ON UTUMI'S RING OF QUOTIENTS

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The purpose of this note is to establish and exploit the fact that Utumi's maximal ring of right quotients **(6)** of an associative ring R (let us say with 1) is the bicommutator of the minimal injective extension of R regarded as a right R-module. Nothing new will be said about Johnson's ring of quotients **(4)**, which is still the most important case.

While earlier papers on rings of quotients are referred to, the present note is self-contained, except that the reader is expected to be familiar with the concept "injective module" and with the following result by Eckmann and Schopf **(2)**:

Every R-module M_R possesses an extension E_R, unique up to isomorphism over M, satisfying any one of the following three equivalent properties:

I. E_R is a maximal essential extension of M_R.

II. E_R is an essential extension of M_R and is injective.

III. E_R is a minimal injective extension[1] of M_R.

Here E_R is called an *essential* extension of M_R, or M_R is called a *large* submodule of E_R, provided every non-zero submodule of E_R has a non-zero intersection with M_R.

In what follows, R will be an associative ring with 1, and all R-modules are understood to be unitary.

1. Let I_R be the minimal injective extension of the right R-module R_R associated with the ring R, and let $H = \mathrm{Hom}_R\,(I, I)$ be the ring of endomorphisms of I_R. We write these endomorphisms on the left of their arguments and obtain a bimodule ${}_HI_R$. Again, let $Q = \mathrm{Hom}_H\,(I, I)$ be the ring of endomorphisms of the left H-module ${}_HI$. We write these endomorphisms on the right of their arguments and obtain a bimodule ${}_HI_Q$. The letters R, I, H, and Q will retain their meaning throughout this paper.[2]

The obvious mapping of R into Q is faithful, since $Ir = 0$ implies $r = 1r = 0$ for all $r \in R$. We shall regard R as a subring of Q.

We also have a canonical mapping $h \to h1$ of H into I. This is clearly an H-homomorphism.

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[1]As a matter of historical record, the minimal injective extension of a module is a special case of the "algebraic closure" of an algebraic system considered by K. Shoda in his paper *Zur Theorie der algebraischen Erweiterungen*, Osaka Math. J., *4* (1952), 133–144.

[2]In the terminology of Bourbaki (**1**, Vol. 23, § 1), Q is the *bicommutator* of I_R; at least this is how I think the French word "bicommutant" should be rendered in English. The dual ring (or mirror image) of H would be called the *commutator*.

LEMMA. *The canonical mapping of ${}_HH$ into ${}_HI$ is an epimorphism, that is* $H1 = I$.

Proof. For any $i \in I$, the mapping $r \to ir$ is a homomorphism of R_R into I_R and may be extended to some $h \in \text{Hom}_R(I, I)$, by injectivity of I_R; hence $h1 = i$.

2. Consider now the canonical homomorphism $q \to 1q$ of Q_R into I_R. We observe that its kernel is 0, since $1q = 0$ implies $Iq = (H1)q = H(1q) = 0$, by the above lemma. We might therefore identify Q with the subset $1Q$ of I; but we prefer to keep up the notational distinction, at the risk of being pedantic.

PROPOSITION. *The canonical image of Q_R in I_R consists precisely of those elements of I which are annihilated by all elements of H which annihilate R.*

Proof. Let $h \in H$, $hR = 0$, then $h(1Q) = (h1)Q = 0$. Thus $1Q$ is contained in the indicated submodule of I_R. Conversely, assume that $i \in I$ has the property: $hR = 0 \Rightarrow hi = 0$ for all $h \in H$. We shall find $q \in Q$ such that $i = 1q$, and this will complete the proof.

By Lemma 1, any element of I can be written in the form $h1$, where $h \in H$. Write $(h1)q = hi$, then q is a well-defined mapping of I into itself; for if $h'1 = h1$ with $h' \in H$ then $(h - h')R = 0$ and so $(h - h')i = 0$, by the assumed property of i. One easily verifies that $q \in \text{Hom}_H(I, I) = Q$. Finally, by taking h to be the identity element of H, we obtain $1q = i$.

It follows from the above and a known result[3] that Q is Utumi's ring of right quotients of R. However, we shall establish this fact more directly in § 8. In the meantime, we shall feel free to call Q the *ring of right quotients* of R.

Looking at the inverse image of $1Q$ in H we find the following:[4]

COROLLARY. $Q \cong P/K$, *where* $K = \{h \in H \mid h1 = 0\}$ *and* $P = \{h \in H \mid Kh \subset K\}$.

3. If we repeat the procedure of § 1 with Q in place of R, we obtain nothing new:

PROPOSITION. *I_Q is the minimal injective extension of Q_Q and* $\text{Hom}_Q(I, I) = H$.

Proof. Let A_Q be a submodule of B_Q and $\phi \in \text{Hom}_Q(A, I)$. Since I_R is injective, ϕ can be extended to $\psi \in \text{Hom}_R(B, I)$. It will follow that I_Q is injective, if we show that ψ is a Q-homomorphism.

[3]See (**3**, 2.7). A proof of this result appears in (**4**, Theorem 1). Incidentally, the proof of (**3**, Theorem 2.6) fell short in failing to verify that the ordered system considered there was a set; but this can easily be remedied.

[4]Findlay and the present author had originally used this formula to construct Q, but later favoured another approach. The formula was again observed by Utumi, who mentioned it in a letter.

For any $a \in A$ consider $\psi_a q = \psi(aq) - (\psi a)q$. Clearly $\psi_a \in \text{Hom}_R(Q, I)$ and $\psi_a R = 0$. But ψ_a can be extended to an element of H; hence, by the proposition in § 2, $\psi_a Q = 0$. This shows that $\psi \in \text{Hom}_Q(B, I)$, as required.

Now, since I_R is an essential extension of Q_R, I_Q is an essential extension of Q_Q. Being injective, it is therefore the minimal injective extension of Q_Q.

Clearly $\text{Hom}_Q(I, I) \subset \text{Hom}_R(I, I) = H$. We have equality, since ${}_H I_Q$ is a bimodule.

COROLLARY. *Q is its own ring of right quotients.*[5]

4. One may ask what happens if the above construction of Q is generalized as follows. Let I_R' be any (not necessarily minimal) injective extension of R_R, and let $H' = \text{Hom}_R(I', I')$ and $Q' = \text{Hom}_{H'}(I', I')$. It can be then shown that

$$R \subset 1Q' \subset 1Q \subset I \subset I',$$

where I_R is the minimal injective extension of R_R contained in I_R'. The following example shows that, by appropriate choice of I', we can arrange to have Q' small.

Example. Let K_R be the minimal injective extension of the R-module Q/R. Take $I_R' = I_R \oplus K_R$; then $Q' = R$.

I do not know whether every ring between R and Q can be obtained in this way.

5. PROPOSITION. *The following conditions are equivalent:*

(1) ${}_H H \cong {}_H I$ *canonically.*

(2) $I_R \cong Q_R$ *canonically.*

(3) $H \cong Q$ *canonically as rings.*

(4) Q_R *is injective.*

(5) $I_Q \cong Q_Q$ *canonically.*

(6) Q *is right self-injective.*

Proof. (1) $\Leftrightarrow$ (2). Assume (1), then the mapping $h \to h1$ has kernel 0. Thus $hR = 0$ implies $hI = 0$, and so $I = 1Q$, by the proposition in § 2. Thus (1) $\Rightarrow$ (2), and this argument may be reversed.

(1) and (2) $\Rightarrow$ (3). Tracing the given isomorphisms $H \to I \leftarrow Q$, we find that $h \in H$ corresponds to $q \in Q$ if and only if $h1 = 1q$. Suppose also that $h'1 = 1q'$; then $(hh')1 = h(h'1) = h(1q') = (h1)q' = (1q)q' = 1(qq')$.

(3) $\Rightarrow$ (2). Assume (3), then the relation $h1 = 1q$ is an isomorphism between H and Q. Now for any $i \in I$ there exists $h \in H$ such that $h1 = i$, by Lemma 1. Hence, by assumption, there exists $q \in Q$ such that $i = 1q$. Therefore (2) is true.

[5]This was first shown in **(6, (1.15))**.

(1) $\Leftrightarrow$ (5). In view of the proposition in § 3, this is a special case of the statement (1) $\Leftrightarrow$ (2), which has already been shown.

(2) $\Leftrightarrow$ (4), (5) $\Leftrightarrow$ (6). Assume Q_R is injective; then so is its canonical image in I_R. But I_R is an essential extension of this image, hence $I = 1Q$.

6. It is known **(6,** (1.3)**)** that an element of Q belongs to the centre of Q if and only if it commutes with every element of R. (Given $q \in Q$, let $\phi_q q' = qq' - q'q$, for any $q' \in Q$. Then $\phi_q R = 0$ and so $\phi_q Q = 0$.) Applying this criterion twice, one sees that if R is commutative, then so is Q **(3,** 7.2**)**.

LEMMA. *The centres of H and Q coincide with $H \cap Q = \mathrm{Hom}_{H,Q}(I, I)$.*

Proof. An endomorphism of ${}_H I_Q$ is an endomorphism h of I_Q such that $h(h'i) = h'(hi)$, for all $h' \in H$ and $i \in I$. This means that h lies in the centre of H. By writing $hi = ih$, we see that this can also be interpreted to mean that $h \in H \cap Q$ or that h lies in the centre of Q.

COROLLARY. *If R is commutative, then Q is isomorphic to the centre of H.*

Proof. Since Q is commutative, it coincides with its own centre, hence with the centre of H.

An example by Utumi **(6,** (1.1)**)** can be used to show that in general H is not commutative even if R is.

Example. Let F be any field, $S = F[x]/(x^4)$, R the subring of S generated by 1, $\bar{x}^2$ and $\bar{x}^3$; thus

$$S = F + F\bar{x}^2 + F\bar{x}^3,$$

where $\bar{x}$ is the image of x in S. As Utumi pointed out, S_R is an essential extension of R_R but, when S_R is regarded as a submodule of I_R, $S \not\subset 1Q$. The endomorphism $s \to \bar{x}s$ of S_R may be extended to $h_1 \in H$. The mapping $\phi: S \to S$ defined by

$$\phi(f_0 + f_1\bar{x} + f_2\bar{x}^2 + f_3\bar{x}^3) = f_0\bar{x} + (f_1 + f_2)\bar{x}^3$$

is an endomorphism of S_R, as is easily verified, and may be extended to $h_2 \in H$. Now $h_1(h_2 1) = \bar{x}\phi 1 = \bar{x}^2$ and $h_2(h_1 1) = \phi\bar{x} = \bar{x}^3$; hence $h_1 h_2 \neq h_2 h_1$.

7. A submodule F_R of Q_R will be called *dense* if $hF = 0$ implies $h1 = 0$ for all $h \in H$.

PROPOSITION. *If F_R and G_R are submodules of Q_R and F is dense, then* $\mathrm{Hom}_R(F, G)$ *is canonically isomorphic to the "residual quotient"*

$$G : F = \{q \in Q \mid qF \subset G\}.$$

Proof.[6] The canonical homomorphism of $G : F$ into $\mathrm{Hom}_R(F, G)$ is of course the mapping which associates with each $q \in G : F$ the homomorphism

[6]This proposition could also have been deduced from **(6)** or **(3)**.

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$f \to qf$ of F into G. It is a monomorphism, since $qF = 0$ implies $q = 0$. (Recall that $1q = h1$, for some $h \in H$, and F is dense.) It will follow that it is an isomorphism if to each $\phi \in \mathrm{Hom}_R(F, G)$ we can find $q \in Q$ such that $\phi f = qf$ for all $f \in F$.

Indeed, extend ϕ to $h \in H$, in the sense that $h1f = 1\phi f$ for all $f \in F$. Now consider any $h' \in H$ such that $h'R = 0$. Then $h'h1F \subset h'1Q = 0$, and so $h'h1Q = 0$, since F is dense. Therefore $h1Q \subset 1Q$, by Proposition 1. In particular, $h1 = 1q$ for some $q \in Q$. Thus $1\phi f = h1f = 1qf$, and hence $\phi f = qf$, for all $f \in F$, as required.

As an application we may mention the following corollary.

COROLLARY A. *Two dense submodules F_R and G_R of Q_R are isomorphic if and only if there exists an invertible element $q \in Q$ such that $qF = G$.*

Now let R be a commutative ring; then so is Q (see § 6). Assume that F_R is *invertible*[7] in the sense that $FF^{-1} = R$ for some submodule F^{-1} of Q. It is not difficult to see that then $F^{-1} = R : F$. By Corollary A, $F_R \cong G_R$ if and only if there is an invertible element $q \in Q$ such that $qR = GF^{-1}$. Thus, as in classical ideal theory, we have the following corollary.

COROLLARY B. *Let R be commutative. The group of isomorphism types of invertible submodules of Q_R is isomorphic to Γ/Π, where Γ is the group of invertible submodules and Π is the subgroup of principal invertible submodules of Q_R.*

8. PROPOSITION. *If Δ is the set of dense right ideals of R, then*

$$Q = \bigcup_{D \in \Delta} R : D.$$

Proof. Let $q \in Q$, then $q \in R : D$, where $D = \{d \in R \mid qd \in R\}$. It remains to show that D is dense.[8] Given $h \in H$ and $hD = 0$, we want to show that $h1 = 0$. Consider the mapping $\phi : R + 1qR \to I$ defined by $\phi(r+1qr') = hr'$. (That this is well-defined follows from the fact that $hD = 0$.) Extend ϕ to $h' \in H$; then $h'R = 0$, and hence $h'1Q = 0$ and $h1 = \phi 1q1 = h'1q1 = 0$.

COROLLARY. *Let $\equiv$ be the equivalence relation that holds between $\phi \in \mathrm{Hom}_R(D, R)$ and $\phi' \in \mathrm{Hom}_R(D', R)$ if and only if $(\phi - \phi')(D \cap D') = 0$, where $D, D' \in \Delta$. Then*

$$Q \cong \bigcup_{D \in \Delta} \mathrm{Hom}_R(D, R)/\equiv.$$

[7]Findlay and the present author first became interested in Utumi's ring of quotients upon observing that an ideal A of R (let us say R is commutative) is invertible in Q if and only if A is projective, finitely generated, and dense. Looking at Bourbaki's treatment of invertible ideals (**1**, Vol. 27, Chapter 2, § 5), one is tempted to ask whether an abstract module M_R is isomorphic to an invertible submodule of Q_R if and only if M_R is finitely generated and projective of rank 1.

[8]The density of D is a special case of (**3**, Proposition 1.2).

Proof. This follows from the last two propositions if we observe that $\phi \equiv \phi'$ if and only if $q = q'$ for the corresponding elements of $R:D$ and $R:D'$.

This formula may be interpreted as a direct limit of groups. It is Utumi's original construction of Q **(6)**, based on Johnson's original construction **(4)**, to which it reduces in the case considered in § 9 below.

Actually, Utumi had considered a slightly more general situation, namely any associative ring S (without unity) satisfying the condition: for all $s \in S$, $sS = 0 \Rightarrow s = 0$. Now let R be the ring of endomorphisms of the right module S_S, then R is a ring with 1 and a faithful extension of S. Moreover Utumi's maximal ring of right quotients of S coincides with that of R.[9]

9. Johnson's *singular submodule* of any module M_R consists of all those elements of M which annihilate some large right ideal of R. Let J_R be the singular submodule of I_R; then $J \cap R$ is an ideal, called the right *singular ideal* of R. Let N be the inverse image of J in H under the canonical epimorphism of ${}_HH$ onto ${}_HI$; then N is an ideal of H. The ring $H/N \cong I/J$ is a regular ring.[10] Utumi[11] used this to show that N is the Jacobson radical of H. It follows immediately that J is the radical (i.e. intersection of maximal submodules) of ${}_HI$.

As in all important applications up to date it has always happened that $J = 0$, we record here, without offering any new ideas, the following known set of equivalent conditions:

(1) R has zero right singular ideal.
(2) I_R has zero singular submodule.
(3) H is semi-simple in the sense of Jacobson.
(4) ${}_HI$ is semi-simple (i.e. has zero radical).
(5) Q is regular in the sense of Von Neumann.
(6) All large right ideals of R are dense.

It is known that if $J = 0$,then Q will be right self-injective. This may also be deduced from the following lemma.

LEMMA. *If $i \in I$ and $Hi \cap J = 0$, then $i \in 1Q$.*

Proof. Let $L = \{r \in R \mid ir \in R\}$. Since I_R is an essential extension of R_R, therefore L_R is a large submodule of R_R. (By a standard argument: if A is a right ideal of R and $L \cap A = 0$, then $iA \cap R = 0$ and so $iA = 0$. But then $A \subset L \cap A = 0$.) Now suppose $h \in H$ and $hR = 0$; then $(hi)L = 0$, and so $hi \in Hi \cap J = 0$. Thus $i \in 1Q$.

[9]In **(3)** rings of quotients of arbitrary associative rings were also considered. It seems less clear how to fit these into the present framework.

[10]See **(8**, Theorem 2**)**, where essentially this result is established in a more general set-up. I_R being the minimal injective extension of any module M_R.

[11]See **(6**, Lemma 1**)**. He considered the more general set-up mentioned in the preceding footnote.

Appendix. A number of people have asked: What is the relation between Utumi's ring of quotients and (I) the classical ring of quotients, say as described in the book by Jacobson (*Theory of rings*, New York, 1943, Chapter 6), or (II) the constructions recently advocated by Bourbaki?

I. Let R be an associative ring with 1, Q_c an extension of R (with the same 1). Then Q_c is called a *classical* ring of right quotients of R if all regular elements of R are invertible in Q_c and every element of Q_c has the form ab^{-1} with a, $b \in R$ and b regular. (b is called *regular* if it is neither a left nor a right zero-divisor.) Q_c will exist (and be uniquely determined up to isomorphism over R) if and only if R satisfies the following condition.

Condition (Ore). For every pair of elements $a, b \in R$, b regular, there exists a common right multiple $ab' = ba'$ such that $a', b' \in R$ and b' is regular.

Now let R satisfy this condition and let b be a regular element of R. Then bR is a dense right ideal of R (in the sense of § 7). For from $hbR = 0$, $h \in H$, we deduce, for any $a \in R$, that $hab' = hba' = 0$, with b' regular, and hence that $hR = 0$. Let $\phi \in \mathrm{Hom}_R(bR, R)$ be defined by $\phi br = r$, $r \in R$. (Note that $br = 0$ implies $r = 0$). Then by the corollary appearing in § 8, ϕ gives rise to an element q of Q such that $qb = 1$. By the regularity of b, also $bq = 1$, and hence b is invertible in Q. In view of the usual addition and multiplication of fractions, we see that the elements of Q of the form ab^{-1}, with $a, b \in R$, b regular, form a subring of Q. We may therefore write $R \subset Q_c \subset Q$.

These facts were observed by Findlay and the present author, but only the commutative case was treated in (**3**, § 7). It was shown there that in general $Q_c \neq Q$. In all known examples for which $Q_c \neq Q$, R fails to satisfy the maximum condition for right ideals. On the other hand, Goldie has proved that $Q_c = Q$ for any semi-prime ring with maximum condition (A. W. Goldie, *Semi-prime rings with maximum condition*, Proc. London Math. Soc., *10* (1960), 201–220). This is where the problem rests today.

II. Bourbaki gives a general construction of what one might call "rings of quotients" in a set of exercises attributed to P. Gabriel (**1**, Vol. 27, pp. 157–165), but which actually contains several results by Johnson and Utumi. Bourbaki considers a set Φ of right ideals of R satisfying the following conditions:

(1) Every right ideal of R containing a member of Φ belongs to Φ.

(2) Φ is closed under finite intersection.

(3) If $A \in \Phi$ and $r \in R$, then $r^{-1}A = \{x \in R \mid rx \in A\} \in \Phi$.

He forms the direct limit R_Φ of the modules $\mathrm{Hom}_R(A, R)$ for all $A \in \Phi$, and shows that this can be naturally turned into a ring under the further condition:

(4) If $B \in \Phi$ and A is a right ideal such that $b^{-1}A \in \Phi$ for all $b \in B$, then $A \in \Phi$.

The canonical mapping $R \to R_\Phi$ is a monomorphism if and only if:

(5) For all $A \in \Phi$, $0 : A = 0$.

It is not diffiuclt to verify that the set Δ of dense ideals satisfies conditions (1) to (5) and is in fact the largest set of right ideals satisfying these conditions. Thus $Q = R_\Delta$ is the largest of those Bourbaki–Gabriel rings of right quotients which faithfully extend R.

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For further references see **(5)**.

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