) solution of (1.1) such that $g'(0) = 0$ and $g(0) = b > 0$. Indeed, $g = e^h$, where h is the unique smooth global solution of (2.1) with $h'(0) = 0$ and $h(0) = \ln b$. In this section, we shall study the asymptotic behaviour of $g(y)$ as $y \to \pm\infty$.

https://doi.org/10.1017/S1446181100012049 Published online by Cambridge University Press

Set

$$P = \int_0^\infty g(z)\,dz, \quad Q = \int_{-\infty}^0 g(z)\,dz. \tag{3.1}$$

Then from (2.2)

$$\rho(y) = \exp\left\{-\gamma \int_0^y g(z)\,dz\right\},$$

from which we obtain that $\rho(y) \to \exp(-\gamma P)$ as $y \to \infty$. Hence from (2.6) we conclude that

$$\lim_{y\to\infty} \frac{g'(y)}{g(y)} = \lim_{y\to\infty} h'(y) = \frac{1}{\gamma}[1 - \exp(\gamma P)]. \tag{3.2}$$

Similarly, we have

$$\lim_{y\to-\infty} \frac{g'(y)}{g(y)} = \frac{1}{\gamma}[1 - \exp(-\gamma Q)].$$

Denote $A = [\exp(\gamma P) - 1]/\gamma$ and $B = [1 - \exp(-\gamma Q)]/\gamma$. Note that $A > 0$, $B > 0$.

THEOREM 3.1. *The limit*

$$\lim_{y\to\infty} \left[g(y)e^{Ay}\right] = C_+$$

exists and $C_+ > 0$.

PROOF. From (3.2) it follows that

$$g(y) \leq Ce^{-Ay/2} \tag{3.3}$$

for all y sufficiently large and for some positive constant C. We claim that for any $\lambda > 0$

$$\lim_{y\to\infty} y^\lambda \left[\frac{g'(y)}{g(y)} + A\right] = 0. \tag{3.4}$$

Indeed, from (2.6) we have

$$\lim_{y\to\infty} y^\lambda \left[\frac{g'(y)}{g(y)} + A\right] = \lim_{y\to\infty} \frac{\exp(\gamma P) - \exp\left(\gamma \int_0^y g(z)\,dz\right)}{\gamma y^{-\lambda}} = 0,$$

using L'Hôpital's rule and (3.3). Hence the theorem follows by integrating (3.4).

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Similarly, the limit

$$\lim_{y \to -\infty} \left[g(y) e^{-By} \right] = C_-$$

exists and $C_- > 0$. This gives the asymptotic behaviour of $g(y)$ as $y \to \pm\infty$.

We shall now study the dependence of functions P, Q, A, B on the initial value b.

PROPOSITION 3.2. *As a function of b, P is one-to-one from* $(0, \infty)$ *onto* $(0, \infty)$ *such that* $P(0^+) = 0$ *and* $P(\infty) = \infty$. *The same holds for* Q.

PROOF. Integrating (1.1) from 0 to ∞, we obtain

$$P = -\gamma b - [1 - \exp(\gamma P)]/\gamma. \tag{3.5}$$

Differentiating (3.5) with respect to b, we get

$$P'(b) = \frac{\gamma}{\exp(\gamma P(b)) - 1} > 0. \tag{3.6}$$

Since P is monotone in $(0, \infty)$, $P(0^+)$ exists and is nonnegative. From (3.5) it follows that $P(0^+)$ satisfies the relation

$$\gamma P(0^+) = \exp(\gamma P(0^+)) - 1.$$

Thus $P(0^+) = 0$. Also, from (3.5) it is clear that $P(b) \to \infty$ as $b \to \infty$.

Similarly, we have

$$Q = \gamma b + [1 - \exp(-\gamma Q)]/\gamma. \tag{3.7}$$

The same reasoning shows that Q has the same properties as P. Hence the proposition follows.

The following corollary is a direct consequence of Proposition 3.2.

COROLLARY 3.3. *As a function of b, A is one-to-one from* $(0, \infty)$ *onto* $(0, \infty)$ *such that* $A(0^+) = 0$ *and* $A(\infty) = \infty$. *However, as a function of b, B is one-to-one from* $(0, \infty)$ *onto* $(0, 1/\gamma)$ *such that* $B(0^+) = 0$ *and* $B(\infty) = 1/\gamma$.

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