ON THE MAXIMUM PRINCIPLE OF KY FAN

M. MARCUS AND B. N. MOYLS

1. Introduction. In 1951 Fan **(1)** proved the following interesting extreme value result: *Let $A_1, \ldots, A_m$ be completely continuous operators on a Hilbert space $\mathfrak{H}$. For $\sigma = 1, 2, \ldots, m$ let $\lambda_{\sigma i} \geqslant \lambda_{\sigma, i+1}$ be the characteristic roots of $A_\sigma^* A_\sigma$. Then, for any positive integer k,*

$$\max\left|\sum_{i=1}^{k} (U_1 A_1 \ldots U_m A_m x_i, x_i)\right| = \sum_{i=1}^{k}\left(\prod_{\sigma=1}^{m} \lambda_{\sigma i}\right)^{\frac{1}{2}}, \tag{1}$$

$$\max_{i, j=1,2,\ldots,k} |\det\{(U_1 A_1 \ldots U_m A_m x_i, x_j)\}|^2 = \prod_{i=1}^{k}\prod_{\sigma=1}^{m} \lambda_{\sigma i}, \tag{2}$$

where both maxima are taken over all unitary operators $U_1, \ldots, U_m$ and all sets of k orthonormal (o.n.) *vectors.*

Fan proved (1) for $m = 1$ and then applied an inequality of Pólya **(7)** and a recent result of Horn **(3)** to obtain the theorem for $m \geqslant 2$. This result is a generalization of a result of von Neumann **(10)**, which states that *if A and B are n-square complex matrices with singular values* **(12)** $\alpha_i \geqslant \alpha_{i+1}$, $\beta_i \geqslant \beta_{i+1}$ $(i = 1, 2, \ldots, n-1)$, *then*

$$\max|\mathrm{tr}(UAVB)| = \sum_{i=1}^{n} \alpha_i \beta_i, \tag{3}$$

where the maximum is taken over all unitary U and V.

In this paper we shall confine our attention to the case of finite matrices. We shall show that both (1) and (2) are special cases of a general maximum result for compound operators (Theorem 3). As applications we obtain inequalities analogous to those of Horn **(3)** and Ostrowski on the singular values of a product. An inequality of S. N. Roy **(8)** (later published with a different proof by B. Sz. Nagy **(9)**) is a special case of Theorem 3 and Ostrowski's inequalities **(6)** connecting Schur-convex functions of singular values and characteristic roots.

2. Fan's first result. Before proceeding, we point out that for finite matrices (1) follows immediately from (3). Following Fan we need only prove (1) in the case $m = 1$.

Let $x_1, \ldots, x_k$ be an o.n. set of vectors, $k \leqslant n$, and let P denote the orthogonal projection into the subspace spanned by $x_1, \ldots, x_k$. Then

Received May 23, 1956. This research was supported by the U.S. Air Force under contract No. AF 18 (603)–83.

https://doi.org/10.4153/CJM-1957-038-6 Published online by Cambridge University Press

$$
\begin{aligned}
\left|\sum_{i=1}^{k}(U_1A_1x_i,x_i)\right| &= \left|\sum_{i=1}^{k}(U_1A_1x_i,Px_i)\right| \\
&= \left|\sum_{i=1}^{k}(PU_1A_1x_i,x_i)\right| \\
&= |\mathrm{tr}(PU_1A_1)| \\
&\leqslant \max_{U_1,V}|\mathrm{tr}(VPU_1A_1)| \\
&\leqslant \sum_{i=1}^{n}\rho_i\sqrt{\lambda_i},
\end{aligned}
$$

by (3), where the ρ_j are the singular values of P. Since $\rho_j = 1$ for $j = 1, \ldots, k$, and $\rho_j = 0$ for $j \geqslant k+1$, (1) follows for $m = 1$.

3. Properties of the Grassmann algebra. Let $x_i (i=1, \ldots, r)$, $1 \leqslant r \leqslant n$ be vectors in unitary n-space V_n. Then

$$z = x_1 \wedge x_2 \wedge \ldots \wedge x_r \tag{4}$$

denotes the Grassmann exterior product of the x_i; it is a vector in V_m where $m = \binom{n}{r}$. If A is a linear transformation of V_n, the rth induced compound of A is a linear transformation of V_m, defined by

$$C_r(A)\, x_1 \wedge \ldots \wedge x_r = Ax_1 \wedge \ldots \wedge Ax_r. \tag{5}$$

The following properties, which we list for later reference, can be found in **(11**, pp. 63–67**)**:

(i) $(x_1 \wedge \ldots \wedge x_r,\ y_1 \wedge \ldots \wedge y_r) = \det\{(x_i, y_j)\}_{i,j=1,\ldots,r}$.

(ii) $C_r(AB) = C_r(A)\,C_r(B)$, $C_r(A^{-1}) = C_r^{-1}(A)$, $C_r(A^*) = [C_r(A)]^*$, where A^* is the transposed conjugate of A.

(iii) The characteristic roots of $C_r(A)$ are the $\binom{n}{r}$ products of the characteristic roots of A taken r at a time.

(iv) If A has any of the following properties, so does $C_r(A)$: non-singular, normal, Hermitian, non-negative Hermitian (n.n.h.), unitary.

To prune some of the foliage of indices usually necessary in discussing these objects we introduce some notation. The set of $\binom{k}{r}$ distinct sequences of positive integers $i_1, \ldots, i_r$ satisfying $1 \leqslant i_1 < \ldots < i_r \leqslant k$ will be denoted by Q_{kr}, and a typical such sequence will be called ω. If $x_1, \ldots, x_k$ is a set of k vectors in V_n, and ω is the set $\{i_1, \ldots, i_r\}$ in Q_{kr}, we set

$$x_\omega = x_{i_1} \wedge \ldots \wedge x_{i_r}.$$

If (a) is a set of n numbers $a_1, \ldots, a_n$, $E_r(a) = E_r(a_1, \ldots, a_n)$ will denote the rth elementary symmetric function of (a).

4. Results. The following theorem is a generalization of the result in **(5)**:

THEOREM 1. *Let H be* n.n.h. *with characteristic roots $h_j \geqslant h_{j+1}$, $j = 1, \ldots, n-1$. Let $f(t) = t^s$, s real, and*

https://doi.org/10.4153/CJM-1957-038-6 Published online by Cambridge University Press

$$\phi(x_1, \ldots, x_k) = \sum_{\omega \in Q_{kr}} f[(C_r(H)x_\omega, x_\omega)],$$

where $1 \leqslant r \leqslant k \leqslant n$. *Then, if* $s \geqslant 1$,

(6) $$\max \phi = E_r\,[f(h_1), \ldots, f(h_k)];$$

and if $0 \leqslant s \leqslant 1$,

(7) $$\min \phi = E_r\,[f(h_{n-k+1}), \ldots, f(h_n)];$$

where the maximum and minimum are taken over all sets of k o.n. *vectors* $x_1, \ldots, x_k$ *in* V_n.

Proof. For the case $s = 1$, the theorem is proved in **(5)**. The present proof is an application of this result to $f(H)$.

First note that[1] if $||x|| = 1$,

(8) $$f[(Hx, x)] \leqslant (f(H)\,x, x).$$

For, if $x_1, \ldots, x_n$ are o.n. characteristic vectors of H corresponding to $h_1, \ldots, h_n$, then

$$\begin{aligned} f[(Hx, x)] &= f\left[\sum_{j=1}^{n} h_j\,|(x, x_j)|^2\right] \\ &\leqslant \sum_{j=1}^{n} f(h_j)\,|(x, x_j)|^2 \\ &= (f(H)x, x). \end{aligned}$$

Secondly,[2]

(9) $$f(C_r(H)) = C_r(f(H)).$$

For, if U is a unitary matrix such that $U^*HU = D$, diagonal,

$$\begin{aligned} f(C_r(H)) &= f(C_r(UDU^*)) = C_r(U)\,f(C_r(D))\,C_r(U^*) \\ &= C_r(U)\,C_r\,(f(D))\,C_r(U^*) = C_r\,(f(UDU^*)) = C_r(f(H)). \end{aligned}$$

To establish (6) we observe that by (8), (9) and the remark at the beginning of the proof,

$$\begin{aligned} \phi &\leqslant \sum_{\omega \in Q_{kr}} (f(C_r(H))x_\omega, x_\omega) = \sum_{\omega \in Q_{kr}} (C_r(f(H))x_\omega, x_\omega) \\ &\leqslant E_r(f(h_1), \ldots, f(h_k)). \end{aligned}$$

Equality can be achieved by choosing $x_1, \ldots, x_k$ to be an o.n. set of characteristic vectors of H corresponding to $h_1, \ldots, h_k$, since by (i),

$$\begin{aligned} \phi &= \sum_{\omega \in Q_{kr}} f(\det\{(Hx_{i_s}, x_{i_t})\}), \qquad s, t = 1, \ldots, r, \\ &= \sum_{\omega \in Q_{kr}} f(\det\{h_{i_s}\,\delta_{i_s i_t}\}) = \sum_{\omega \in Q_{kr}} f\left(\prod_{s=1}^{r} h_{i_s}\right) \\ &= E_r(f(h_1), \ldots, f(h_k)). \end{aligned}$$

(7) is proved similarly.

[1]This inequality holds for any continuous convex function f defined on the spectrum of H. The inequality is reversed if f is concave.

[2]This equality holds for any function f defined on the spectrum of H and satisfying the relation $f(xy) = f(x)\,f(y)$.

https://doi.org/10.4153/CJM-1957-038-6 Published online by Cambridge University Press

Theorem 3 is our main result. Theorem 2 is a special case. However, it is the first step in the proof of the general theorem, and its proof appears to be of some interest in itself.

THEOREM 2. *Let A be an arbitrary n-square complex matrix with singular values $\alpha_i \geqslant \alpha_{i+1} (i = 1, \ldots, n-1)$. Let*

$$\phi(x_1, \ldots, x_k; U) = \left| \sum_{\omega \in Q_{kr}} (C_r(UA)x_\omega, x_\omega) \right| ,$$

where $1 \leqslant r \leqslant k \leqslant n$. Then

$$\max \phi = E_r(\alpha_1, \ldots, \alpha_k), \tag{10}$$

where the maximum is taken over all sets of k o.n. vectors $x_1, \ldots, x_k$ and all unitary U.

Proof. Since $A = VH$, where V is unitary and $H = (A^*A)^{\frac{1}{2}}$ is n.n.h., we may without loss in generality replace A by H. Set $y_j = U^{-1}x_j$, $(j = 1, \ldots, k)$. By the notation $\sum \oplus C_r(A)$ we mean the direct sum of $C_r(A)$ taken $\binom{k}{r}$ times; and similarly for $\sum \oplus x_\omega$. Since $\sum \oplus C_r(H)$ is n.n.h. by (iv),

$$\begin{aligned}
\phi &= \left| \sum_{\omega \in Q_{kr}} (\cup_r(H)x_\omega, C_r(\cup^{-1})x_\omega) \right| \\
&= \left| \left(\sum \oplus C_r(H) \sum \oplus x_\omega, \sum \oplus y_\omega\right) \right| \\
&\leqslant \left(\sum \oplus C_r(H) \sum \oplus x_\omega, \sum \oplus x_\omega\right)^{\frac{1}{2}} \\
&\qquad\qquad \left(\sum \oplus C_r(H) \sum \oplus y_\omega, \sum \oplus y_\omega\right)^{\frac{1}{2}} \\
&= \left[\sum_{\omega \in Q_{kr}} (C_r(H)x_\omega, x_\omega) \right]^{\frac{1}{2}} \left[\sum_{\omega \in Q_{kr}} (C_r(H)y_\omega, y_\omega) \right]^{\frac{1}{2}} \\
&\leqslant E_r(\alpha_1, \ldots, \alpha_k).
\end{aligned}$$

The last inequality follows from (6) applied to each of the square roots. Equality holds when the x_j are suitable eigenvectors of H and U is the identity matrix.

THEOREM 3. *Let $A_1, \ldots, A_m$ be arbitrary n-square complex matrices with singular values $\alpha_{\sigma i}$, where $\alpha_{\sigma i} \geqslant \alpha_{\sigma, i+1}$ $(i = 1, \ldots, n-1;\ \sigma = 1, \ldots, m)$. Let*

$$\phi(x_1, \ldots, x_k; U_1, \ldots, U_m) = \left| \sum_{\omega \in Q_{kr}} (C_r(U_1A_1 \ldots U_mA_m)x_\omega, x_\omega) \right|,$$

where $1 \leqslant r \leqslant k \leqslant n$. Then

$$\max \phi = E_r\left(\prod_{\sigma=1}^{m} \alpha_{\sigma 1}, \ldots, \prod_{\sigma=1}^{m} \alpha_{\sigma k} \right), \tag{11}$$

where the maximum is taken over all sets of k o.n. vectors $x_1, \ldots, x_k$, and all unitary matrices $U_1, \ldots, U_m$.

https://doi.org/10.4153/CJM-1957-038-6 Published online by Cambridge University Press

Proof. For a fixed $U_2, \ldots, U_m$, let $\beta_1 \geqslant \beta_2 \geqslant \ldots \geqslant \beta_n$ be the singular values of $B = A_1(U_2A_2) \ldots (U_mA_m)$. By Theorem 2

$$\phi \leqslant E_r\,(\beta_1, \ldots, \beta_k). \tag{12}$$

The singular values of $U_\sigma A_\sigma$ are the $\alpha_{\sigma i}$ $(i = 1, \ldots, n)$. By a result of Ostrowski (**6**, p. 283, equation 106a),

$$E_r(\beta_1, \ldots, \beta_k) \leqslant E_r\left(\prod_{\sigma=1}^{m} \alpha_{\sigma 1}, \ldots, \prod_{\sigma=1}^{m} \alpha_{\sigma k}\right).$$

There remains to show that the right side of (11) is taken on for a suitable choice of the x's and U's. As in Theorem 2 we may replace each A_σ by $H_\sigma = (A_\sigma^* A_\sigma)^{\frac{1}{2}}$. Let $x_{\sigma 1}, \ldots, x_{\sigma n}$ $(\sigma = 1, \ldots, m)$ be an o.n. set of characteristic vectors of H_σ corresponding to $\alpha_{\sigma 1}, \ldots, \alpha_{\sigma n}$. Choose the U_σ so that

$$U_\sigma\, x_{\sigma i} = x_{\sigma-1,i}, \qquad \sigma = 2, \ldots, m,$$
$$U_1\, x_{1i} = x_{mi},$$

for $i = 1, \ldots, n$. Set

$$x_{\sigma\omega} = x_{\sigma i_1} \wedge \ldots \wedge x_{\sigma i_r}$$

and

$$\alpha_{\sigma\omega} = \prod_{j=1}^{r} \alpha_{\sigma i_j}$$

where ω is the set $\{i_1, \ldots, i_r\}$. Then

$$\begin{aligned}
&\left|\sum_{\omega \in Q_{kr}} (C_r(U_1H_1 \ldots U_mH_m)\, x_{m\omega}, x_{m\omega})\right| \\
&= \left|\sum \alpha_{m\omega}\, C_r(U_1H_1 \ldots U_{m-1}H_{m-1})\, x_{m-1,\omega}, x_{m\omega})\right| \\
&= \sum \alpha_{m\omega} \ldots \alpha_{1\omega}(x_{m\omega}, x_{m\omega}) \\
&= \sum \left(\prod_{\sigma=1}^{m} \alpha_{\sigma i_1} \ldots \prod_{\sigma=1}^{m} \alpha_{\sigma i_r}\right) \\
&= E_r\left(\prod_{\sigma=1}^{m} \alpha_{\sigma 1}, \ldots, \prod_{\sigma=1}^{m} \alpha_{\sigma k}\right).
\end{aligned}$$

This completes the proof of the theorem.

Remarks. I. By setting $r = 1$ in (11) we obtain (1); by setting $r = k$ we obtain (2).

II. Theorem 3 does not follow immediately from (1) applied to the compound because the lexicographic ordering of the eigenvalues does not necessarily correspond to the ordering by magnitude.

III. Let A_σ, $\sigma = 1, \ldots, m$, have eigenvalues $\lambda_{\sigma i}$, ordered so that $|\lambda_{\sigma i}| \geqslant |\lambda_{\sigma, i+1}|$ $(i = 1, \ldots, n-1)$. We can find unitary V_σ such that $V_\sigma^* A_\sigma V_\sigma = T_\sigma$, triangular, with the diagonal elements ordered by absolute magnitude. Then if $U_\sigma = V_{\sigma-1} V_\sigma^*$ $(\sigma = 2, \ldots, m)$, $U_1 = V_m U V_1^*$ where U is the diagonal matrix with

$$\exp\left(-\sum_{\sigma=1}^{m} \arg \lambda_{\sigma i}\right)$$

in the ith row and ith column; and if $x_j = V_m e_j$ $(j = 1, \ldots, k)$, where e_j is the unit vector with 1 in the ith position; then

$$\left|\sum_{\omega \in Q_{kr}} (C_r(U_1 A_1 \ldots U_m A_m) x_\omega, x_\omega)\right| = E_r\left(\prod_{\sigma=1}^{m} |\lambda_{\sigma 1}|, \ldots, \prod_{\sigma=1}^{m} |\lambda_{\sigma k}|\right).$$

It follows from Theorem 3 that

$$E_r\left(\prod_{\sigma=1}^{m} |\lambda_{\sigma 1}|, \ldots, \prod_{\sigma=1}^{m} |\lambda_{\sigma k}|\right) \leqslant E_r\left(\prod_{\sigma=1}^{m} \alpha_{\sigma 1}, \ldots, \prod_{\sigma=1}^{m} \alpha_{\sigma k}\right). \tag{13}$$

The case $m = 1$, $r = k$ is Weyl's inequality. Actually, (13) also follows easily from Ostrowski's discussion of Schur-convex functions.

5. Applications. Let $X = AB$, where A and B are arbitrary n-square complex matrices. Consider the convex sum

$$Y = \sigma X^* X + \delta X X^*,$$

where $\sigma + \delta = 1$, $\sigma \geqslant 0$, $\delta \geqslant 0$. Let $\phi_i \geqslant \phi_{i+1}$, $\alpha_i \geqslant \alpha_{i+1}$, $\beta_i \geqslant \beta_{i+1}$ $(i = 1, \ldots, n-1)$ be respectively the non-negative square roots of the characteristic roots of Y and the singular values of A and B. In the Theorem 4 we shall use the following concavity property of the elementary symmetric functions:[3]

If (a) *and* (b) *are sets of* n *non-negative numbers and* $1 \leqslant r \leqslant n$, *then*

$$E_r^{1/r}(a + b) \geqslant E_r^{1/r}(a) + E_r^{1/r}(b).$$

THEOREM 4. *For* $0 \leqslant s \leqslant 1$ *and* $1 \leqslant r \leqslant k \leqslant n$,

$$E_r(\phi_n^{2s}, \ldots, \phi_{n-k+1}^{2s})$$

$$\geqslant (2\sigma^\sigma \delta^\delta)^{r(s-1)} \left(\prod_{j=1}^{r} \alpha_{n-j+1}^{\sigma} \beta_{n-j+1}^{\delta}\right)^{2s} E_r^{\sigma}(\beta_n^{2s}, \ldots, \beta_{n-k+1}^{2s})\, E_r^{\delta}(\alpha_n^{2s}, \ldots, \alpha_{n-k+1}^{2s}).$$

Proof. Let $x_1, \ldots, x_k$ be an o.n. set of characteristic vectors of Y corresponding respectively to

$$\phi_n^2, \ldots, \phi_{n-k+1}^2.$$

Then

[3]This property follows from a concavity property proved by W. Fenchel **(2)**. This inequality also follows directly as a corollary of a similar property for symmetric functions proved by Lopes and Marcus in **(4)**.

$$
\begin{aligned}
E_r^{1/r}(\phi_n^{2s}, \ldots, \phi_{n-k+1}^{2s}) &= E_r^{1/r}[(Yx_j, x_j)^s] \\
&= E_r^{1/r}[\{\sigma(X^*X\, x_j, x_j) + \delta(XX^* x_j, x_j)\}^s] \\
&\geqslant E_r^{1/r}[2^{s-1}\{\sigma^s(X^*X\, x_j, x_j)^s + \delta^s(XX^*\, x_j, x_j)^s\}] \\
&\geqslant 2^{s-1}\{E_r^{1/r}[\sigma^s(X^*X\, x_j, x_j)^s] + E_r^{1/r}[\delta^s(XX^* x_j, x_j)^s]\} \\
&= 2^{s-1}\{\sigma^s\, E_r^{1/r}[(X^*X\, x_j, x_j)^s] + \delta^s\, E_r^{1/r}[(XX^*\, x_j, x_j)^s]\}.
\end{aligned}
$$

It follows from the Hadamard determinant theorem that

$$
\begin{aligned}
E_r[(X^*X\, x_j, x_j)^s] &= \sum_{1\leqslant i_1<\ldots<i_r\leqslant k} \left(\prod_{t=1}^{r} (X^*X\, x_{i_t}, x_{i_t})\right)^s \\
&\geqslant \sum_{\omega\in Q_{kr}} (C_r(X^*X)\, x_\omega, x_\omega)^s.
\end{aligned}
$$

Thus

$$
\begin{aligned}
E_r^{1/r}(\phi_n^{2s}, \ldots, \phi_{n-k+1}^{2s}) &\geqslant 2^{s-1}\, \sigma^s \left\{ \sum (C_r(X^*X)\, x_\omega, x_\omega)^s \right\}^{1/r} \\
&\qquad + 2^{s-1}\, \delta^s \left\{ \sum (C_r(XX^*)\, x_\omega, x_\omega)^s \right\}^{1/r} \\
&= 2^{s-1}\, \sigma^s \left\{ \sum (C_r(A^*A)\, C_r(B)\, x_\omega, C_r(B)\, x_\omega)^s \right\}^{1/r} \\
&\quad + 2^{s-1}\, \delta^s \left\{ \sum (C_r(BB^*)\, C_r(A^*)\, x_\omega, C_r(A^*)\, x_\omega)^s \right\}^{1/r} \\
&\geqslant 2^{s-1}\, \sigma^s \left(\prod_{j=1}^{r} \alpha_{n-j+1}^s \right)^{2/r} \left\{ \sum (C_r(B^*B)\, x_\omega, x_\omega)^s \right\}^{1/r} \\
&\quad + 2^{s-1}\, \delta^s \left(\prod_{i=1}^{r} \beta_{n-j+1}^s \right)^{2/r} \left\{ \sum (C_r(AA^*)\, x_\omega, x_\omega)^s \right\}^{1/r} \\
&\geqslant \sigma \left\{ 2^{s-1}\, \sigma^{s-1} \left(\prod_{j=1}^{r} \alpha_{n-j+1}^s \right)^{2/r} E_r^{1/r}(\beta_n^{2s}, \ldots, \beta_{n-k+1}^{2s}) \right\} \\
&\quad + \delta \left\{ 2^{s-1}\, \delta^{s-1} \left(\prod_{j=1}^{r} \beta_{n-j+1}^s \right)^{2/r} E_r^{1/r}(\alpha_n^{2s}, \ldots, \alpha_{n-k+1}^{2s}) \right\}.
\end{aligned}
$$

The last inequality follows by Theorem 1. The result follows by a classical inequality and by taking rth powers of both sides.

We remark that when σ is zero, σ^σ is to be taken as 1.

Theorem 4 may be used to relate the characteristic polynomials of the matrices involved. For example,

$$
|p_r((ABB^*A^*)^s)| \geqslant 2^{r(s-1)} \prod_{j=1}^{r} \alpha_{n-j+1}^{2s}\, |p_r((B^*B)^s)|,
$$

where

$$
x^n + \sum_{r=1}^{n} p_r(M)\, x^{n-r}
$$

is the characteristic polynomial of the matrix M.

In **(9)** B. Sz. Nagy proves that if A and B are complex n-square matrices, then

$$\alpha_n \beta_n \leqslant |\lambda_i| \leqslant \alpha_1 \beta_1, \tag{14}$$

where $\alpha_i \geqslant \alpha_{i+1}$, $\beta_i \geqslant \beta_{i+1}$ are the singular values of A and B, and λ_i, $|\lambda_i| \geqslant |\lambda_{i+1}|$, are the characteristic roots of AB. By Theorem 3,

$$E_r(|\lambda_1|, \ldots, |\lambda_k|) \leqslant E_r(\alpha_1 \beta_1, \ldots, \alpha_k \beta_k), \tag{15}$$

for $1 \leqslant r \leqslant k \leqslant n$. By setting $r = k = 1$, we obtain the upper inequality of (14). If $\alpha_n \beta_n = 0$, the lower inequality is trivial. Otherwise A and B are non-singular. The characteristic roots of $(AB)^{-1}$ are the λ_i^{-1}, and the singular values of A^{-1} and B^{-1} are the α_i^{-1} and β_i^{-1}. Hence

$$E_r(|\lambda_n|^{-1}, \ldots, |\lambda_{n-k+1}|^{-1}) \leqslant E_r(\alpha_n^{-1} \beta_n^{-1}, \ldots, \alpha_{n-k+1}^{-1} \beta_{n-k+1}^{-1}). \tag{16}$$

Again setting $r = k = 1$, we obtain the lower inequality of (14).

Both (15) and (16) have immediate generalizations in two directions: first, to a product of more than two matrices, and second, to the more general class of Schur-convex functions.

References

1. K. Fan, *Maximum properties and inequalities for the eigenvalues of completely continuous operators*, Proc. N.A.S. (U.S.A.), *37* (1951), 760–766.

2. W. Fenchel, *Généralisation du théorème de Brunn-Minkowski concernant les corps convexes*, C.R. des Sci. de l'Acad. des Sci., *203* (1936), 764–766.

3. A. Horn, *On the singular values of a product of completely continuous operators*, Proc. N.A.S. (U.S.A.), *36* (1950), 374–375.

4. M. D. Marcus and L. Lopes, *Inequalities for symmetric functions and Hermitian matrices*, Can. J. Math., *9* (1957), 305–312.

5. M. D. Marcus and J. L. McGregor, *Extremal properties of Hermitian matrices*, Can. J. Math., *8* (1956), 524–531.

6. A. Ostrowski, *Sur quelques applications des fonctions convexes et concaves au sens de I. Schur*, J. Math. pures et appl. (9), *31* (1952), 253–292.

7. G. Pólya, *Remark on Weyl's note: Inequalities between the two kinds of eigenvalues of a linear transformation*, Proc. N.A.S. (U.S.A.), *36* (1950), 49–51.

8. S. N. Roy, *A useful theorem in matrix theory*, Proc. Amer. Math. Soc., *5* (1954), 635–638.

9. B. Sz. Nagy, *Remark on S. N. Roy's paper: A useful theorem in matrix theory*, Proc. Amer. Math. Soc., *7* (1956), 1.

10. J. von Neumann, *Some matrix-inequalities and metrization of matrix-space*, Tomsk Univ. Rev., *1* (1937), 286–300.

11. J. H. M. Wedderburn, *Lectures on Matrices*, Amer. Math. Soc. Colloq. Publications, *17* (1934).

12. H. Weyl, *Inequalities between the two kinds of eigenvalues of a linear transformation*, Proc. N.A.S. (U.S.A.), *35* (1949), 408–411.

University of British Columbia