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DISCREPANCY BOUNDS FOR THE DISTRIBUTION OF RANKIN–SELBERG *L*-FUNCTIONS

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Abstract

We investigate the discrepancy between the distributions of the random variable $\log L(\sigma, f \times f, X)$ and that of $\log L(\sigma + it, f \times f)$, that is,

$$D_\sigma(T) := \sup_{\mathcal{R}} |\mathbb{P}_T(\log L(\sigma + it, f \times f) \in \mathcal{R}) - \mathbb{P}(\log L(\sigma, f \times f, X) \in \mathcal{R})|,$$

where the supremum is taken over rectangles $\mathcal{R}$ with sides parallel to the coordinate axes. For fixed $T > 3$ and $2/3 < \sigma_0 < \sigma < 1$, we prove that

$$D_\sigma(T) \ll \frac{1}{(\log T)^\sigma}.$$

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1. Introduction

Let $X(p)$ be independent random variables uniformly distributed on the unit circle, where p runs over the prime numbers. The random Euler product of the Riemann zeta-function is defined by $\zeta(\sigma, X) = \prod_p (1 - (X(p)/p^\sigma))^{-1}$. The behaviour of p^{-it} is almost like the independent random variables $X(p)$, which indicates that $\zeta(\sigma, X)$ should be a good model for the Riemann zeta-function.

Bohr and Jessen [1] suggested that $\log \zeta(\sigma + it)$ converges in distribution to $\log \zeta(\sigma, X)$ for $\sigma > 1/2$. In 1994, Harman and Matsumoto [4] studied the discrepancy between the distribution of the Riemann zeta-function and that of its random model. For fixed σ with $1/2 < \sigma \leq 1$ and any $\varepsilon > 0$, they proved that the discrepancy

$$D_{\sigma,\zeta}(T) := \sup_{\mathcal{R}} |\mathbb{P}_T(\log \zeta(\sigma + it) \in \mathcal{R}) - \mathbb{P}(\log \zeta(\sigma, X) \in \mathcal{R})|,$$

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where the supremum is taken over rectangles $\mathcal{R}$ with sides parallel to the coordinate axes, satisfies the bound $D_{\sigma,\zeta}(T) \ll 1/(\log T)^{(4\sigma-2)/(21+8\sigma)-\varepsilon}$. Here, $\mathbb{P}_T(f(t) \in \mathcal{R}) := T^{-1}\text{meas}\{T \le t \le 2T : f(t) \in \mathcal{R}\}$. Lamzouri *et al.* [8] improved the result by showing that $D_{\sigma,\zeta}(T) \ll 1/(\log T)^{\sigma}$.

Dong *et al.* [3] analysed the discrepancy between the distribution of values of Dirichlet L-functions and the distribution of values of random models for Dirichlet L-functions in the q-aspect. Lee [9] investigated the upper bound on the discrepancy between the joint distribution of L-functions on the line $\sigma = 1/2 + 1/G(T), t \in [T, 2T]$, and that of their random models, where $\log\log T \le G(T) \le (\log T)/(\log\log T)^2$.

Let f be a primitive holomorphic cusp form of weight k for $\mathrm{SL}_2(\mathbb{Z})$. The normalised Fourier expansion at the cusp ∞ is $f(z) = \sum_{n\ge1} \lambda_f(n) n^{(k-1)/2} e^{2\pi i n z}$, where $\lambda_f(n) \in \mathbb{R}$, $n = 1, 2, \ldots,$ are normalised eigenvalues of Hecke operators $T(n)$ with $\lambda_f(1) = 1$, that is, $T(n)f = \lambda_f(n) f$.

According to Deligne [2], for all prime numbers p, there are complex numbers $\alpha_f(p)$ and $\beta_f(p)$, satisfying

$$\begin{cases} |\alpha_f(p)| = \alpha_f(p)\beta_f(p) = 1, \\ \lambda_f(p^\nu) = \displaystyle\sum_{0\le j\le\nu} \alpha_f(p)^{\nu-j}\beta_f(p)^j \quad (\nu \ge 1). \end{cases} \tag{1.1}$$

The function $\lambda_f(n)$ is multiplicative. Moreover, $\lambda_f(p)$ is real and satisfies Deligne's inequality $|\lambda_f(n)| \le d(n)$ for $n \ge 1$, where $d(n)$ is the divisor function. In particular, $|\lambda_f(p)| \le 2$. For $\mathrm{Re}\, s > 1$, the L-function attached to f is defined by

$$L(s, f) = \sum_{n\ge1} \frac{\lambda_f(n)}{n^s} = \prod_p \Big(1 - \frac{\alpha_f(p)}{p^s}\Big)^{-1}\Big(1 - \frac{\beta_f(p)}{p^s}\Big)^{-1}.$$

For $\mathrm{Re}\, s > 1$, the Rankin–Selberg L-function associated to f is defined by

$$L(s, f\times f) := \prod_p \Big(1 - \frac{\alpha_f(p)^2}{p^s}\Big)^{-1}\Big(1 - \frac{\beta_f(p)^2}{p^s}\Big)^{-1}\Big(1 - \frac{1}{p^s}\Big)^{-2} = \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_f(n)^2}{n^s}.$$

According to [6], for $\mathrm{Re} s > 1$,

$$\log L(s, f\times f) = \sum_{n=2}^{\infty} \frac{\Lambda_{f\times f}(n)}{n^s \log n},$$

where

$$\Lambda_{f\times f}(n) := \begin{cases} |\alpha_f(p)^\nu + \beta_f(p)^\nu|^2 \log p & \text{for } n = p^\nu, \\ 0 & \text{otherwise.} \end{cases}$$

For automorphic L-functions, from [10],

$$\sum_{p\le x} \lambda_f^4(p) \sim C_f \frac{x}{\log x}. \tag{1.2}$$

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Recently, Xiao and Zhai [12] studied the discrepancy between the distributions of $\log L(\sigma + it, f)$ and its corresponding random variable $\log L(\sigma, f, X)$. In this article, we investigate the discrepancy between the distribution of the random variable $\log L(\sigma, f \times f, X)$ and that of $\log L(\sigma + it, f \times f)$. Define the Euler product

$$L(\sigma, f \times f, X) = \prod_p \Big(1 - \frac{\alpha_f(p)^2 X(p)}{p^\sigma}\Big)^{-1}\Big(1 - \frac{\beta_f(p)^2 X(p)}{p^\sigma}\Big)^{-1}\Big(1 - \frac{X(p)}{p^\sigma}\Big)^{-2},$$

which converges almost surely for $\sigma > \frac{1}{2}$. Consider

$$D_\sigma(T) := \sup_{\mathcal{R}} |\mathbb{P}_T(\log L(\sigma + it, f \times f) \in \mathcal{R}) - \mathbb{P}(\log L(\sigma, f \times f, X) \in \mathcal{R})|,$$

where the supremum is taken over rectangles $\mathcal{R}$ with sides parallel to the coordinate axes. We prove the following theorem.

THEOREM 1.1. *Let $T > 3$ and $2/3 < \sigma_0 < \sigma < 1$, where T and σ_0 are fixed. Then,*

$$D_\sigma(T) \ll \frac{1}{(\log T)^\sigma},$$

where the implied constant depends on f and σ.

The proof follows the method in [8]. The range of σ depends on the zero density theorem of $L(s, f \times f)$ and $L(s, \mathrm{sym}^2 f)$ by noticing that $L(s, f \times f) = \zeta(s)L(s, \mathrm{sym}^2 f)$. Unfortunately, the zero density of $L(s, \mathrm{sym}^2 f)$ can only be obtained nontrivially when $2/3 < \sigma \leq 1$ (see [5]).

2. Preliminaries

This section gathers several preliminary results. Since several proofs are essentially the same as those in [8], we omit their details. For any prime number p and integer $\nu > 0$, we define $b_f(p^\nu) = |\alpha_f(p)^\nu + \beta_f(p)^\nu|^2$. Thanks to (1.1),

$$|b_f(p^\nu)| \leq 4.$$

From probability theory, if the characteristic functions of two real-valued random variables are close, then the corresponding probability distributions are also close. The key to proving Theorem 1.1 is to demonstrate that the joint distribution characteristic function of $\mathrm{Re}\log L(\sigma + it)$ and $\mathrm{Im}\log L(\sigma + it)$ can be well estimated. For $u, v \in \mathbb{R}$, we define

$$\Phi_{\sigma,T}(u, v) := \frac{1}{T}\int_T^{2T} \exp(iu\,\mathrm{Re}\log L(\sigma + it, f \times f) + iv\,\mathrm{Im}\log L(\sigma + it, f \times f))\,dt \tag{2.1}$$

and

$$\Phi_\sigma^{\mathrm{rand}}(u, v) := \mathbb{E}(\exp(iu\,\mathrm{Re}\log L(\sigma, f \times f, X) + iv\,\mathrm{Im}\log L(\sigma, f \times f, X))). \tag{2.2}$$

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LEMMA 2.1 [7, Lemma 4.3]. *Let $y > 2$ and $|t| \geq y + 3$ be real numbers. Let $\frac{1}{2} < \sigma_0 < \sigma \leq 1$ and suppose that the rectangle $\{s : \sigma_0 < \mathrm{Re}(s) \leq 1, |\mathrm{Im}(s) - t| \leq y + 2\}$ does not contain zeros of $L(s, f \times f)$. Then,*

$$\log L(s, f \times f) = \sum_{p^\nu \leq y} \frac{b_f(p^\nu)}{\nu p^{\nu(\sigma+it)}} + O\Big(\frac{\log |t|}{(\sigma_1 - \sigma_0)^2} y^{\sigma_1 - \sigma}\Big),$$

where $\sigma_1 = \min(\sigma_0 + 1/\log y, (\sigma + \sigma_0)/2)$.

LEMMA 2.2. *Define $N(\sigma_0, T)$ as the number of zeros $\rho_f = \beta_f + i\gamma_f$ of $L(s, f \times f)$ with $\sigma_0 \leq \beta_f \leq 1$ and $|\gamma_f| \leq T$. Then,*

$$N(\sigma_0, T) = \begin{cases} T^{5(1-\sigma_0)/(3-2\sigma_0)+\epsilon} & \text{for } 1/2 < \sigma_0 < 23/32, \\ T^{26(1-\sigma_0)/(11-4\sigma_0)+\epsilon} & \text{for } 23/32 \leq \sigma_0 < 3/4, \\ T^{2(1-\sigma_0)/\sigma_0+\epsilon} & \text{for } 3/4 \leq \sigma_0 < 1. \end{cases}$$

PROOF. Here, $L(s, f \times f)$ can be written as $L(s, f \times f) = \zeta(s)L(s, \mathrm{sym}^2 f)$. The result is easily obtained from the zero density of the Riemann zeta-function [13] and symmetric square L-functions [5]. □

LEMMA 2.3. *Let $2/3 < \sigma < 1$ and $3 \leq Y \leq T/2$. Then, for all $t \in [T, 2T]$,*

$$\log L(s, f \times f) = \sum_{p^\nu \leq Y} \frac{b_f(p^\nu)}{\nu p^{\nu(\sigma+it)}} + O_f(Y^{-(\sigma-2/3)/2} \log^3 T)$$

except for a set $\mathcal{D}(T)$ with meas$(\mathcal{D}(T)) \ll_f T^{(10/3-5/2\sigma)/(7/3-\sigma)+\epsilon} Y$.

PROOF. Take $\sigma_0 = \frac{1}{2}(\frac{2}{3} + \sigma)$ in Lemma 2.1. The result follows easily from Lemma 2.2. □

The details of the next three results can be found in [12].

LEMMA 2.4. *Let $2/3 < \sigma < 1$, $128 \leq y \leq z$ and $\{b(p)\}$ be any real sequence with $|b(p)| \leq 4$. For any positive integer $k \leq \log T/20 \log z$,*

$$\frac{1}{T} \int_T^{2T} \Big| \sum_{y \leq p \leq z} \frac{b(p)}{p^{\sigma+it}} \Big|^{2k} dt \ll k! \Big(\sum_{y \leq p \leq z} \frac{(b(p))^2}{p^{2\sigma}} \Big)^k + T^{-1/3}.$$

Moreover,

$$\mathbb{E}\Big(\Big| \sum_{y \leq p \leq z} \frac{b(p)X(p)}{p^\sigma} \Big|^{2k} \Big) \ll k! \Big(\sum_{y \leq p \leq z} \frac{(b(p))^2}{p^{2\sigma}} \Big)^k.$$

PROPOSITION 2.5. *Let $2/3 < \sigma < 1$ and $Y = (\log T)^A$ for a fixed $A \geq 1$. There exist $a_1 = a_1(\sigma, A) > 0$ and $a_1' = a_1'(\sigma, A) > 0$ such that*

$$\mathbb{P}_T\Big(\Big| \sum_{p^\nu \leq Y} \frac{b_f(p^\nu)}{\nu p^{\nu(\sigma+it)}} \Big| \geq \frac{(\log T)^{1-\sigma}}{\log \log T} \Big) \ll \exp\Big(-a_1 \frac{\log T}{\log \log T} \Big)$$

and

$$\mathbb{P}\Big(\Big|\sum_{p^\nu \le Y} \frac{b_f(p^\nu)X(p)^\nu}{\nu p^{\nu\sigma}}\Big| \ge \frac{(\log T)^{1-\sigma}}{\log\log T}\Big) \ll \exp\Big(-a_1' \frac{\log T}{\log\log T}\Big).$$

LEMMA 2.6. *Let Y be a large positive real number and $|z| \le Y^{\sigma-1/2}$. Then,*

$$\mathbb{E}(|L(\sigma, f\times f, X)|^z) = \mathbb{E}\Big(\exp\Big(z\,\mathrm{Re}\Big(\sum_{p^\nu \le Y} \frac{b_f(p^\nu)X(p)^\nu}{\nu p^{\nu\sigma}}\Big)\Big)\Big) + O\Big(\mathbb{E}(|L(\sigma, f\times f, X)|^{\mathrm{Re}(z)})\frac{|z|}{Y^{\sigma-1/2}}\Big).$$

Moreover, if u, v are real numbers such that $|u|+|v| \le Y^{\sigma-1/2}$, then

$$\Phi_\sigma^{rand}(u,v) = \mathbb{E}\Big(\exp\Big(iu\,\mathrm{Re}\Big(\sum_{p^\nu \le Y} \frac{b_f(p^\nu)X(p)^\nu}{\nu p^{\nu\sigma}}\Big) + iv\,\mathrm{Im}\Big(\sum_{p^\nu \le Y} \frac{b_f(p^\nu)X(p)^\nu}{\nu p^{\nu\sigma}}\Big)\Big)\Big) + O\Big(\frac{|u|+|v|}{Y^{\sigma-1/2}}\Big).$$

LEMMA 2.7. *Let $2/3 < \sigma < 1$ and $Y = (\log T)^A$ for a fixed $A \ge 1$. For any positive integer $k \le \log T/(20A\log\log z)$, there exist $a_2(\sigma) > 0$ and $a_2'(\sigma) > 0$ such that*

$$\frac{1}{T}\int_T^{2T} \Big|\sum_{p^\nu \le Y} \frac{b_f(p^\nu)}{\nu p^{\nu(\sigma+it)}}\Big|^{2k} dt \ll \Big(\frac{a_2(\sigma)k^{1-\sigma}}{(\log k)^\sigma}\Big)^{2k}$$

and

$$\mathbb{E}\Big(\Big|\sum_{p^\nu \le Y} \frac{b_f(p^\nu)X(p)^\nu}{\nu p^{\nu\sigma}}\Big|^{2k}\Big) \ll \Big(\frac{a_2'(\sigma)k^{1-\sigma}}{(\log k)^\sigma}\Big)^{2k}.$$

Here the implied constants are absolute.

PROOF. By using Lemma 2.4, the lemma follows easily from the method in [8, Lemma 3.3]. □

LEMMA 2.8 [11, Lemma 6]. *Let $2/3 < \sigma < 1$ and $Y = (\log T)^A$ for a fixed $A \ge 1$. For any positive integers u, v such that $u + v \le \log T/(6A\log\log T)$,*

$$\frac{1}{T}\int_T^{2T} \Big(\sum_{p^\nu \le Y} \frac{b_f(p^\nu)}{\nu p^{\nu(\sigma+it)}}\Big)^u \Big(\sum_{p^\nu \le Y} \frac{b_f(p^\nu)}{\nu p^{\nu(\sigma-it)}}\Big)^v dt = \mathbb{E}\Big(\Big(\sum_{p^\nu \le Y} \frac{b_f(p^\nu)X(p)^\nu}{\nu p^{\nu\sigma}}\Big)^u \Big(\sum_{p^\nu \le Y} \frac{\overline{b_f(p^\nu)X(p)^\nu}}{\nu p^{\nu\sigma}}\Big)^v\Big) + O\Big(\frac{Y^{u+v}}{\sqrt{T}}\Big),$$

with an absolute implied constant.

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PROPOSITION 2.9. *Let $2/3 < \sigma < 1$ and $Y = (\log T)^A$ for a fixed $A \geq 1$. For all complex numbers z_1, z_2, there exist positive constants $a_3 = a_3(\sigma, A) > 0$ and $a_4 = a_4(\sigma, A) > 0$ with $|z_1|, |z_2| \leq a_3(\log T)^\sigma$ such that*

$$\frac{1}{T}\int_{\mathcal{A}(T)} \exp\Big(z_1 \sum_{p^\nu \leq Y} \frac{b_f(p^\nu)}{\nu p^{\nu(\sigma+it)}} + z_2 \sum_{p^\nu \leq Y} \frac{b_f(p^\nu)}{\nu p^{\nu(\sigma-it)}}\Big)\,dt$$
$$= \mathbb{E}\Big(\exp\Big(z_1 \sum_{p^\nu \leq Y} \frac{b_f(p^\nu)X(p)^\nu}{\nu p^{\nu\sigma}} + z_2 \sum_{p^\nu \leq Y} \frac{\overline{b_f(p^\nu)X(p)^\nu}}{\nu p^{\nu\sigma}}\Big)\Big) + O\Big(\exp\Big(-a_4 \frac{\log T}{\log\log T}\Big)\Big),$$

with an absolute implied constant. Here, $\mathcal{A}(T)$ is the set of those $t \in [T, 2T]$ such that

$$\Big|\sum_{p^\nu \leq Y} \frac{b_f(p^\nu)}{\nu p^{\nu(\sigma+it)}}\Big| \leq \frac{(\log T)^{1-\sigma}}{\log\log T}.$$

PROOF. The proof is the same as that of [8, Proposition 2.3] by using Lemma 2.7, Proposition 2.5 and Lemma 2.8. □

PROPOSITION 2.10. *Let $2/3 < \sigma_0 < \sigma < 1$ and $A \geq 1$ be fixed. There exists a constant $a_5 = a_5(\sigma, A)$ such that for $|u|, |v| \leq a_5(\log T)^\sigma$,*

$$\Phi_{\sigma,T}(u,v) = \Phi_\sigma^{\mathrm{rand}}(u,v) + O\Big(\frac{1}{(\log T)^A}\Big),$$

with the implied constant depending on σ_0 only.

PROOF. Follow the general idea of the proof of [8, Theorem 2.1]. Let $B = B(A)$ be a large enough constant. Let $Y = (\log T)^{B/(\sigma-2/3)}$. By Lemma 2.3,

$$\log L(s, f \times f) = \sum_{p^\nu \leq Y} \frac{b_f(p^\nu)}{\nu p^{\nu(\sigma+it)}} + O\Big(\frac{1}{(\log T)^{B/2-3}}\Big)$$

for all $t \in [T, 2T]$, except for a set $\mathcal{D}(T)$ of measure $T^{1-d(\sigma)}$ for some constant $d(\sigma) > 0$. Define $C(T) = \{t \in [T, 2T], t \notin \mathcal{D}(T)\}$. Then,

$$\Phi_{\sigma,T}(u,v)$$
$$= \frac{1}{T}\int_{C(T)} \exp(iu \operatorname{Re}\log L(\sigma+it, f\times f) + iv \operatorname{Im}\log L(\sigma+it, f\times f))\,dt + O(T^{-d(\sigma)})$$
$$= \frac{1}{T}\int_{C(T)} \exp\Big(iu \operatorname{Re}\sum_{p^\nu \leq Y} \frac{b_f(p^\nu)}{\nu p^{\nu(\sigma+it)}} + iv \operatorname{Im}\sum_{p^\nu \leq Y} \frac{b_f(p^\nu)}{\nu p^{\nu(\sigma+it)}}\Big)\,dt + O\Big(\frac{1}{(\log T)^{B/2-4}}\Big)$$
$$= \frac{1}{T}\int_{T}^{2T} \exp\Big(iu \operatorname{Re}\sum_{p^\nu \leq Y} \frac{b_f(p^\nu)}{\nu p^{\nu(\sigma+it)}} + iv \operatorname{Im}\sum_{p^\nu \leq Y} \frac{b_f(p^\nu)}{\nu p^{\nu(\sigma+it)}}\Big)\,dt + O\Big(\frac{1}{(\log T)^{B/2-4}}\Big).$$

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Let $\mathcal{A}(T)$ be defined as in Proposition 2.9 and take $z_1 = i(u - iv)/2$ and $z_2 = i(u + iv)/2$ in Proposition 2.9. From Proposition 2.5 and Lemma 2.6, the integral above is

$$= \frac{1}{T}\int_{\mathcal{A}(T)} \exp\Big(iu\,\mathrm{Re}\sum_{p^\nu\le Y}\frac{b_f(p^\nu)}{\nu p^{\nu(\sigma+it)}} + iv\,\mathrm{Im}\sum_{p^\nu\le Y}\frac{b_f(p^\nu)}{\nu p^{\nu(\sigma+it)}}\Big)\,dt + O\Big(\frac{1}{(\log T)^B}\Big)$$
$$= \mathbb{E}\Big(\exp\Big(iu\,\mathrm{Re}\sum_{p^\nu\le Y}\frac{b_f(p^\nu)X(p)^\nu}{\nu p^{\nu(\sigma+it)}} + iv\,\mathrm{Im}\sum_{p^\nu\le Y}\frac{b_f(p^\nu)X(p)^\nu}{\nu p^{\nu(\sigma+it)}}\Big)\Big) + O\Big(\frac{1}{(\log T)^B}\Big)$$
$$= \Phi_\sigma^{\mathrm{rand}}(u,v) + O\Big(\frac{1}{(\log T)^{B-1}}\Big). \qquad \square$$

LEMMA 2.11 [8, Lemma 7.2]. *Let $\lambda > 0$ be a real number. Let $\chi(y) = 1$ if $y > 1$ and 0 otherwise. For any $c > 0$,*

$$\chi(y) \le \frac{1}{2\pi i}\int_{(c)} y^s \frac{e^{\lambda s}-1}{\lambda s}\frac{ds}{s} \quad \text{for } y > 0.$$

We cite the following smooth approximation [8] for the indicator function.

LEMMA 2.12. *Let $\mathcal{R} = \{z = x + iy \in \mathbb{C} : m_1 < x < m_2, n_1 < y < n_2\}$ for real numbers m_1, m_2, n_1, n_2. Let $K > 0$ be a real number. For any $z = x + iy \in \mathbb{C}$, we denote the indicator function of $\mathcal{R}$ by*

$$\mathbf{1}_{\mathcal{R}}(z) = W_{K,\mathcal{R}}(z) + O\Big(\frac{\sin^2(\pi K(x-m_1))}{(\pi K(x-m_1))^2} + \frac{\sin^2(\pi K(x-m_2))}{(\pi K(x-m_2))^2} + \frac{\sin^2(\pi K(y-n_1))}{(\pi K(y-n_1))^2} + \frac{\sin^2(\pi K(y-n_2))}{(\pi K(y-n_2))^2}\Big),$$

where

$$W_{K,\mathcal{R}}(z) = \frac{1}{2}\mathrm{Re}\int_0^K\int_0^K G\Big(\frac{u}{K}\Big)G\Big(\frac{v}{K}\Big)\big(e^{2\pi i(ux-vy)}f_{m_1,m_2}(u)\overline{f_{n_1,n_2}(v)} - e^{2\pi i(ux+vy)}f_{m_1,m_2}(u)f_{n_1,n_2}(v)\big)\,\frac{du}{u}\frac{dv}{v}.$$

Here,

$$G(u) = \frac{2u}{\pi} + 2(1-u)u\cot(\pi u) \quad \text{for } u \in [0,1],$$

and

$$f_{\alpha,\beta}(u) = \frac{e^{-2\pi i\alpha u} - e^{-2\pi i\beta u}}{2} \quad \text{for } \alpha,\beta \in \mathbb{R}.$$

LEMMA 2.13. *Let $2/3 < \sigma < 1$. Let u be a large positive real number. There exist constants $a_6 = a_6(f,\sigma)$ and $a_6' = a_6'(f,\sigma)$ such that*

$$\mathbb{E}(\exp(iu\,\mathrm{Re}\log L(\sigma, f\times f, X))) \ll \exp\Big(-a_6\frac{u^{2/\sigma-2}}{\log u}\Big)$$

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and

$$\mathbb{E}(\exp(iu\,\mathrm{Im}\log L(\sigma, f\times f, X))) \ll \exp\Big(-a_6'\frac{u^{2/\sigma-2}}{\log u}\Big).$$

PROOF. Follow the general idea of the proof of [8, Lemma 6.3]. We denote the Bessel function of order 0 by $J_0(s)$ for all $s\in\mathbb{R}$. Note that for any prime p, $\mathbb{E}(e^{is\mathrm{Re}X(p)}) = \mathbb{E}(e^{is\mathrm{Im}X(p)}) = J_0(s)$. Since $\log(1+t) = t + O(t^2)$ for $|t|<1$,

$$\begin{aligned}
&|\mathbb{E}(\exp(iu\,\mathrm{Re}\log L(\sigma, f\times f, X)))|\\
&\quad= \Big|\mathbb{E}\Big(\exp\Big(iu\,\mathrm{Re}\log\Big(\prod_p\Big(1-\frac{\alpha_f(p)^2X(p)}{p^\sigma}\Big)^{-1}\Big(1-\frac{\beta_f(p)^2X(p)}{p^\sigma}\Big)^{-1}\Big(1-\frac{X(p)}{p^\sigma}\Big)^{-2}\Big)\Big)\Big)\Big|\\
&\quad\le \prod_{p>u^{2/\sigma}}\mathbb{E}\Big(\exp\Big(\frac{iu\lambda_f^2(p)}{p^\sigma}\mathrm{Re}X(p) + O\Big(\frac{u}{p^{2\sigma}}\Big)\Big)\Big) = \exp(O(u^{2/\sigma-3}))\prod_{p>u^{2/\sigma}}\Big|J_0\Big(\frac{u\lambda_f^2(p)}{p^\sigma}\Big)\Big|.
\end{aligned}$$

For $|s|<1$, we have $J_0(s) = 1-(s/2)^2+O(s^4)$. By using (1.2), for some constant $a_6 = a_6(f,\sigma), c>0$, the product above is

$$= \exp\Big\{-\frac{u^2}{4}\sum_{p>u^{2/\sigma}}\Big(\frac{\lambda_f^4(p)}{p^{2\sigma}} + O\Big(\frac{u^2}{p^{4\sigma}}\Big)\Big)\Big\} \le \exp\Big(-a_6\frac{u^{2/\sigma-2}}{\log u}\Big).$$

The second inequality can be derived similarly. □

3. Proof of the main theorem

Let $\mathcal{R}$ be a rectangle with sides parallel to the coordinate axes. Define $\Psi_T(\mathcal{R}) = \mathbb{P}(\log L(\sigma+it, f\times f)\in\mathcal{R})$ and $\Psi(\mathcal{R}) = \mathbb{P}(\log L(\sigma, f\times f, X)\in\mathcal{R})$. Let

$$\widetilde{\mathcal{R}} = \mathcal{R}\cap[-(\log T)^3, (\log T)^3]\times[-(\log T)^3, (\log T)^3].$$

According to Lemma 2.3 and Proposition 2.5, for some constant $a_7>0$,

$$\Psi_T(\mathcal{R}) = \Psi_T(\widetilde{\mathcal{R}}) + O\Big(\exp\Big(-a_7\frac{\log T}{\log\log T}\Big)\Big).$$

Similarly to [12], by using Lemmas 2.6 and 2.11, we can obtain the relationship between $\Psi(\mathcal{R})$ and $\Psi(\widetilde{\mathcal{R}})$: for some constant $a_7'>0$,

$$\Psi(\mathcal{R}) = \Psi(\widetilde{\mathcal{R}}) + O\Big(\exp\Big(-a_7'\frac{\log T}{\log\log T}\Big)\Big).$$

Let $\mathcal{S}$ be the set of rectangles $\mathcal{R}\subset[-(\log T)^3, (\log T)^3]\times[-(\log T)^3, (\log T)^3]$ with sides parallel to the coordinate axes. Then,

$$D_\sigma(T) = \sup_{\mathcal{R}\subset\mathcal{S}}|\Psi_T(\mathcal{R}) - \Psi(\mathcal{R})| + O\Big(\exp\Big(-a_7\frac{\log T}{\log\log T}\Big)\Big).$$

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In light of Lemma 2.12, choose $K = a_8(\log T)^{\sigma}$, for some $a_8 > 0$, and $|m_1|, |m_2|, |n_1|, |n_2| \le (\log T)^3$. Then it follows that

$$\Psi_T(\mathcal{R}) = \frac{1}{T}\int_T^{2T} W_{K,\mathcal{R}}(\log L(\sigma+it, f\times f))\,dt + E_1 \tag{3.1}$$

and, in addition,

$$E_1 \ll I_T(K,m_1) + I_T(K,m_2) + J_T(K,n_1) + J_T(K,n_2),$$

where

$$I_T(K,m) = \frac{1}{T}\int_T^{2T} \frac{\sin^2(\pi K(\operatorname{Re}\log L(\sigma+it, f\times f) - m))}{(\pi K(\operatorname{Re}\log L(\sigma+it, f\times f) - m))^2}\,dt \tag{3.2}$$

and

$$J_T(K,n) = \frac{1}{T}\int_T^{2T} \frac{\sin^2(\pi K(\operatorname{Im}\log L(\sigma+it, f\times f) - n))}{(\pi K(\operatorname{Im}\log L(\sigma+it, f\times f) - n))^2}\,dt.$$

First, we treat the main term of (3.1):

$$\frac{1}{T}\int_T^{2T} W_{K,\mathcal{R}}(\log L(\sigma+it, f\times f))\,dt = \frac{1}{2}\operatorname{Re}\int_0^K\int_0^K G\Big(\frac{u}{K}\Big)G\Big(\frac{v}{K}\Big)$$
$$\times\,(\Phi_{\sigma,T}(2\pi u, -2\pi v) f_{m_1,m_2}(u)\overline{f_{n_1,n_2}(v)} - \Phi_{\sigma,T}(2\pi u, 2\pi v) f_{m_1,m_2}(u) f_{n_1,n_2}(v))\,\frac{du}{u}\,\frac{dv}{v},$$

where $\Phi_{\sigma,T}$ is defined by (2.1). Since $0 \le G(u) \le 2/\pi$ and $|f_{\alpha,\beta}(u)| \le \pi u|\beta-\alpha|$, by Proposition 2.10,

$$\frac{1}{T}\int_T^{2T} W_{K,\mathcal{R}}(\log L(\sigma+it, f\times f))\,dt = \mathbb{E}(W_{K,\mathcal{R}}(\log L(\sigma, f\times f, X))) + O\Big(\frac{1}{(\log T)^2}\Big).$$

Moreover,

$$\Psi(\mathcal{R}) = \mathbb{E}(W_{K,\mathcal{R}}(\log L(\sigma, f\times f, X))\,dt) + E_2.$$

Here,

$$E_2 \ll I_{\text{rand}}(K,m_1) + I_{\text{rand}}(K,m_2) + J_{\text{rand}}(K,n_1) + J_{\text{rand}}(K,n_2),$$

where

$$I_{\text{rand}}(K,m) = \mathbb{E}\Big(\frac{\sin^2(\pi K(\operatorname{Re}\log L(\sigma, f\times f, X) - m))}{(\pi K(\operatorname{Re}\log L(\sigma, f\times f, X) - m))^2}\Big),$$

and

$$J_{\text{rand}}(K,n) = \mathbb{E}\Big(\frac{\sin^2(\pi K(\operatorname{Im}\log L(\sigma, f\times f, X) - n))}{(\pi K(\operatorname{Im}\log L(\sigma, f\times f, X) - n))^2}\Big).$$

Hence,

$$\Psi_T(\mathcal{R}) = \Psi(\mathcal{R}) + E_3, \tag{3.3}$$

where

$$E_3 = E_1 + E_2 + O\Big(\frac{1}{(\log T)^2}\Big).$$

Notice that

$$\frac{\sin^2(\pi K x)}{(\pi K x)^2} = \frac{2(1-\cos(2\pi K x))}{K^2(2\pi x)^2} = \frac{2}{K^2}\int_0^K (K-v)\cos(2\pi x v)\,dv. \tag{3.4}$$

To bound E_1, we use (3.4) to rewrite (3.2):

$$\begin{aligned} I_T(K,m) &= \mathrm{Re}\Big(\frac{1}{T}\int_T^{2T}\frac{2}{K^2}\int_0^K (K-v)\exp(2\pi i v(\mathrm{Re}\log L(\sigma+it,f\times f)-m))\,dv\,dt\Big) \\ &= \mathrm{Re}\frac{2}{K^2}\int_0^K (K-v)e^{-2\pi i v m}\Phi_{\sigma,T}(2\pi v,0)\,dv. \end{aligned}$$

From Proposition 2.10,

$$I_T(K,m) = \mathrm{Re}\frac{2}{K^2}\int_0^K (K-v)e^{-2\pi i v m}\Phi_\sigma^{rand}(2\pi v,0)\,dv + O\Big(\frac{1}{(\log T)^9}\Big),$$

uniformly for all $m \in \mathbb{R}$. Lemma 2.13 implies that

$$I_T(K,m) \ll \frac{1}{K}.$$

The bound $J_T(K,n) \ll 1/K$ can be obtained using the same method. Therefore,

$$E_1 \ll \frac{1}{K}. \tag{3.5}$$

Then, using (2.2), (3.4) and Lemma 2.13,

$$\begin{aligned} I_{\mathrm{rand}}(K,m) &= \mathbb{E}\Big(\frac{2}{K^2}\int_0^K (K-v)\cos(2\pi v[\mathrm{Re}\log L(\sigma,f\times f,X)-m])\Big)\,dv \\ &= \mathrm{Re}\frac{2}{K^2}\int_0^K (K-v)e^{-2\pi i v m}\Phi_\sigma^{\mathrm{rand}}(2\pi v,0)\,dv \ll \frac{1}{K}, \end{aligned}$$

uniformly for all $m \in \mathbb{R}$. Similarly, we can obtain $J_{\mathrm{rand}}(K,n) \ll 1/K$, uniformly for all $n \in \mathbb{R}$. Thus,

$$E_2 \ll \frac{1}{K}. \tag{3.6}$$

Combining the estimates with (3.3), (3.5) and (3.6),

$$D_\sigma(T) \ll \frac{1}{(\log T)^\sigma},$$

which completes the proof.

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