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FIRST CHERN CLASS AND HOLOMORPHIC TENSOR FIELDS

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1. Introduction

Let M be an n-dimensional compact Kaehler manifold, TM its (holomorphic) tangent bundle and T^*M its cotangent bundle. Given a complex vector bundle E over M, we denote its m-th symmetric tensor power by S^mE and the space of holomorphic sections of E by $\Gamma(E)$. In [4] we have shown that $\Gamma(S^mTM) = 0$ (resp. $\Gamma(S^mT^*M) = 0$) if $c_1(M) \leq 0$ (resp. $c_1(M) \geq 0$) and if M is simply connected. (For the precise statement of a little stronger result, see [4]).

In this paper we consider more general tensor bundles. Our results may be summarized as follows:

THEOREM A. *Let M be a compact Kaehler manifold with $c_1(M) < 0$ (i.e., with ample canonical line bundle K_M). Let*

$$T^r_sM = \left(\overset{r}{\otimes} TM\right) \otimes \left(\overset{s}{\otimes} T^*M\right) .$$

If $r > s$, then $\Gamma(T^r_sM) = 0$, i.e., there is no holomorphic tensor fields of contravariant degree r and covariant degree s.

The theorem above is an immediate consequence of a theorem of Bochner [7] and a recent result of Aubin [1] and Yau [8].

COROLLARY A.1. *Let M be as above. Let m be a non-negative integer and q a (possibly negative) integer. Then*

(1) $\Gamma(S^mTM \otimes K^q_M) = 0$ *for* $m - qn > 0$,
(2) $\Gamma(S^mT^*M \otimes K^q_M) = 0$ *for* $-m - qn > 0$.

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For applications, the cases $\pm m - qn = 0$ are of some interest.

THEOREM B. *Let M be a compact Kaehler manifold with $c_1(M) < 0$. Then*

$$\Gamma(S^{nq}TM \otimes K_M^q) = \Gamma(S^{nq}T^*M \otimes K_M^{-q}) = 0 \qquad \text{for } q > 0\,,$$

unless the universal covering space $\tilde{M}$ of M is biholomorphic to a product $D \times N$ of a symmetric bounded domain D and a complex manifold N with $\dim D > 0$ *and* $\dim N \geq 0$.

COROLLARY B.1. *Let M be a compact Kaehler manifold with $c_1(M) < 0$ and finite fundamental group. Then*

$$\Gamma(S^{nq}TM \otimes K_M^q) = \Gamma(S^{nq}T^*M \otimes K_M^{-q}) = 0 \qquad \text{for } q > 0\,.$$

COROLLARY B.2. *Let M be a compact Kaehler surface with $c_1(M) < 0$. Then*

$$\Gamma(S^{2q}TM \otimes K_M^q) = \Gamma(S^{2q}T^*M \otimes K_M^{-q}) = 0 \qquad \text{for } q > 0\,,$$

unless the universal covering space $\tilde{M}$ of M is biholomorphic to the bidisk $\{(z, w) \in \boldsymbol{C}^2; |z| < 1, |w| < 1\}$.

The proof of Theorem B is a little more involved; as in our previous paper [4], it requires Berger's holonomy classification theorem [2]. By a similar method, we obtain also the following

THEOREM C. *Let M be a compact Kaehler manifold with $c_1(M) = 0$ and finite fundamental group. Then*

$$\Gamma(S^{m}TM \otimes K_M^q) = \Gamma(S^{m}T^*M \otimes K_M^q) = 0 \qquad \text{for } m > 0\,.$$

A vector bundle E of rank r over M is said to be *instable* in the sense of Bogomolov if there exists a representation ρ of $GL(r; \boldsymbol{C})$ with determinant 1 (i.e., a representation which factors through $PGL(r; \boldsymbol{C})$) such that the associated vector bundle $E^{(\rho)}$ has a nonzero section s which vanishes at some point of M (see [5]).

The following result is a simple consequence of Bochner's theorem and the representation theory of $GL(n; \boldsymbol{C})$.

THEOREM D. *Let M be a compact Kaehler-Einstein manifold. Then its tangent bundle TM is not instable in the sense of Bogomolov.*

COROLLARY D.1. *Let M be a compact Kaehler manifold with $c_1(M) < 0$ or $c_1(M) = 0$. Then TM is not instable in the sense of Bogomolov.*

In deriving Corollary D.1 we use again the result of Aubin (for $c_1(M) < 0$) and Yau (for $c_1(M) < 0$ and $c_1(M) = 0$). By a purely algebraic method, a result similar to Corollary D.1 has been obtained by Reid [6].

COROLLARY D.2. *Let M be a simply connected compact homogeneous Kaehler manifold. Then TM is not instable in the sense of Bogomolov.*

In the last section, we shall discuss possibility of generalizing Theorem D to other vector bundles.

I want to thank F. Sakai and M. Reid for explaining to me the concept of instable vector bundle.

2. Proof of Theorem A

Let M be a compact Kaehler manifold and $\xi \in \Gamma(T^r_s M)$. Let $\|\xi\|$ denote the length of ξ and set $f = \|\xi\|^2$. If M is Kaehler-Einstein, i.e., $R_{i\bar{j}} = cg_{i\bar{j}}$, then a simple calculation shows

$$\Delta f = \|\nabla\xi\|^2 - c(r - s)\|\xi\|^2$$

and we obtain the following theorem of Bochner ([7; p. 142]):

THEOREM 1. *Let M be a compact Kaehler-Einstein manifold. Then*

(1) *If the Ricci tensor is positive, then*

$$\Gamma(T^r_s M) = 0 \qquad \textit{for } r < s ,$$

and $\Gamma(T^r_r M)$ consists of parallel tensor fields;

(2) *If the Ricci tensor is negative, then*

$$\Gamma(T^r_s M) = 0 \qquad \textit{for } r > s ,$$

and $\Gamma(T^r_r M)$ consists of parallel tensor fields;

(3) *If the Ricci tensor is zero, then $\Gamma(T^r_s M)$, for all r, s, consists of parallel tensor fields.*

If $c_1(M) < 0$, then by Aubin [1] and Yau [8], there exists a Kaehler-Einstein metric with negative Ricci tensor on M. Hence, Theorem A follows from (2) of Theorem 1. Since K_M is a subbundle of $T^0_n M$ and $ST^m M$ is a subbundle of $T^m_0 M$, Corollary A.1 is a special case of Theorem A.

3. Proof of Theorem B

Again, by the theorem of Aubin and Yau, we may assume that M is a Kaehler-Einstein with negative Ricci tensor. Let $\tilde{M}$ be the universal covering space of M and let

$$\tilde{M} = M_1 \times \cdots \times M_r$$

be the de Rham decomposition of $\tilde{M}$ into Kaehler manifolds $M_1, \cdots, M_r$ with irreducible holonomy group. (Since the Ricci tensor is negative definite, there is no Euclidean factor in the decomposition). Under a natural identification, we have

$$S^m T\tilde{M} \otimes K^q_{\tilde{M}} = \sum (S^{m_1} TM_1 \otimes K^q_{M_1}) \otimes \cdots \otimes (S^{m_r} TM_r \otimes K^q_{M_r}) ,$$

where the summation is taken over all partitions $m = nq = m_1 + \cdots + m_r$.

Since every $\xi \in \Gamma(S^{nq} T\tilde{M} \otimes K^q_{\tilde{M}})$ is parallel by (2) of Theorem 1, we have only to show that if $V(=M_i)$ is a non-symmetric Kaehler manifold with irreducible holonomy group, then the bundle $S^k TV \otimes K^q_V$ has no parallel sections (other than the zero section). But this is an algebraic problem. Set $U = C^p$ where $p = \dim V$. Then the problem is to show that no element of $S^k U \otimes (\Lambda^p U)^q$ is invariant under the natural action of the holonomy group. By the holonomy classification theorem of Berger [2] (see also [4] where Berger's theorem is explained in our special situation), the holonomy group of a p-dimensional non-symmetric irreducible Kaehler manifold V with nonzero Ricci tensor must be either $U(p)$ or $Sp(p/2) \times U(1)$. Since these groups act irreducibly on $S^k U$ and since $\dim (\Lambda^p U) = 1$, this completes the proof. The case of $S^{nq} T^* M \otimes K^{-q}_M$ is similar.

The assumption in Corollary B.1 eliminates the possibility of symmetric domain for M_i in the above decomposition.

In Corollary B.2, if M is reducible, then $\tilde{M}$ is a bidisk. If M is irreducible, its holonomy group is $U(2)$ even when M is a symmetric domain. As long as the group is as large as $U(2)$, the proof is still valid.

The proof above is also valid when M is a compact Kaehler-Einstein manifold with positive Ricci tensor.

THEOREM 2. *If M is a compact Kaehler-Einstein manifold with positive Ricci tensor, then*

$$\Gamma(S^{nq}TM \otimes K_M^q) = \Gamma(S^{nq}T^*M \otimes K_M^{-q}) = 0 \qquad \textit{for } q > 0\,,$$

unless M *decomposes into a product* $M' \times M''$ *of a hermitian symmetric space* M' *and a Kaehler manifold* M'' *with* $\dim M' > 0$ *and* $\dim M'' \geq 0$.

We remark that a compact Kaehler manifold with positive Ricci tensor is simply connected [3] so that we can apply the de Rham decomposition theorem directly to M.

4. Proof of Theorem C

By the theorem of Yau, there is a Kaehler metric with zero Ricci tensor on M. By (3) of Theorem 1, every holomorphic tensor field is parallel. The rest of the argument is similar to that in the proof of Theorem B.

We note that Theorem C is applicable to the Kaehler K3 surfaces.

5. Proof of Theorem D

Let $U = C^n$. Let $1 \leq n_1 < n_2 \cdots < n_s \leq n$ and $q_1, q_2, \cdots, q_s, q$ be positive integers such that

$$q_1 n_1 + \cdots + q_s n_s = qn\,.$$

Then the representation λ of $GL(n; C)$ on

$$W = (\Lambda^{n_1} U)^{\otimes q_1} \otimes \cdots \otimes (\Lambda^{n_s} U)^{\otimes q_s} \otimes (\Lambda^n U^*)^{\otimes q}$$

has determinant 1 and, conversely, every irreducible representation ρ of $GL(n; C)$ with determinant 1 appears as an irreducible factor of such a representation λ.

But W is a subspace of $T^{nq}_{nq} = (\otimes^{nq} U) \otimes (\otimes^{nq} U^*)$. Set $E = TM$. If ρ is an irreducible representation of $GL(n; C)$ with determinant 1, then the associated vector bundle $E^{(\rho)}$ is a subbundle of $T^{nq}_{nq}M$. From Theorem 1 it follows that every holomorphic section ξ of $E^{(\rho)}$ is parallel and hence cannot vanish unless it vanishes identically. Since every representation of $PGL(n; C)$ is fully reducible, this completes the proof of Theorem D.

Corollary D.2 follows from the well known fact (due to H. C. Wang and A. Borel) that every simply connected, compact homogeneous complex manifold admits a Kaehler-Einstein metric (of positive Ricci tensor).

Although a compact Kaehler manifold M with $c_1(M) > 0$ may not admit a Kaehler-Einstein metric, it may be still true that TM is not in-stable.

6. Hermitian-Einstein vector bundle

Let E be a holomorphic vector bundle of rank r over a complex manifold M of dimension n. Let h be a Hermitian fibre-metric in E. Let $s_1, \cdots, s_r$ be linearly independent local holomorphic sections of E. We set

$$h_{\alpha\bar{\beta}} = h(s_\alpha, \bar{s}_\beta) .$$

Let $z^1, \cdots, z^n$ be a local coordinate system in M. With respect to $(s_1, \cdots, s_r)$ and $(z^1, \cdots, z^n)$ the curvature of the Hermitian connection in the bundle E is given by

$$R_{\alpha\bar{\beta}j\bar{k}} = \frac{\partial^2 h_{\alpha\bar{\beta}}}{\partial z^j \partial \bar{z}^k} - \sum h^{\varepsilon\bar{\gamma}} \frac{\partial h_{\alpha\bar{\gamma}}}{\partial z^j} \frac{\partial h_{\varepsilon\bar{\beta}}}{\partial \bar{z}^k} .$$

Let $ds^2 = 2\sum g_{j\bar{k}} dz^j d\bar{z}^k$ be a Hermitian metric on M. Then we can define the "Ricci tensor" $(R_{\alpha\bar{\beta}})$ of h with respect to ds^2 by

$$R_{\alpha\bar{\beta}} = \sum g^{j\bar{k}} R_{\alpha\bar{\beta}j\bar{k}} .$$

We say that (E, M, h, ds^2) is a *Hermitian-Einstein vector bundle* if the Ricci tensor $(R_{\alpha\bar{\beta}})$ is proportional to h, i.e.,

$$R_{\alpha\bar{\beta}} = \varphi h_{\alpha\bar{\beta}} ,$$

where φ is a function which does not change its sign. (It is not clear whether we should assume that φ is constant. But, for our present purpose the assumption above suffices.).

Without any change in their proofs, Theorems 1 and D generalize to Hermitian-Einstein vector bundles. We have therefore

THEOREM 3. *Let (E, M, h, ds^2) be a Hermitian-Einstein vector bundle over a compact complex manifold M. Then E is not instable in the sense of Bogomolov.*

Remark. For the tangent bundle $E = TM$, the existence of h and ds^2 such that (TM, M, h, ds^2) is a Hermitian-Einstein vector bundle is probably a much weaker condition than the existence of a Kaehler-Einstein metric on M. (The latter condition means the existence of a

Hermitian-Einstein structure (h, ds^2) such that $h = ds^2$).

Many homogeneous vector bundles carry Hermitian-Einstein structures. But it is important to find other (preferably, algebraic-geometric) conditions for the existence of Hermitian-Einstein structure.

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