ON SOME CLASSES OF AC-OPERATORS

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Abstract. AC-operators generalise normal operators on Hilbert space in the context of well-boundedness. In this paper we study AC-operators $T = U + iV$, where U and V are commuting well-bounded operators with decomposition of the identity of bounded variation. We also explore some properties of AC-operators by applying the theory of (Foiaş) decomposable operators.

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1. Introduction. The class of well-bounded operators on a Banach space was introduced by Smart [**14**] as a natural analogue of selfadjoint operators on Hilbert space, and was first studied by Smart and Ringrose ([**12**], [**13**], [**14**]). Well-bounded operators are defined as those which possess a functional calculus for absolutely continuous functions on some compact interval $[a, b]$ of the real line. Smart and Ringrose [**12**] proved that on a reflexive Banach space a well-bounded operator can always be written as an integral with respect to a spectral family of projections. Ringrose showed that the dual of a well-bounded operator always admits an integral representation with respect to a family of projections. This family of projections was called a decomposition of the identity (a definition will be given in section 2). Well-bounded operators of type (B) were characterised by Berkson and Dowson [**2**] and by Spain [**15**] as being those for which the absolutely continuous functional calculus is weakly compact. Berkson and Dowson [**2**] introduced the class of well-bounded operators with decomposition of the identity of bounded variation. They showed that there exists a well-bounded operator with decomposition of the identity of bounded variation which is not "decomposable in X" in that the projections in the decomposition of the identity are not the adjoints of projections on X.

In [**4**] Berkson and Gillespie introduced the concept of an AC-operator as an operator which possesses a functional calculus for the absolutely continuous functions on some rectangle in $\mathbb{C}$ (more detailed definitions will be given in section 3). Berkson and Gillespie showed that these operators can be characterised by the fact that they possess a splitting into real and imaginary parts, $T = U + iV$, where U and V are commuting well-bounded operators. They showed [**4**] that if U and V are well-bounded of type (B) this splitting is unique, and that if $S \in L(X)$ commutes with $U + iV$ then S commutes with U and V. Berkson, Doust and Gillespie later showed that neither result is guaranteed if the type (B) hypothesis is omitted [**3**].

In Section 3 we study operators $U + iV$, where U and V are commuting well-bounded with decomposition of the identity of bounded variation. In this case also, if $S \in L(X)$ commutes with $U + iV$ then S commutes with U and V. It is shown if $T = U + iV$, where U and V are commuting well-bounded operators with decomposition of the identity of bounded variation, and if either X does not contain a copy of c_0, or if U and V are type (A) then such a representation is unique. An important tool is the fact that if $S' \in L(X')$ is a scalar-type operator of class X then $S \in L(X)$ is strongly normal-equivalent ([**9**, Theorem 2.4]). In Section 4, by applying the theory of (Foiaş) decomposable operators, we explore some properties of AC-operators $T = U + iV$ where U, V are type (B) well-bounded operators or well-bounded operators with decomposition of the identity of bounded variation.

2. Well-bounded operators.

Let X be a complex Banach space with dual space X'. The Banach algebra of all bounded linear operators on X is denoted by $L(X)$. Given $T \in L(X)$, let $T' \in L(X')$ be its adjoint, and let

$$||T|| = \sup\{||Tx|| : x \in X, ||x|| \leq 1\}.$$

The theory of well-bounded operators is given in more detail in [**8**]. We shall give some of the basic definitions regarding well-bounded operators.

DEFINITION 2.1. An operator $T \in L(X)$ is said to be *well-bounded* if there exist a constant K and a closed interval $J = [a, b] \subseteq \mathbb{R}$ such that

$$||p(T)|| \leq K\left\{|p(a)| + \int_a^b |p'(t)|\, dt\right\}$$

for all complex polynomials p.

DEFINITION 2.2. A *decomposition* of the identity for X (on J) is a family

$$\{E(s) : s \in \mathbb{R}\}$$

of projections on X' such that
(1) $E(s) = 0$ for $s < a$ and $E(s) = I$ for $s \geq b$,
(2) $E(s)E(t) = E(t)E(s) = E(s)$ for $s \leq t$,
(3) there is a real constant K such that $||E(s)|| \leq K$ for $s \in \mathbb{R}$,
(4) the function $s \mapsto \langle x, E(s)y \rangle$ is Lebesgue measurable for $x \in X$, and $y \in X'$,
(5) for each $x \in X$, the map $y \mapsto \langle x, E(s)y \rangle$ from X' into $L^\infty[a, b]$ is continuous when X' and $L^\infty[a, b]$ are given their weak* topologies as the duals of X and $L^1[a, b]$ respectively,
(6) if $x \in X$, $y \in X'$, $s \in [a, b)$, and if the function $t \mapsto \int_a^t \langle x, E(u)y \rangle\, du$ is right differentiable at s, then the right derivative at s is $\langle x, E(s)y \rangle$.

Given a decomposition of the identity $\{E(s) : s \in \mathbb{R}\}$ there exists a unique well-bounded operator $T \in L(X)$ such that

$$\langle Tx, y \rangle = b\langle x, y \rangle - \int_a^b \langle x, E(u)y \rangle\, du.$$

Every well-bounded operator has such a representation, but in general the decomposition of the identity is not uniquely determined by T.

DEFINITION 2.3. Let T be a well-bounded operator and let $\{E(s) : s \in \mathbb{R}\}$ be a decomposition of the identity for T. Then T is *well-bounded of type* (A) if there exists a family of projections $F(s)$ on X such that $F(s)' = E(s)$ for all $s \in \mathbb{R}$.

This definition is equivalent to Definition 16.7 of [8], by ([5, Theorem 3.2]).

DEFINITION 2.4. T is said to be *of type* (B) if T is of type (A) and, in addition, for each real s, $\lim_{t \to s^-} F(t)x$ exists for every $x \in X$.

DEFINITION 2.5. Let $T \in L(X)$ be a well-bounded operator on X. A decomposition of the identity $\{E(s) : s \in \mathbb{R}\}$ for T is said to be of *bounded variation* if the function $s \mapsto \langle x, E(s)y' \rangle$ is of bounded variation on $\mathbb{R}$ for every $x \in X$ and $y' \in X'$.

Let $T \in L(\ell^2)$ denote the well-bounded operator constructed in ([10, Lemma 1]). T is a well-bounded operator of type (B) which is not a well-bounded operator with decomposition of the identity of bounded variation. ([8, Example 16.19]) shows that there exists a well-bounded operator of type (A) with decomposition of the identity of bounded variation which is not well-bounded of type (B). This shows that neither of the classes of well-bounded operators with decomposition of the identity of bounded variation and of well-bounded operators of type (B) includes the other.

DEFINITION 2.6. An operator $T \in L(X)$ is *hermitian* if

$$\|\exp(itT)\| = 1 \qquad (t \in \mathbb{R}).$$

DEFINITION 2.7. An operator $R \in L(X)$ is *hermitian-equivalent* if and only if there exists an equivalent norm on X with respect to which R is hermitian.

Equivalently, R is hermitian-equivalent if and only if there is an $M \ (\geq 1)$ such that

$$\|\exp(itR)\| \leq M \qquad (t \in \mathbb{R}).$$

If this condition is satisfied, then

$$|x| = \sup\{\|\exp(itR)x\| : t \in \mathbb{R}\}$$

defines a norm on X, equivalent to $\|\cdot\|$, with respect to which R is hermitian.

More generally, a set $\Lambda \subseteq L(X)$ is hermitian-equivalent if and only if there is an equivalent norm on X with respect to which every operator in Λ is hermitian. It is known ([8, Theorem 4.17]) that when Λ is a commutative subset of $L(X)$, then Λ is hermitian-equivalent if and only if each operator in the closed real linear span of Λ is hermitian-equivalent.

DEFINITION 2.8. The operator $T = R + iJ$ is *normal-equivalent* if $RJ = JR$ and the set $\{R, J\}$ is hermitian-equivalent.

If $T \in L(X)$ is normal-equivalent then T can be expressed uniquely in the form $R + iJ$, where $RJ = JR$ and the set $\{R, J\}$ is hermitian-equivalent ([9, Lemma 1.10]).

THEOREM 2.9. *If T is a well-bounded operator with decomposition of the identity of bounded variation then the set $\{T^n : n = 0, 1, 2, \cdots\}$ is hermitian-equivalent.*

Proof. If T is a well-bounded operator with decomposition of the identity of bounded variation, then, by [**8**, Theorem 16.15], T' is real scalar-type of class X with resolution of the identity $E(\cdot)$. Then by [**9**, Theorem 2.5] there are operators $H, K \in L(X)$ such that $\{H^n K^m : n, m = 0, 1, 2, \cdots\}$ is hermitian-equivalent and $H' = \int_{\sigma(T)} Re\lambda\, E(d\lambda)$ and $K' = \int_{\sigma(T')} Im\lambda\, E(d\lambda)$. Note that $\sigma(T) \subseteq \mathbb{R}$ so that $\sigma(T') \subseteq \mathbb{R}$: then $K' = 0$: so that $T = H$. $\square$

DEFINITION 2.10. An operator $T \in L(X)$ is said to have the *single-valued extension property* if whenever $f: D_f \to X$ is analytic in an open set $D_f \subseteq \mathbb{C}$ and satisfies

$$(\lambda I - T)f(\lambda) = 0, \quad (\lambda \in D_f)$$

it follows that $f = 0$ in D_f.

Let $T \in L(X)$ have the single-valued extension property. Given $x \in X$ we denote by $\varrho_T(x)$ the set of elements $\alpha \in \mathbb{C}$ such that there exists an X-valued function $x(\lambda)$ analytic in a neighbourhood V_α of α, such that

$$(\lambda I - T)x(\lambda) = x \quad (\lambda \in V_\alpha).$$

In particular, $\varrho(T) \subseteq \varrho_T(x)$. The complement $\sigma_T(x) = \mathbb{C} \backslash \varrho_T(x)$ is the *local spectrum* of T at x; it is a compact subset of $\sigma(T)$, and is non-empty for $x \neq 0$.

For $F \subseteq \mathbb{C}$, let $X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}$. This is a T-invariant manifold.

LEMMA 2.11. *Let U and U_1 be commuting well-bounded operators on X, where U is well-bounded of type* (A) *and $U - U_1$ is quasinilpotent. Then $U = U_1$.*

Proof. Let $\{E(s) : s \in \mathbb{R}\}$ be the family of projections on X whose adjoints form a decomposition of the identity of U. By [**8**, Theorem 15.19 (iii) and 16.3] there exist a decomposition of the identity $\{F(s) : s \in \mathbb{R}\}$ for U_1 such that $E(s)'F(s) = F(s)E(s)'(s \in \mathbb{R})$. Observe that U' and U_1' have the single valued extension property ([**8**, Proposition 5.28]). Given $x' \in X'$ denote by $\sigma_{U_1'}(x')$ the local spectrum of x' relative to U_1'. Since $U - U_1$ is quasinilpotent, so also is $U' - U_1'$. Hence $\sigma_{U_1'}(x')$ is equal to the local spectrum x' relative to U' by [**6**, Theorem 1.2.4]. Hence by ([**2**, Theorem 5.6]) the projections $E(s)'$ and $F(s)$ have the same range, and hence are equal, since they commute. This is true for all $s \in \mathbb{R}$, and so $U = U_1$. $\square$

COROLLARY 2.12. *If U is a quasinilpotent well-bounded operator, then $U = 0$.*

3. AC-operators One of the major complications one encounters when trying to extend this theory to operators with complex spectra is deciding upon the correct concept of an *absolutely continuous* function of two variables to use. In the discussion that follows we shall identify the subsets of $\mathbb{R}^2$ with subsets of $\mathbb{C}$ in the usual way. Let m denote Lebesgue measure on $\mathbb{R}^2$. Let $J = [a, b]$ and $K = [c, d]$ be two compact intervals in $\mathbb{R}$. Let Λ be a rectangular partition of $J \times K$:

$$a = s_0 < s_1 < \cdots < s_n = b, \qquad c = t_0 < t_1 < \cdots < t_m = d.$$

For a function $f: J \times K \to \mathbb{C}$, define

$$V_\Lambda(f) = \sum_{i=1}^{n} \sum_{j=1}^{m} |f(s_i, t_j) - f(s_i, t_{j-1}) - f(s_{i-1}, t_j) + f(s_{i-1}, t_{j-1})|.$$

The *variation* of f is defined to be

$$\mathrm{var}_{J \times K}(f) = \sup\{V_\Lambda(f) : \Lambda \text{ is a rectangular partition of } J \times K\}.$$

We shall say that the function f is of *bounded variation* if $\mathrm{var}_{J \times K} f$, $\mathrm{var}_J f(\cdot, d)$ and $\mathrm{var}_K f(b, \cdot)$ are all finite. The set $BV(J \times K)$ of all functions $f: J \times K \to \mathbb{C}$ of bounded variation is a Banach algebra under the norm

$$\|f\| = |f(b, d)| + \mathrm{var}_J f(\cdot, d) + \mathrm{var}_K f(b, \cdot) + \mathrm{var}_{J \times K} f.$$

A function $f: J \times K \to \mathbb{C}$ is said to be *absolutely continuous* if
(1) for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{R \in \mathfrak{R}} \mathrm{var}_R f < \epsilon$$

whenever $\mathfrak{R}$ is a finite collection of non-overlapping subrectangles of $J \times K$ with $\sum_{R \in \mathfrak{R}} m(R) < \delta$;
(2) The marginal functions $f(\cdot, d)$ and $f(b, \cdot)$ are absolutely continuous functions on J and K respectively.
The set $\mathrm{AC}(J \times K)$ of all absolutely continuous functions $f: J \times K \to \mathbb{C}$ is a Banach subalgebra of $\mathrm{BV}(J \times K)$, and is the closure in $\mathrm{BV}(J \times K)$ of the polynomials in two real variables on $J \times K$. Equivalently, one can consider $\mathrm{AC}(J \times K)$ to be the closure of the polynomial functions $p(z, \bar{z})$ on $J \times K \subseteq \mathbb{C}$.
Define the functions $u, v \in \mathrm{AC}(J \times K)$ by $u(x, y) = x$ and $v(x, y) = y$.

DEFINITION 3.1. An operator $T \in L(X)$ is said to be an AC-*operator* if there exists a continuous unital Banach algebra homomorphism $\Theta : \mathrm{AC}(J \times K) \to L(X)$ for which $\Theta(u + iv) = T$.

Berkson and Gillespie ([**4**, Theorem 5]) proved that this is equivalent to the condition that T can be written as $T = U + iV$ where U and V are commuting well-bounded operators on X. They showed that if U and V are well-bounded of type (B) the representation in the form $T = U + iV$ is unique and if $S \in L(X)$ commutes with T then S commutes with U and V. We generalise this result in section 4. However, neither result is guaranteed if the type (B) hypothesis is omitted as is shown by [**3**, Examples 3.1 and 3.4].

THEOREM 3.2. *Let U and V be commuting well-bounded operators with decompositions of the identity of bounded variation on X and let $S \in L(X)$ commute with $U + iV$. Then S commutes with U and V.*

Proof. By Theorem 2.9 the operators U, V are hermitian-equivalent. Since $UV = VU$ it follows that $U + iV$ is normal-equivalent. By [**8**, Theorem 4.22] we have $SU = US$, $SV = VS$. $\quad\square$

By applying the theory of (Foiaş) decomposable operators we generalise Theorem 3.2 in Section 4.

THEOREM 3.3. *Let*

$$T = U + iV = U_1 + iV_1,$$

where

(1) *U and V are commuting well-bounded operators of type (A) with decompositions of the identity of bounded variation;*

(2) *U_1 and V_1 are commuting well-bounded operators on X.*

Then $U = U_1$ and $V = V_1$.

Proof. U_1 commutes with $U_1 + iV_1$ and hence U_1 commutes with $U + iV$. By Theorem 3.2, U_1 commutes with U and V. Similarly V_1 commutes with U and V. Hence the set $\{U, V, U_1, V_1\}$ is commutative. Since well-bounded operators have real spectra, we can apply standard Gelfand theory to deduce that $U - U_1$ and $V - V_1$ are quasinilpotent. Now, by Lemma 2.11, $U = U_1$ and $V = V_1$. □

When X does not contain a copy of c_0 we need not assume that the real and imaginary parts are decomposable in X.

THEOREM 3.4. *Suppose that X does not contain a copy of c_0. Let*

$$T = U + iV = U_1 + iV_1,$$

where

(1) *U and V are commuting well-bounded operators on X with decomposition of the identity of bounded variation,*

(2) *U_1 and V_1 are commuting well-bounded operators on X.*

Then $U = U_1$ and $V = V_1$.

Proof. U and V are real scalar-type spectral operators ([**7**, Theorem 2]). Now by [**8**, Theorem 16.17] U and V are type (A) well-bounded operators. The result follows from Theorem 3.3. □

4. AC-operators and (Foiaş) decomposable operators.

DEFINITION 4.1. Let X be a Banach space and $T \in L(X)$. A closed linear subspace $\mathcal{Y}$ of X is called a *spectral maximal space* of T if

(1) $\mathcal{Y}$ is invariant under T,

(2) if $\mathcal{Z}$ is another closed linear subspace of X, invariant under T, such that $\sigma(T|\mathcal{Z}) \subseteq \sigma(T|\mathcal{Y})$, then $\mathcal{Z} \subseteq \mathcal{Y}$.

A spectral maximal space of $T \in L(X)$ is ultra-invariant under T; that is it is invariant under any operator commuting with T.

DEFINITION 4.2. An operator $T \in L(X)$ is called *decomposable* if any open cover $\mathbb{C} = G_1 \cup G_2$ of the complex plane $\mathbb{C}$ by two sets G_1 and G_2 yields a splitting of the

spectrum $\sigma(T)$ and of the space X in the sense that there exist closed T-invariant linear subspaces Y and Z of X for which $\sigma(T|Y) \subseteq G_1, \sigma(T|Y) \subseteq G_2$ and $X = Y + Z$.

The definition of decomposability has been simplified considerably since Colojoară and Foiaş wrote their book [6]. The original definition of a decomposable operator as developed by Foiaş was somewhat complicated and involved the notion of a spectral maximal space. See [11], [6] and [16] for an account of the classical theory of decomposable operators.

Let Ω be a subset of the complex plane. An algebra $\mathcal{A}$ of complex functions defined on Ω is called *normal* if for every open finite covering $\{G_i\}_{1 \leq i \leq n}$ of $\bar{\Omega}$ there exist functions $f_i \in \mathcal{A}$ such that

(1) $f_i(\Omega) \subseteq [0, 1]$, $(1 \leq i \leq n)$,
(2) $\text{supp}(f_i) \subseteq G_i$, $(1 \leq i \leq n)$ where $\text{supp}(f_i) = \overline{\{\lambda \in \Omega | f(\lambda \neq 0)\}}$
(4) $\sum_{i=1}^{n} f_i = 1$ *on* Ω.

DEFINITION 4.3. Let $\mathcal{A}$ be an algebra of complex functions defined on the closed set $\Omega \subseteq \mathbb{C}$. $\mathcal{A}$ will be called *topologically admissible* if

(1) $\lambda \in \mathcal{A}$, $1 \in \mathcal{A}$ (where λ is the function $f(\lambda) = \lambda$, and 1 is the function $f(\lambda) = 1$),
(2) $\mathcal{A}$ is normal,
(3) $\mathcal{A}$ is endowed with a locally convex topology τ such that if $\{f_n\} \subseteq \mathcal{A}$ is a Cauchy sequence in τ and $f_n(\lambda) \to 0$ for every $\lambda \in \Omega$, then $f_n \to 0$ in τ,
(4) for every $f \in \mathcal{A}$ and every $\xi \notin \text{supp}(f)$, the function

$$f_\xi(\lambda) = \begin{cases} \dfrac{f(\lambda)}{\xi - \lambda} & \text{if } \lambda \in \Omega \backslash \{ \xi \} \ , \\ 0 & \text{if } \lambda \in \Omega \cap \{ \xi \} \end{cases}$$

belongs to $\mathcal{A}$, and the mapping $\xi \to f_\xi$ of $\mathbb{C} \backslash \text{supp}(f)$ into $\mathcal{A}$ is continuous.

LEMMA 4.4. *Let* $J = [a, b]$, $K = [c, d]$, *and* $\Omega = J \times K$; *let* $\mathcal{A} = \text{AC}(\Omega)$ *and* $f \in \mathcal{A}$ *and let* f_ξ *be as in Definition 4.3. Then*

(1) $f_\xi \in \text{AC}(J \times K)$,
(2) *the mapping* $\xi \to f_\xi$ *of* $\mathbb{C} \backslash \text{supp}(f)$ *into* $\mathcal{A}$ *is continuous.*

Proof. Let $\xi \notin \text{supp}(f)$. We can find a C^∞ function $\kappa_{f,\xi}$ and a closed disc D_ξ containing ξ such that $\kappa_{f,\xi}|_{\text{supp}(f)} = 1$ and $\kappa_{f,\xi}|_{D_\xi} = 0$; then $\lambda \mapsto \dfrac{\kappa_{f,\xi}}{\xi - \lambda} \in C^\infty$ and $f_\xi(\lambda) = \dfrac{f(\lambda)\kappa_{f,\xi}(\lambda)}{\xi - \lambda}$, so is in $\text{AC}(J \times K)$.

The mapping $\xi \mapsto f_\xi$ is clearly continuous. $\square$

COROLLARY 4.5. $\text{AC}(J)$ *and* $\text{AC}(J \times K)$ *are topologically admissible algebras.*

DEFINITION 4.6 Let $\mathcal{A}$ be a topologically admissible algebra. A mapping $\Theta : \mathcal{A} \to L(X)$ is called a *continuous $\mathcal{A}$-spectral function* if

(1) $\Theta : \mathcal{A} \to L(X)$ is an algebraic homomorphism, and $\Theta(1) = I$,
(2) $\Theta : \mathcal{A} \to L(X)$ is continuous ([6, Definition 3.5.3]).

REMARK 4.7. By [**6**, Theorem 3.5.4] every continuous $\mathcal{A}$-spectral function is an $\mathcal{A}$-spectral function in the sense of [**6**, Definition 3.1.3].

DEFINITION 4.8. An operator $S \in L(X)$ is called $\mathcal{A}$-*scalar* if there exists an $\mathcal{A}$-spectral function Θ such as $\Theta(\lambda) = S$. Such an $\mathcal{A}$-spectral function will be called an $\mathcal{A}$-spectral function of S.

By Corollary 4.5 well-bounded operators are $\mathcal{A}$-scalar and hence by [**6**, Theorem 3.1.16] they are decomposable operators. If $T \in L(X)$ is a decomposable operator then for any closed subset F of $\sigma(T)$ the subspace $X_T(F)$ is a *spectral maximal space* of T [**6**, Theorem 2.1.5].

THEOREM 4.9. *Let $T \in L(X)$ be a well-bounded operator, $x_0 \in X$ and* $\lim_{n \to \infty} \|T^n x_0\|^{1/n} = 0$. *Then $Tx_0 = 0$.*

Proof. T is decomposable. Hence $X_T(\{0\})$ is a closed subspace of X which is invariant for T and satisfies $\sigma(T|X_T(\{0\})) = \{0\} \cap \sigma(T)$ ([**6**, Theorem 2.1.5]). Therefore, $T|X_T(\{0\})$ is a quasinilpotent well-bounded operator. Hence, by Corollary 2.12, $T|X_T(\{0\} = 0$. Now by [**6**, Lemma 4.4.4] we have $X_T(\{0\}) = \{x \in X : \lim_{n \to \infty} \|T^n x\|^{1/n} = 0\}$. Thus $x_0 \in X_T(\{0\})$ and $Tx_0 = 0$. $\qquad \square$

COROLLARY 4.10. *Let $T = U + iV$ where U, V are commuting well-bounded operators. If $\lim_{n \to \infty} \|T^n x\|^{1/n} = 0$ for some $x \in X$, then $Ux = Vx = 0$.*

Proof. There is a continuous homomorphism $\Theta : \mathrm{AC}(J \times K) \to L(X)$ such that $\Theta(u) = U$, $\Theta(v) = V$ and $\Theta(u + iv) = U + iV$.

$$\lim_{n \to \infty} \|T^n x\|^{1/n} = 0.$$

Hence $x \in X_T(\{0\})$ ([**6**, Lemma 4.4.4]). This gives

$$x \in X_T(\{0\}) \subseteq X_T(u^{-1}(\{0\})) = X_{\Theta(u)}(\{0\}) = X_U(\{0\}),$$

where the inclusion is by [**6**, Theorem 1.1.2] and the first equality is by [**6**, Theorem 3.2.4]. Hence

$$\lim_{n \to \infty} \|U^n x\|^{1/n} = 0.$$

By Theorem 4.9, $Ux = 0$. Similarly we can show that $Vx = 0$. $\qquad \square$

Lemma 4.11 is due to Gillespie (private communication).

LEMMA 4.11. *Let $T = U + iV$, where U, V are commuting type (B) well-bounded operators on a Banach space X. Fix $a, b \in \mathbb{R}$ with $a < b$ and let $F = \{z \in \mathbb{C} : a \leq \mathrm{Re} z \leq b\}$. Then*

$$X_T(F) = [E(b) - E(a^-)]X,$$

where $E(\cdot)$ is the spectral family of U.

Proof. Standard Gelfand theory shows that the spectrum of the restriction of T to $[E(b) - E(a^-)]X$ is contained in F. Hence $[E(b) - E(a^-)]X \subseteq X_T(F)$. Now U and V commute with T, and hence $X_T(F)$ is invariant under both U and V ([**6**, Theorem 2.3.3]). Again, standard Gelfand theory implies that the spectrum of the restriction of U to $X_T(F)$ is contained in $[a, b]$. It now follows from [**8**, Theorem 19.3] that

$$X_T(F) \subseteq [E(b) - E(a^-)]X.$$

Let X and Y be two Banach spaces. $L(X, Y)$ will denote the collection of all bounded linear mappings of X into Y. For $T \in L(X)$ and $S \in L(Y)$ we define

$$L(S), R(T), C(S, T) : L(X, Y) \to L(X, Y)$$

by $L(S)A = SA$, $R(T)A = AT$ (where $A \in L(X, Y)$) and $C(T, S) = L(S) - R(T)$, respectively.

We can now prove the following theorem, which generalises that of Berkson and Gillespie ([**4**, Lemma 4]).

THEOREM 4.12. *Suppose X and Y are Banach spaces. Let $T_1 = U_1 + iV_1 \in L(X)$ and $T_2 = U_2 + iV_2 \in L(Y)$ be AC-operators where $U_i, V_i, (i = 1, 2)$ are type (B) wellbounded operators. Let $S \in L(X, Y)$ be an operator such that*

$$\lim_{n \to \infty} \|C(T_2, T_1)^n S\|^{1/n} = 0.$$

Then $U_2 S = S U_1$, $V_2 S = S V_1$ and $T_2 S = S T_1$.

Proof. It is sufficient to show that $E_2(a)S = SE_1(a)$, $(a \in \mathbb{R})$, where $\{E_1(s) : s \in \mathbb{R}\}$ and $\{E_2(s) : s \in \mathbb{R}\}$ are the spectral families of U_1 and U_2 respectively. Let $a, b \in \mathbb{R}$, $a < b$ and $F = \{z \in \mathbb{C} : a \leq \mathrm{Re}z \leq b\}$. By [**6**, Theorem 2.3.3] we have $SX_{T_1}(F) \subseteq Y_{T_2}(F)$ and hence $S[E_1(b) - E_1(a^-)]X \subseteq [E_2(b) - E_2(a^-)]Y$ (Lemma 4.11). If $\{a_n\}$ is a sequence decreasing to a, then $E_i(a_n) \to E_i(a)(i = 1, 2)$ strongly and $S[E_1(b) - E_1(a_n^-)]X \subseteq [E_2(b) - E_2(a_n^-)]Y$ then gives that

$$S[E_1(b) - E_1(a)]X \subseteq [E_2(b) - E_2(a)]Y.$$

Taking b sufficiently large and positive we get $S[I - E_1(a)]X \subseteq [I - E_2(a)]Y$ and taking a sufficiently large and negative we get that $SE_1(b)X \subseteq E_2(b)Y$. Hence for $a \in \mathbb{R}$ we have $SE_1(a)X \subseteq E_2(a)Y$, $S[I - E_1(a)]X \subseteq [I - E_2(a)]Y$ and therefore $E_2(a)S = SE_1(a)$. □

Recall that two operators $T, S \in L(X)$ are said to be *quasinilpotent equivalent*, $T \overset{q}{\sim} S$ ([**6**, Definition 1.21]), if and only if

$$\lim_{n \to \infty} \|C(T, S)^n I\|^{1/n} = 0 = \lim_{n \to \infty} \|C(S, T)^n I\|^{1/n}.$$

COROLLARY 4.13. *Suppose X and Y are Banach spaces. Let $T_1 = U_1 + iV_1 \in L(X)$ and $T_2 = U_2 + iV_2 \in L(X)$ be AC-operators where $U_i, V_i, (i = 1, 2)$ are type (B) wellbounded operators. Suppose $T_1 \overset{q}{\sim} T_2$. Then $U_1 = U_2$, $V_1 = V_2$ and $T_1 = T_2$.*

Proof. With $Y = X$ and $S = I$ in Theorem 4.12 we obtain $U_1 = U_2$, $V_1 = V_2$ and $T_1 = T_2$. $\qquad\square$

The following results are immediate corollaries of Theorem 2.9 and [1, Theorem 1].

COROLLARY 4.14. *Suppose X and Y are Banach spaces. Let $T_1 = U_1 + iV_1 \in L(X)$ and $T_2 = U_2 + iV_2 \in L(Y)$ be* AC-*operators, where U_i, V_i ($i = 1, 2$) are commuting well-bounded operators with decompositions of the identity of bounded variation on X, Y. Suppose further that*

$$\lim_{n\to\infty} \|C(T_2, T_1)^n S\|^{1/n} = 0.$$

Then $U_2 S = S U_2$, $V_2 S = S V_1$, *and* $T_2 S = S T_1$.

COROLLARY 4.15. *Let X be a Banach space and let $T_i = U_1 + iV_1 \in L(X)$ ($i = 1, 2$) be* AC-*operators with $U_i, V_i, (i = 1, 2)$ commuting well-bounded operators with decompositions of the identity of bounded variation. Suppose $T_1 \overset{q}{\sim} T_2$. Then $U_1 = U_2$, $V_1 = V_2$ and $T_1 = T_2$.*

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