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Bilinear Kloosterman sums in function fields and the distribution of irreducible polynomials

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Abstract. Inspired by the work of Bourgain and Garaev (2013), we provide new bounds for certain weighted bilinear Kloosterman sums in polynomial rings over a finite field. As an application, we build upon and extend some results of Sawin and Shusterman (2022). These results include bounds for exponential sums weighted by the Möbius function and a level of distribution for irreducible polynomials beyond 1/2, with arbitrary composite modulus. Additionally, we can do better when averaging over the modulus, to give an analogue of the Bombieri-Vinogradov Theorem with a level of distribution even further beyond 1/2.

1 Introduction

1.1 Background

Motivated by a range of applications, in recent years, there has been notable effort dedicated to studying certain bilinear forms of Kloosterman sums. One important example are those of the form

$$\sum_{\substack{1\le x_1<N_1 \\ (x_1,m)=1}} \sum_{\substack{1\le x_2<N_2 \\ (x_2,m)=1}} \alpha_{x_1}\beta_{x_2} e_m(a\overline{x_1}\,\overline{x_2}) \tag{1.1}$$

for $a, m \in \mathbb{Z}$ and complex weighs $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, where $e_m(x) = \exp(2\pi i x/m)$ and where $\overline{x}$ denotes the inverse of x modulo m. Perhaps the most well-known application of bounds for (1.1) has been to estimate exponential sums over primes

$$\sum_{\substack{1\le x<N \\ (x,m)=1}} \Lambda(x) e_m(a\overline{x}) \tag{1.2}$$

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as in [5, 13, 9, 15, 12, 18, 21], where Λ denotes the von Mangoldt function over $\mathbb{Z}$ (although later, by abuse of notation, this will denote the von Mangoldt function in a different setting). Bounds for (1.1) have also found applications to the Brun-Titchmarsh theorem [14, 9, 10] and the distribution of fractional parts of fractions with modular inverses [20]. Higher dimensional analogues were also considered in [22, 28].

In their recent groundbreaking work [25], Sawin and Shusterman consider analogues of (1.1) and (1.2) in polynomial rings over finite fields. They establish highly nontrivial bounds and apply them to a number of cornerstone problems regarding irreducible polynomials. First, they establish a level of distribution beyond 1/2 for irreducible polynomials to square-free modulus (for details, see the discussion in Section 2.2). We note that even under the assumption of the Generalized Riemann Hypothesis, this is not known over the integers but is implied by the famous Elliot-Halberstam conjecture. Furthermore, Sawin and Shusterman establish a strong and explicit form of the twin prime conjecture in that setting.

Motivated by these applications, here we also consider (1.1) and (1.2) in function fields but focus on working with arbitrary composite modulus. This includes improving some bounds from [25] on sums of the form (1.1) and (1.2). Additionally, we extend their results regarding the level of distribution of irreducible polynomials, from square-free to arbitrary modulus. Furthermore, we establish a function field version of the Bombieri-Vinogradov Theorem with a level of distribution even further beyond 1/2.

1.2 General notation

We fix an odd prime power $q = p^\ell$ and let $\mathbb{F}_q$ denote the finite field of order q. Let $\mathbb{F}_q[T]$ denote the ring of univariate polynomials with coefficients from $\mathbb{F}_q$. Throughout, $F \in \mathbb{F}_q[T]$ will always denote an arbitrary polynomial of degree r.

Next, we denote by $\mathbb{F}_q(T)_\infty$ the field of Laurent series in $1/T$ over $\mathbb{F}_q$. That is,

$$\mathbb{F}_q(T)_\infty = \left\{ \sum_{i=-\infty}^{n} a_i T^i \ : \ n \in \mathbb{Z},\ a_i \in \mathbb{F}_q,\ a_n \neq 0 \right\}.$$

We note that, of course, $\mathbb{F}_q[T] \subseteq \mathbb{F}_q(T)_\infty$. On $\mathbb{F}_q(T)_\infty$, we have a nontrivial additive character

$$e\left(\sum_{i=-\infty}^{n} a_i T^i \right) = \exp\left(\frac{2\pi i}{p} \mathrm{Tr}(a_{-1}) \right),$$

where $\mathrm{Tr} : \mathbb{F}_q \to \mathbb{F}_p$ is the absolute trace. Further, for any $F \in \mathbb{F}_q[T]$, we note that

$$e_F(x) = e(x/F)$$

defines a nontrivial additive character of $\mathbb{F}_q[T]/\langle F(T)\rangle$. See [17] for additional details.

We will let $\mathcal{M}$ and $\mathcal{P}$ be the set of all monic and all monic irreducible polynomials, respectively. For a positive integer n, we will let $\mathcal{M}_n$ be the set of monic polynomials of degree n.

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We can also define an analogue of the Möbius function in $\mathbb{F}_q[T]$, as

$$\mu(x) = \begin{cases} 0, & x \text{ is not square-free} \\ (-1)^k, & x = hP_1^{e_1}\cdots P_k^{e_k} \text{ for some positive integers } e_1, \dots, e_k, \\ & \text{some } h \in \mathbb{F}_q\backslash\{0\} \text{ and some distinct } P_i \in \mathcal{P}. \end{cases}$$

Similarly, we can define the von Mangoldt function

$$\Lambda(x) = \begin{cases} \deg P, & x = hP^k \text{ for some } P \in \mathcal{P}, \text{ positive integer } k \\ & \text{and } h \in \mathbb{F}_q\backslash\{0\}, \\ 0, & \text{otherwise.} \end{cases}$$

Finally, given some $x \in \mathbb{F}_q[T]$, $\bar{x}$ will denote the inverse of x modulo F (unless it is specified that the inverse should be taken to a different modulus). Also, ε will denote some small constant (unless otherwise specified).

2 Results

2.1 Bilinear Kloosterman sums

Given positive integers m and n, sequences of complex weights

$$\boldsymbol{\alpha} = (\alpha_{x_1})_{\deg x_1 < m},\ \boldsymbol{\beta} = (\boldsymbol{\beta}_{x_2})_{\deg x_2 < n} \text{ with } \|\boldsymbol{\alpha}\|_\infty, \|\boldsymbol{\beta}\|_\infty < q^{o(r)}, \tag{2.1}$$

and $a \in \mathbb{F}_q[T]$, we define the bilinear Kloosterman sum

$$\mathcal{W}_{F,a}(m, n; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_{\substack{\deg x_1 < m \\ (x_1,F)=1}} \sum_{\substack{\deg x_2 < n \\ (x_2,F)=1}} \alpha_{x_1}\beta_{x_2} e_F(a\overline{x_1 x_2}).$$

We will be interested in improving upon the trivial bound $q^{m+n+\varepsilon r}$. As mentioned previously, bounds on sums of this form are used as tools to establish some of the main results in [25]. Here, we take a different approach to bounding these sums which can hold for arbitrary F, based on the ideas of Bourgain and Garaev [9], Garaev [15], Fouvry and Shparlinski [13], Banks, Harcharras, and Shparlinski [6] and Irving [18].

More flexible bounds, given explicitly in terms of additive energies of modular inversions, are stated in Section 4. These would imply function field analogues of most of the bounds in [9]. But the following will be the most useful for our purposes.

Theorem 2.1 *Let $\varepsilon > 0$, and let $a, F \in \mathbb{F}_q[T]$ be coprime with $\deg F = r$. Then for any positive integers n and m satisfying*

$$n \geq r\varepsilon \text{ and } m \geq r(1/4 + \varepsilon)$$

and weights as in (2.1), we have

$$\mathcal{W}_{F,a}(m, n; \boldsymbol{\alpha}, \boldsymbol{\beta}) \ll_\varepsilon q^{m+n-r\delta}$$

for some $\delta = \delta(\varepsilon) > 0$.

The proof of this result could be carried out in the integer setting and would give a direct improvement on [10, Theorem 7]. Although, our approach is modeled heavily after [10] and additionally incorporates ideas from [21].

2.2 Kloosterman sums with the Möbius function

As in [26, 25], we next consider sums of the form

$$\sum_{\substack{\deg x<n \\ (x,F)=1}} \mu(x)e_F(a\bar{x}) \tag{2.2}$$

and seek improvement over the trivial bound q^n, for n as small as possible in comparison to r. We note that we are working with the Möbius function as opposed to the Von Mangoldt function as in (1.2), but the similarity between Vaughan's identity [19, Propositions 13.4 and 13.5] for μ and Λ allows for both of these to be treated very similarly.

Analogous results dealing with sums as in (2.2) over the integers always require $\gcd(a,F)=1$. But because the analogue of the Generalized Riemann Hypothesis (GRH) holds in $\mathbb{F}_q[T]$, we can drop this condition (with some additional analytic effort).

A special case of [26, Theorem 1.13] is the following: let $\varepsilon>0$ and suppose F is irreducible. If

$$q > 4e^2\left(1+\frac{3}{2p}\right)^{p/\varepsilon} p^2, \tag{2.3}$$

then, for $n > r\varepsilon$, we have

$$\sum_{\substack{\deg x<n \\ (x,F)=1}} \mu(x)e_F(a\bar{x}) \ll_\varepsilon q^{n(1-\delta)}$$

for some $\delta=\delta(\varepsilon)>0$. In summary, this implies that for any $\varepsilon>0$, one obtains a power savings over the trivial bound for any $n>r\varepsilon$ (for sufficiently large q in terms of p and ε). This achievement of a power savings in arbitrarily small intervals far surpasses any previous work in this area.

Here, we consider what can be said without these restrictions on q, and for arbitrary composite modulus F. Using Theorem 2.1 together with classical ideas regarding Vaughan's identity, we show the following. This is analogous to [8, Theorem A.9], which holds for prime modulus.

Theorem 2.2 *Let $\varepsilon>0$ and $a,F\in\mathbb{F}_q[T]$ with $\deg F=r$. For any positive integer n satisfying $n>r(1/2+\varepsilon)$,*

$$\sum_{\substack{\deg x<n \\ (x,F)=1}} \mu(x)e_F(a\bar{x}) \ll_\varepsilon q^{n(1-\delta)}$$

for some $\delta=\delta(\varepsilon)>0$.

For a comparison with [26, Theorem 1.13], the most important point is that this result holds for arbitrary modulus F as opposed to only irreducible modulus. But

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this also does not require the restriction on q as in (2.3). Thus, Theorem 2.2 gives an improvement for irreducible modulus when $q = p^\ell$ and

$$\begin{aligned}
&p \in \{3\} \text{ and } \ell < 8,\\
&p \in \{5,7\} \text{ and } \ell < 6,\\
&p \in \{11,\dots,23\} \text{ and } \ell < 5,\\
&p \in \{29,\dots,587\} \text{ and } \ell < 4,\\
&p \in \{593,\dots\} \text{ and } \ell < 3.
\end{aligned}$$

Another important avenue to pursue with regard to these sums is obtaining more explicit (and larger) savings over the trivial bound. In [25], these are required for applications. For square-free modulus F, [25, Theorem 1.8] demonstrates

$$\sum_{\substack{\deg x<n\\(x,F)=1}} \mu(x)e_F(a\bar{x}) \ll_\varepsilon q^{3r/16+25n/32+\varepsilon n}, \tag{2.4}$$

which is nontrivial when $n > 6r/7$.

This can be improved and again can be extended to arbitrary modulus. This is analogous to the main result in [15] which holds for prime modulus, but we can do better in $\mathbb{F}_q[T]$ and extend to arbitrary modulus. The proof also makes use of some ideas of Fouvry and Shparlinski [13],

Theorem 2.3 *Let $a, F \in \mathbb{F}_q[T]$ with $\deg F = r$ and let n denote a positive integer. Then for any $\varepsilon > 0$,*

$$\sum_{\substack{\deg x<n\\(x,F)=1}} \mu(x)e_F(a\bar{x}) \ll_\varepsilon q^{15n/16+\varepsilon n} + q^{2n/3+r/4+\varepsilon n}.$$

This is nontrivial when $n > 3r/4$, and gives a savings of $q^{r/16}$ over the trivial bound when $n \approx r$. Also, this always improves on (2.4).

For applications, we will also make use of the following variant of a result of Irving [18] which gives an improvement on average over the modulus.

Theorem 2.4 *For any positive integers n and r and any $\varepsilon > 0$,*

$$\sum_{\deg F=r} \max_{a\in\mathbb{F}_q[T]} \left| \sum_{\substack{\deg x<n\\(x,F)=1}} \mu(x)e_F(a\bar{x}) \right| \ll_\varepsilon q^{r+n\varepsilon}\left(q^{9n/10} + q^{r/6+13n/18} + q^{13n/8-5r/6}\right).$$

This is again nontrivial when $n > 3r/4$, with a savings of $q^{r/10}$ over the trivial bound when $n \approx r$.

2.3 Level of distribution of irreducible polynomials

The main application in [25] of the sums considered in the previous section is to obtain a level of distribution beyond 1/2 for irreducible polynomials in aritheoremetic

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progressions. In particular, that means nontrivial bounds for

$$\left| \sum_{\substack{x\in\mathcal{M}_n \\ x\equiv a \pmod F}} \Lambda(x) - \frac{q^n}{\phi(F)} \right|$$

when $n > r/2$. The start of [19, section 17.1] gives a good background on this problem over $\mathbb{Z}$, but in summary, it is a classical problem in number theory to show

$$\sum_{\substack{x\in\mathcal{M}_n \\ x\equiv a \pmod F}} \Lambda(x) \sim \frac{q^n}{\phi(F)} \text{ uniformly for } r \le n\omega \tag{2.5}$$

for $\omega < 1$ as large as possible. The strongest analogous results over $\mathbb{Z}$ only imply that under the assumption of GRH, (2.5) holds for $\omega < 1/2$, although it is conjectured that this should hold for any $\omega < 1$; again, see [19, section 17.1].

In $\mathbb{F}_q[T]$, Sawin and Shusterman [25, Theorem 1.9] move beyond this barrier of $1/2$ for square-free modulus F by showing (for sufficiently large but fixed q in terms of ω and p) that

$$\text{(2.5) holds for any } \omega < 1/2 + 1/126 \text{ and square-free } F. \tag{2.6}$$

Sawin subsequently gives another ground-breaking improvement in [24, Theorem 1.2] to achieve the conjectured value of ω for square-free modulus, by showing (for sufficiently large but fixed q in terms of only ω) that

$$\text{(2.5) holds for any } \omega < 1 \text{ and square-free } F. \tag{2.7}$$

Again, one may ask whether we can move past the barrier of $\omega < 1/2$ for arbitrary modulus. The methods used to show (2.7) are very specialized to square-free modulus, and it is probably infeasible to make these work more generally. But, by inserting our Theorem 2.3 into the proof of (2.6), we have the following.

Theorem 2.5 *Fix $\omega < 1/2 + 1/62$, and suppose*

$$q > p^2 e^2 \left(\frac{16-\omega}{16-31\omega} \right)^2 .$$

Then for any coprime $a, F \in \mathbb{F}_q[T]$ with $\deg F = r$, and any positive integer n satisfying $r \le \omega n$, we have

$$\sum_{\substack{x\in\mathcal{M}_n \\ x\equiv a \pmod F}} \Lambda(x) - \frac{q^n}{\phi(F)} \ll_\omega q^{n-r(1+\delta)}$$

for some $\delta = \delta(\omega) > 0$.

While this holds for arbitrary modulus F, we do note that for square-free modulus, Sawin's result [24, Theorem 1.2] always gives a more relaxed condition on q.

We can also use Theorem 2.4 to do better on average – that is, when considering an analogue of the Bombieri-Vinogradov Theorem.

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Theorem 2.6 *Fix $\omega < 1/2 + 1/38$, and suppose*

$$q > p^2 e^2 \left(\frac{10 - \omega}{10 - 19\omega} \right)^2 .$$

Then for any positive integers R and n satisfying $R \le \omega n$, we have

$$\sum_{\deg F < R} \max_{(a,F)=1} \left| \sum_{\substack{x \in \mathcal{M}_n \\ x \equiv a \pmod F}} \Lambda(x) - \frac{q^n}{\phi(F)} \right| \ll_\omega q^{n - R\delta}$$

for some $\delta = \delta(\omega) > 0$.

We note that it may be possible to adapt the ideas from Sawin in [24] to either directly improve upon Theorem 2.6 or to obtain a result similar to (2.7) for moduli whose square-full part has low degree, which in turn could improve upon Theorem 2.6.

3 Preliminaries

Throughout this section, F always denotes an arbitrary polynomial of degree r, and $\varepsilon > 0$ is always some small positive constant.

As a general preliminary, we will repeatedly make use of the following from [11, Lemma 1].

Lemma 3.1 *The number of divisors of any $x \in \mathbb{F}_q[T]$ is $O_\varepsilon(q^{\varepsilon \deg x})$.*

3.1 Sums involving the Möbius function

We will need a number of results regarding cancellations in sums of the Möbius function. First, we recall the following elementary result from [23, Chapter 2, Ex 12].

Lemma 3.2 *For any positive integer n,*

$$\sum_{x \in \mathcal{M}_n} \mu(x) = \begin{cases} -q, & n = 1, \\ 0, & n > 1. \end{cases}$$

The next result is found in [7, Theorem 2]. We observe that there is a mistake in the statement of this result in [7], but it is correct as stated here (see the discussion in Section 4.5 of [1]).

Lemma 3.3 *Suppose*

$$r \ge 10^4 \text{ and } \frac{\log r}{\log \log r} \ge \log q.$$

Let χ denote a nonprincipal character modulo F. Then for any positive integer n,

$$\left| \sum_{x \in \mathcal{M}_n} \mu(x)\chi(x) \right| \le q^{\frac{n}{2} + \frac{n \log\log r}{\log r} + 8q^{\frac{r}{\log^2 r}} \log_q e}.$$

We will also make use of the following from [16].

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Lemma 3.4 *Let χ denote a nonprincipal character modulo F. Then for any positive integer n,*

$$\Big|\sum_{x\in\mathcal{M}_n}\mu(x)\chi(x)\Big| \le q^{n/2}\binom{n+r-2}{n}.$$

The previous two results can be combined and simplified for our purposes. This is classical in the literature, but we include brief details for completeness.

Corollary 3.5 *For any positive integer $n \ge r$ and any nonprincipal character χ modulo F, we have*

$$\sum_{\deg x<n}\mu(x)\chi(x) \ll_\varepsilon q^{n(1/2+\varepsilon)}.$$

Proof Let S denote the sum in question. We split our sum into intervals depending on the degree of x and write

$$S \ll \sum_{i=0}^{n-1}\Big|\sum_{x\in\mathcal{M}_n}\mu(x)\chi(x)\Big|.$$

This implies there exists some integer $t < n$ such that

$$S \ll n\Big|\sum_{x\in\mathcal{M}_t}\mu(x)\chi(x)\Big|.$$

First, if $t < n/2$, then the result follows trivially. So suppose $n/2 \le t \le n$. If $r < \log n$, then by Lemma 3.4,

$$S \ll nq^{t/2}\binom{t+\log n-2}{t} \ll_\varepsilon q^{n(1/2+\varepsilon)}.$$

Finally, if $r \ge \log n$, then since $t \ge n/2 \ge r/2$, Lemma 3.3 implies

$$S \ll nq^{t(\frac{1}{2}+\frac{\log\log r}{\log r}+8q\frac{r}{t\log^2 r}\log_q e)} \ll nq^{t(\frac{1}{2}+\frac{\log\log r}{\log r}+16q\frac{1}{\log^2 r}\log_q e)}$$
$$\ll_\varepsilon q^{n(1/2+\varepsilon)}. \qquad \blacksquare$$

This now implies the following, which is again well-known, but we include details for completeness.

Corollary 3.6 *Let $a \in \mathbb{F}_q[T]$ with $\gcd(a,F) = 1$. Then for any positive integer n,*

$$\sum_{\substack{\deg x<n\\ x\equiv a \pmod F}}\mu(x) \ll_\varepsilon q^{n(1/2+\varepsilon)}.$$

Proof Of course, if $n < r$, then this is trivial, so we assume otherwise. Using the orthogonality of multiplicative characters, we may write

$$\sum_{\substack{\deg x<n\\ x\equiv a \pmod F}}\mu(x) = \frac{1}{\phi(F)}\sum_{\chi \pmod F}\overline{\chi(a)}\sum_{\deg x<n}\mu(x)\chi(x).$$

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The trivial character contributes only $O(1)$ by Lemma 3.2. To bound the rest, we can apply the triangle inequality and then Corollary 3.5 to reach the desired result. ∎

The following is a special case of [25, Theorem 4.5], which significantly improves upon the previous result when r is close to n (with some restrictions on the size of q).

Lemma 3.7 *Let $\varepsilon > 0$ and $0 < \beta < 1/2$, and suppose*

$$q > \left(\frac{\varepsilon + 2}{\varepsilon} pe\right)^{\frac{2}{1-2\beta}}.$$

Then for any nonnegative integer $n \geq (1+\varepsilon)r$ and any $a \in \mathbb{F}_q[T]$ coprime to F, we have

$$\sum_{\substack{x \in \mathcal{M}_n \\ x \equiv a \pmod F}} \mu(x) \ll_{\varepsilon,\beta} q^{(n-r)(1-\beta/p)}.$$

Finally, the next result is [25, Proposition 5.2]. Originally, this was only stated for square-free F, but it is actually immediate that this holds for arbitrary F (brief details are given).

Lemma 3.8 *For any positive integer d,*

$$\sum_{k=1}^{d} kq^{-k} \sum_{\substack{x \in \mathcal{M}_k \\ (x,F)=1}} \mu(x) = -\frac{q^r}{\phi(F)} + q^{o(r+d)-d}.$$

Proof Assume that this holds for square-free modulus as in [25, Proposition 5.2]. Let $\operatorname{rad}(F)$ denote the product of the distinct, monic, irreducible factors of F and let $r_0 = \deg \operatorname{rad}(F)$. Then

$$\begin{aligned}\sum_{k=1}^{d} kq^{-k} \sum_{\substack{x \in \mathcal{M}_k \\ (x,F)=1}} \mu(x) &= \sum_{k=1}^{d} kq^{-k} \sum_{\substack{x \in \mathcal{M}_k \\ (x,\operatorname{rad}(F))=1}} \mu(x) \\ &= -\frac{q^{r_0}}{\phi(\operatorname{rad}(F))} + q^{o(r_0+d)-d} \\ &= -\frac{q^r}{\phi(F)} + q^{o(r+d)-d},\end{aligned}$$

where the second line follows from our initial assumption. ∎

3.2 The Weil bound for Kloosterman sums

To effectively bound the bilinear Kloosterman sums introduced in Section 2.1, we will need a few well-known estimates regarding complete and incomplete Kloosterman sums. First, we need the following orthogonality relation (see [2, Corollary 4.2]).

Lemma 3.9 *For any $a \in \mathbb{F}_q[T]$ with $\deg a < r$ and positive integer n,*

$$\sum_{\deg x < n} e_F(ax) = \begin{cases} q^n, & \deg a < r - n \\ 0, & \textit{otherwise}. \end{cases}$$

The following is from [2, Lemma A.13].

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Lemma 3.10 *For any $a, b \in \mathbb{F}_q[T]$,*

$$\left| \sum_{\substack{\deg x<r \\ (x,F)=1}} e_F(ax + b\bar{x}) \right| \ll_\varepsilon q^{r/2+\deg(a,b,F)/2+r\varepsilon}.$$

Next, Lemma 3.9 and Lemma 3.10 imply the following.

Lemma 3.11 *For any $b \in \mathbb{F}_q[T]$ and positive integer $n \leq r$,*

$$\left| \sum_{\substack{\deg x<n \\ (x,F)=1}} e_F(b\bar{x}) \right| \ll_\varepsilon q^{r/2+\deg(b,F)/2+\varepsilon r}.$$

Proof By applying Lemma 3.9 and then rearranging and applying Lemma 3.10,

$$\begin{aligned} \left| \sum_{\substack{\deg x<n \\ (x,F)=1}} e_F(b\bar{x}) \right| &= q^{n-r} \left| \sum_{\substack{\deg x<r \\ (x,F)=1}} e_F(b\bar{x}) \sum_{\deg a<r-n} e_F(ax) \right| \\ &\ll_\varepsilon q^{n-r} \sum_{\deg a<r-n} q^{r/2+\deg(a,b,F)/2+\varepsilon r} \\ &\ll_\varepsilon q^{r/2+\deg(b,F)/2+\varepsilon r}. \end{aligned}$$

■

We will also make use of the following.

Lemma 3.12 *Let $b, u \in \mathbb{F}_q[T]$ and suppose $\deg u = O(r)$. Then*

$$\left| \sum_{\substack{\deg x<r \\ (x,uF)=1}} e_F(b\bar{x}) \right| \ll_\varepsilon q^{r/2+\deg(b,F)/2+\varepsilon r}.$$

Proof Without loss of generality, we may suppose that $(u, F) = 1$. We recall the identity

$$\sum_{\substack{d|x \\ \text{d monic}}} \mu(d) = \begin{cases} 1, & \deg x = 0, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, a typical application of inclusion-exclusion implies

$$\begin{aligned} \sum_{\substack{\deg x<r \\ (x,uF)=1}} e_F(b\bar{x}) &= \sum_{\substack{\deg x<r \\ (x,F)=1}} e_F(b\bar{x}) \sum_{\substack{d|(u,x) \\ d \text{ monic}}} \mu(d) \\ &= \sum_{\substack{d|u \\ d \text{ monic}}} \mu(d) \sum_{\substack{\deg x<r \\ (x,F)=1 \\ d|x}} e_F(b\bar{x}) \\ &= \sum_{\substack{d|u \\ d \text{ monic}}} \mu(d) \sum_{\substack{\deg x<r-\deg d \\ (x,F)=1}} e_F(b\overline{dx}). \end{aligned}$$

Now applying the triangle inequality and Lemmas 3.1 and 3.11 concludes the proof. ■

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3.3 Additive energy of modular inversions

We will repeatedly make use of bounds regarding the number of solutions to certain equations with modular inverses. For positive integers n and k, we define $I_{F,a,k}(n)$ to count the number of solutions to

$$\overline{x_1} + \cdots + \overline{x_k} \equiv a \pmod{F}, \ \deg x_i < n, \tag{3.1}$$

and

$$E^{\mathrm{inv}}_{F,k}(n) = \sum_{a \pmod{F}} I_{F,a,k}(n)^2.$$

This can be considered a measure of the additive energy of the set

$$\{\bar{x} \pmod{F} : \deg x < n\}.$$

First, we will make use of the following from [3].

Lemma 3.13 *Let k be a fixed positive integer. Then for any positive integer $n \le r$,*

$$E^{\mathrm{inv}}_{F,k}(n) \ll_{\varepsilon,k} q^{kn+\varepsilon n} + q^{n(3k-1)-r+\varepsilon n}.$$

In particular, this implies

$$E^{\mathrm{inv}}_{F,k}(n) \ll_{\varepsilon,k} \begin{cases} q^{kn+\varepsilon n}, & n < r/(2k-1), \\ q^{n(3k-1)-r+\varepsilon n}, & r/(2k-1) \le n \le r/k, \\ q^{n(2k-1)+\max\{0,n-r\}}, & r/k < n \end{cases}$$

by using the trivial bound when $r/k < n$.

This can be improved upon when $k = 2$, and the following is a generalization of [4, Theorem 2.5] to arbitrary modulus.

Lemma 3.14 *For any positive integer $n \le r$,*

$$E^{\mathrm{inv}}_{F,2}(n) \ll_{\varepsilon} q^{2n+\varepsilon n} + q^{7n/2-r/2+\varepsilon n}.$$

In particular, this implies

$$E^{\mathrm{inv}}_{F,2}(n) \ll_{\varepsilon} \begin{cases} q^{2n+\varepsilon n}, & n < r/3, \\ q^{7n/2-r/2+\varepsilon n}, & r/3 \le n \le r, \\ q^{4n-r}, & r < n \end{cases}$$

by using the trivial bound when $r < n$.

Proof Recall that we are counting the number of solutions to

$$\overline{x_1} + \overline{x_2} \equiv \overline{x_3} + \overline{x_4} \pmod{F}, \ \deg x_i < n. \tag{3.2}$$

This is trivially satisfied if $x_1 \equiv -x_2 \pmod{F}$ and $x_3 \equiv -x_4 \pmod{F}$. Thus, we can write

$$E^{\mathrm{inv}}_{F,2}(n) = E^{\mathrm{inv}*}_{F,2}(n) + O(q^{2n}),$$

where $E^{\mathrm{inv}*}_{F,2}(n)$ counts the number of solutions to (3.2) where each side is nonzero. Next, we observe

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$$E^{\mathrm{inv}*}_{F,2}(n) = \sum_{0\le \deg a<r} I_{F,a,2}(n)^2.$$

Using [2, Lemma 5.3], we have

$$I_{F,a}(n) \ll_\varepsilon q^{\varepsilon n}(1 + q^{3n/2-r/2} + q^{2n+\deg(a,F)-r}),$$

which implies

$$\begin{aligned} E^{\mathrm{inv}*}_{F,2}(n) &\ll_\varepsilon q^{\varepsilon n} \sum_{0\le \deg a<r} I_{F,a}(n)(1 + q^{3n/2-r/2} + q^{2n+\deg(a,F)-r}) \\ &\ll_\varepsilon q^{\varepsilon n}(q^{2n} + q^{7n/2-r/2}) + q^{2n-r+\varepsilon n} \sum_{0\le \deg a<r} q^{\deg(a,F)} I_{F,a}(n). \end{aligned} \tag{3.3}$$

To deal with the sum in this expression, we write

$$\sum_{0\le \deg a<r} q^{(a,F)} I_{F,a}(n) = \sum_{\substack{d\mid F\\ d \text{ monic}}} q^{\deg d} \sum_{\substack{0\le \deg a<r\\ (a,F)=d}} I_{F,a}(n). \tag{3.4}$$

First, if $\overline{x_1} + \overline{x_2} \equiv a \pmod F$ for some (x_1, x_2), then of course, a is uniquely determined. Next, given some a in the inner sum on the right of (3.4), write $a = a_0 d$ and $F = F_0 d$ where $\gcd(a, F) = d$. Thus, if

$$\overline{x_1} + \overline{x_2} \equiv a_0 d \pmod{F_0 d},$$

then

$$x_1 + x_2 \equiv 0 \pmod d, \tag{3.5}$$

implying

$$\begin{aligned} &\sum_{0\le \deg a<r} q^{(a,F)} I_{F,a}(n) \\ &\qquad \le \sum_{\substack{d\mid F\\ d \text{ monic}}} q^{\deg d} \#\{(x_1, x_2) : \deg x_i < n,\ x_1 + x_2 \equiv 0 \pmod d\}. \end{aligned}$$

If $\deg d \ge n$, then there are no solutions to (3.5) with $\deg x_i < n$ unless $x_1 = -x_2$, but we have already eliminated this case. If $\deg d < n$, then for any choice of x_1, there are at most $q^{n-\deg d}$ possibilities for x_2. Thus, by Equations (3.3) and (3.4) and Lemma 3.1, we can conclude

$$\begin{aligned} E^{\mathrm{inv}*}_{F,2}(n) &\ll_\varepsilon q^{\varepsilon n}(q^{2n} + q^{7n/2-r/2}) + q^{2n-r} \sum_{\substack{d\mid F\\ d \text{ monic}\\ \deg d<n}} q^{\deg d}(q^{2n-\deg d}) \\ &\ll_\varepsilon q^{\varepsilon n}(q^{2n} + q^{7n/2-r/2} + q^{4n-r}),, \end{aligned}$$

as desired. ∎

Also, ideas from [13] show that these can be improved when averaging over the modulus.

Lemma 3.15 *Let n, r, and k be positive integers. Then*

$$\sum_{\deg F=r} E^{\mathrm{inv}}_{F,k}(n) \ll_{\varepsilon,k} q^{r+n(k+\varepsilon)} + q^{n(2k+\varepsilon)}.$$

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Proof By clearing denominators, it suffices to count solutions to

$$\sum_{i=1}^{k}\prod_{\substack{j=1\\ j\neq i}}^{2k} x_i - \sum_{i=k+1}^{2k}\prod_{\substack{j=1\\ j\neq i}}^{2k} x_i \equiv 0 \pmod F, \;\; \deg x_i < n, \;\; \deg F = r.$$

If the left-hand side of the expression is equal to 0, then [27, Lemma 2.6] implies there are at most $O_{\varepsilon,k}(q^{r+nk+n\varepsilon})$ solutions. Otherwise, we must have that F divides the left-hand side, yielding at most $O_\varepsilon(q^{n\varepsilon})$ choices for F, implying at most $O_\varepsilon(q^{2kn+n\varepsilon})$ solutions in total. ∎

4 Bilinear Kloosterman sums

We can now present our results regarding bilinear Kloosterman sums. Before proving Theorem 2.1, we will present a few more general results. The following can give a power-savings over the trivial bound when used in conjunction with Lemma's 3.13 and 3.14, although for flexibility, we do not substitute these bounds yet. We note that the case $k_1 = k_2 = 2$ recovers [4, Theorem 2.5] when Lemma 3.14 is applied, although this generalizes it to composite modulus.

Lemma 4.1 *Let $\varepsilon > 0$. Let k_1 and k_2 denote positive integers and $a, F \in \mathbb{F}_q[T]$ with $\gcd(a,F) = 1$ and $\deg F = r$. Then for any positive integers n and m and weights as in (2.1), we have*

$$\mathcal{W}_{F,a}(m,n;\boldsymbol{\alpha},\boldsymbol{\beta}) \ll_\varepsilon q^{m+n+\varepsilon r}\left(E_{F,k_1}(m)E_{F,k_2}(n)q^{r-2nk_2-2mk_1}\right)^{\frac{1}{2k_1k_2}}.$$

Proof Let $S = |\mathcal{W}_F(m,n;\boldsymbol{\alpha},\boldsymbol{\beta})|$. Applying Hölders inequality yields

$$S^{k_2} \ll_\varepsilon q^{\varepsilon r k_2/2}\, q^{m(k_2-1)} \sum_{\substack{\deg x_1<m\\ (x_1,F)=1}} \left| \sum_{\substack{\deg x_2<n\\ (x_2,F)=1}} \beta_{x_2} e_F(a\bar{x}_1\bar{x}_2)\right|^{k_2}.$$

Expanding the inner sum and rearranging then yields

$$\begin{aligned} S^{k_2} &\ll_\varepsilon q^{\varepsilon r k_2/2}\, q^{m(k_2-1)} \sum_{\substack{\deg x_1<m\\ (x_1,F)=1}} \left| \sum_{\substack{y_1,\dots,y_{k_2}\\ \deg y_i<n\\ (y_i,F)=1}} \beta_{y_1}\dots\beta_{y_{k_2}} e_F(a\bar{x}_1(\bar{y}_1+\dots+\bar{y}_{k_2}))\right| \\ &= q^{\varepsilon r k_2/2}\, q^{m(k_2-1)} \sum_{\substack{\deg x_1<m\\ (x_1,F)=1}} \gamma_{x_1} \sum_{\substack{y_1,\dots,y_{k_2}\\ \deg y_i<n\\ (y_i,F)=1}} \beta_{y_1}\dots\beta_{y_{k_2}} e_F(a\bar{x}_1(\bar{y}_1+\dots+\bar{y}_{k_2})) \\ &\ll q^{\varepsilon r k_2}\, q^{m(k_2-1)} \sum_{\substack{y_1,\dots,y_{k_2}\\ \deg y_i<n\\ (y_i,F)=1}} \left| \sum_{\substack{\deg x_1<m\\ (x_1,F)=1}} \gamma_{x_1} e_F(a\bar{x}_1(\bar{y}_1+\dots+\bar{y}_{k_2}))\right| \end{aligned}$$

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for some $|\gamma_{x_1}| \le 1$. By Applying Hölder's inequality again, we have

$$S^{k_1k_2} \ll_\varepsilon q^{\varepsilon r k_1 k_2} q^{mk_1(k_2-1)+nk_2(k_1-1)} \times \sum_{\substack{y_1,\dots,y_{k_2} \\ \deg y_i<n \\ (y_i,F)=1}} \left| \sum_{\substack{\deg x_1<m \\ (x_1,F)=1}} \gamma_{x_1} e_F(a\overline{x}_1(\overline{y}_1+\dots+\overline{y}_{k_2})) \right|^{k_1}.$$

This can be rewritten as

$$S^{k_1k_2} \ll_\varepsilon q^{\varepsilon r k_1 k_2+mk_1(k_2-1)+nk_2(k_1-1)} \sum_{\deg\lambda<r} I_{F,\lambda,k_2}(n) \left| \sum_{\substack{\deg x_1<m \\ (x_1,F)=1}} \gamma_{x_1} e_F(a\overline{x}_1\lambda) \right|^{k_1}.$$

Applying the Cauchy-Schwarz inequality now yields

$$\begin{aligned} S^{2k_1k_2} &\ll_\varepsilon q^{2\varepsilon r k_1 k_2} q^{2mk_1(k_2-1)+2nk_2(k_1-1)} \times \sum_{\deg\lambda<r} I_{F,\lambda,k_2}(n)^2 \times \sum_{\deg\lambda<r} \left| \sum_{\substack{\deg x_1<m \\ (x_1,F)=1}} \gamma_{x_1} e_F(a\overline{x}_1\lambda) \right|^{2k_1} \\ &\ll q^{2\varepsilon r k_1 k_2} q^{2mk_1(k_2-1)+2nk_2(k_1-1)} E^{\mathrm{inv}}_{F,k_2}(n)\, q^r E^{\mathrm{inv}}_{F,k_1}(m), \end{aligned}$$

and rearranging gives the desired result. ∎

Another useful way to state Lemma 4.1 is

$$\mathcal{W}_{F,a}(m,n;\boldsymbol{\alpha},\boldsymbol{\beta}) \ll_\varepsilon q^{m+n+\varepsilon r} \times \left(E_{F,k_1}(m)q^{r/2-2mk_1}\right)^{\frac{1}{2k_1k_2}} \left(E_{F,k_2}(n)q^{r/2-2nk_2}\right)^{\frac{1}{2k_1k_2}}. \tag{4.1}$$

A simpler result is the following, which is obtained using the argument of [15, Lemma 2.4].

Lemma 4.2 *Let k denote a positive integer and take other notation as in Lemma 4.1. Then*

$$\mathcal{W}_{F,a}(m,n;\boldsymbol{\alpha},\boldsymbol{\beta}) \ll_\varepsilon q^{(m(2k-1)+\max\{r,m\})/2k+\varepsilon r} E^{\mathrm{inv}}_{F,k}(n)^{1/2k}.$$

Proof Again, let $S = |\mathcal{W}_{F,a}(m,n;\boldsymbol{\alpha},\boldsymbol{\beta})|$. Applying Hölders inequality and rearranging yields

$$\begin{aligned} S^{2k} &\ll_\varepsilon q^{k\varepsilon r} q^{m(2k-1)} \sum_{\substack{\deg x_1<m \\ (x_1,F)=1}} \left| \sum_{\substack{\deg x_2<n \\ (x_2,F)=1}} \beta_{x_2} e_F(a\overline{x_1}\overline{x_2}) \right| \\ &\ll_\varepsilon q^{k\varepsilon r} q^{m(2k-1)+\max\{0,m-r\}} \sum_{\deg x_1<r} \left| \sum_{\substack{\deg x_2<n \\ (x_2,F)=1}} \beta_{x_2} e_F(ax_1\overline{x_2}) \right|^{2k}. \end{aligned}$$

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Expanding and using orthogonality then implies

$$S^{2k} \ll_\varepsilon q^{2k\varepsilon r} q^{m(2k-1)+\max\{0,m-r\}+r} E^{\mathrm{inv}}_{F,k}(n), \tag{4.2}$$

as desired. ■

In the case of $k = 2$, this becomes

$$\mathcal{W}_{F,a}(m,n;\boldsymbol{\alpha},\boldsymbol{\beta}) \ll_\varepsilon q^{(3m+\max\{r,m\})/4+\varepsilon r} E^{\mathrm{inv}}_{F,2}(n)^{1/4}. \tag{4.3}$$

which will be used most often.

We can again improve upon this by averaging over the modulus, using the exact same ideas as in the proof of Lemma 4.2 above.

Lemma 4.3 *With notation as in Lemma 4.2,*

$$\sum_{\deg F=r} \max_{(a,F)=1} |\mathcal{W}_{F,a}(m,n;\boldsymbol{\alpha},\boldsymbol{\beta})| \ll_\varepsilon q^{(m+r)\frac{2k-1}{2k}+\max\{m,r\}\frac{1}{2k}+\varepsilon r} \left(\sum_{\deg F=r} E^{\mathrm{inv}}_{F,k}(n)\right)^{1/2k}.$$

Proof We can use Equation (4.2) and then Hölders inequality to see

$$\begin{aligned}\sum_{\deg F=r} \max_{(a,F)=1} |\mathcal{W}_{F,a}(m,n;\boldsymbol{\alpha},\boldsymbol{\beta})| &\ll_\varepsilon q^{\varepsilon r} q^{m\frac{2k-1}{2k}+\max\{m,r\}\frac{1}{2k}} \sum_{\deg F=r} E^{\mathrm{inv}}_{F,k}(n)^{\frac{1}{2k}} \\ &\ll q^{\varepsilon r} q^{m\frac{2k-1}{2k}+\max\{m,r\}\frac{1}{2k}} \left(\sum_{\deg F=r} E^{\mathrm{inv}}_{F,k}(n)\right)^{\frac{1}{2k}} q^{r\frac{2k-1}{2k}},\end{aligned}$$

and rearranging gives the desired result. ■

4.1 Proof of Theorem 2.1

As before, we let $S = |\mathcal{W}_{F,a}(m,n;\boldsymbol{\alpha},\boldsymbol{\beta})|$ and split the discussion into a few cases. Without loss of generality, we may suppose that $n \le m$.

First, we assume that $n \le r/3$, and let $k \ge 2$ denote the largest integer such that $n(k-1) \le r/2$. Note that k is bounded above in terms of ε since n is from below, and $nk > r/2$. Thus, applying (4.1) with $k_1 = 2$ and $k_2 = k$, together with Lemma 3.13 gives

$$\begin{aligned} S &\ll_{\varepsilon'} q^{m+n+\varepsilon' r}\left(E_{F,2}(m) q^{r/2-4m}\right)^{\frac{1}{4k}} \left(q^{r/2-nk} + q^{n(k-1)-r/2}\right)^{\frac{1}{4k}} \\ &\ll q^{m+n+\varepsilon' r}\left(E_{F,2}(m) q^{r/2-4m}\right)^{\frac{1}{4k}}\end{aligned}$$

for some sufficiently small ε'. Since k is bounded from above, it now suffices to show that for any $m > r(1/4+\varepsilon)$,

$$E_{F,2}(m) q^{r/2-4m} < q^{-\delta_1 r}$$

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for some $\delta_1 > 0$. If $r(1/4+\varepsilon) < m < r/3$, then Lemma 3.14 yields

$$E_{F,2}(m)q^{r/2-4m} \ll_\varepsilon q^{r/2-2m+\varepsilon m} < q^{-r\varepsilon},$$

as desired. Similarly, applying Lemma 3.14 in the case $r/3 \le m \le r$ and the case $r \le m$ gives the desired result when $n \le r/3$.

Next, we may assume $m, n \ge r/3$. By Lemma 4.1 with $k_1 = k_2 = 2$, it suffices to show

$$E_{F,k_1}(m)E_{F,k_2}(n)q^{r-4n-4m} < q^{-\delta_2 r}$$

for some $\delta_2 > 0$. If $r/3 \le n \le m \le r$, then Lemma 3.14 gives

$$E_{F,k_1}(m)E_{F,k_2}(n)q^{r-4n-4m} \ll_\varepsilon q^{-m/2-n/2+\varepsilon m},$$

which is sufficient. Similarly, applying Lemma 3.14 in the cases $r/2 \le n \le r$ and $r \le m$, as well as $r \le n \le m$, yields the desired result.

5 Applications

Before proceeding, we will make a few reductions common to each of Theorems 2.2, 2.3, and 2.4. For $a, F \in \mathbb{F}_q[T]$ with $\deg F = r$, we set

$$d = \deg(a,F),\ F_0 = F/(a,F),\ r_0 = \deg F_0,\ a_0 = a/(a,F). \tag{5.1}$$

Lemma 5.1

$$\sum_{\substack{\deg x<n \\ (x,F)=1}} \mu(x)e_F(a\bar{x}) \ll_\varepsilon q^{r_0+n(1/2+\varepsilon)}.$$

Proof This is a direct application of Corollary 3.6 as

$$\begin{aligned}\sum_{\substack{\deg x<n \\ (x,F)=1}} \mu(x)e_F(a\bar{x}) &= \sum_{\substack{\deg x<n \\ (x,F)=1}} \mu(x)e_{F_0}(a_0\bar{x}) \\ &= \sum_{\substack{\deg y<\deg F_0 \\ (y,F)=1}} e_{F_0}(a\bar{y}) \sum_{\substack{\deg x<n \\ x\equiv y \pmod{F_0}}} \mu(x) \ll_\varepsilon q^{r_0+n(1/2+\varepsilon)}.\end{aligned}$$

■

Lemma 5.2 *For any positive integer U satisfying $2U < n$,*

$$\left|\sum_{\substack{\deg x<n \\ (x,F)=1}} \mu(x)e_F(a\bar{x})\right| \ll_\varepsilon S_1 + S_2,$$

where

$$S_1 = q^{n\varepsilon} \sum_{\substack{\deg x\le u \\ (x,F)=1}} \left|\sum_{\substack{\deg y<n-u \\ (y,F)=1}} e_{F_0}(a_0\overline{xy})\right|,\ S_2 = q^{n\varepsilon} \sum_{\substack{\deg x\le v \\ (x,F)=1}} \left|\sum_{\substack{\deg y<n-v \\ (y,F)=1}} \beta_y e_{F_0}(a_0\overline{xy})\right|$$

for some integers $u \le 2U$ and $U < v \le n-U$, and $|\beta_y| \ll_\varepsilon q^{n\varepsilon}$.

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Proof This follows from a standard manipulation of Vaughan's identity as in [15, 13, 9, 18], but we will include a few details for completeness. By applying Vaughan's identity in function fields [25, equation (A.1)], we have

$$\sum_{\substack{\deg x<n\\(x,F)=1}} \mu(x)e_F(a\bar{x}) \ll S_1+S_2,$$

where

$$S_1 = \sum_{\substack{\deg x<n\\(x,F)=1}} \sum_{\substack{\deg g\le k\\ \deg h\le k\\ gh|x}} \mu(g)\mu(h)e_F(a\bar{x}),$$

$$S_2 = \sum_{\substack{\deg x<n\\(x,F)=1}} \sum_{\substack{\deg g> k\\ \deg h> k\\ gh|x}} \mu(g)\mu(h)e_F(a\bar{x}),$$

for any positive integer $k<n$. First to manipulate S_1, we have

$$S_1 \le \sum_{\substack{\deg g\le k\\(g,F)=1}} \sum_{\substack{\deg h\le k\\(h,F)=1}} \left| \sum_{\substack{\deg y<n-\deg g-\deg h\\(y,F)=1}} e_F(a\overline{ghy})\right|.$$

For each pair (g,h), we only ever take into account the value $\overline{gh}$. So by (3.1),

$$S_1 \le q^{o(n)} \sum_{\substack{\deg x\le 2k\\(x,F)=1}} \left| \sum_{\substack{\deg y<n-\deg x\\(y,F)=1}} e_F(a\overline{xy})\right|.$$

Thus, there exists some integer $t_1\le 2k$ such that

$$S_1 \le q^{o(n)} \sum_{\substack{\deg x\le t_1\\(x,F)=1}} \left| \sum_{\substack{\deg y<n-t_1\\(y,F)=1}} e_F(a\overline{xy})\right|.$$

Next, we consider S_2. We treat this sum similarly and obtain

$$S_2 \le \sum_{\substack{k<\deg x<n-k\\(x,F)=1}} \left| \sum_{\substack{\deg y<n-\deg x\\(y,F)=1}} \beta_y e_F(a\overline{xy})\right|,$$

where

$$\beta_y = \begin{cases} \sum\limits_{\substack{k<\deg z<\deg y\\ z|y}} \mu(z), & k<\deg y<n-\deg x\\ 0, & \deg y\le k,\end{cases}$$

which of course implies $|\beta_y|\le q^{o(n)}$ by (3.1). Now again, this implies that

$$S_2 \le q^{o(n)} \sum_{\substack{\deg x\le t_2\\(x,F)=1}} \left| \sum_{\substack{\deg y<n-t_2\\(y,F)=1}} \beta_y e_F(a\overline{xy})\right|$$

for some integer t_2 satisfying $k<t_2<n-k$.

Combining these estimates for S_1 and S_2 gives the desired result. ∎

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5.1 Proof of Theorem 2.2

Recall that we fix $\varepsilon > 0$ and suppose that $r(1/2+\varepsilon) < n < r$. Additionally, recall r_0, F_0, and a_0 from (5.1).

If $r_0 \le n(1/2 - 2\varepsilon)$, then Lemma 5.1 implies

$$\sum_{\substack{\deg x < n \\ (x,F)=1}} \mu(x) e_F(a\overline{x}) \ll_\varepsilon q^{n(1-\varepsilon)},$$

as desired. So we may now assume $r_0 > n(1/2 - 2\varepsilon)$.

By letting

$$U = (n - r/2 - r\varepsilon/2)/2,$$

it suffices to bound the sums S_1 and S_2 in Lemma 5.2.

First, we bound S_1 by applying Lemma 3.11 and Lemma 3.12. If $n - u \ge r_0$, then Lemma 3.12 implies

$$S_1 \ll_\varepsilon q^{u+n-u-r_0+r_0/2+\varepsilon r/2} = q^{n-r_0/2+\varepsilon r/2}.$$

If $n - u \le r_0$, then by $u \le 2U = n - r/2 - r\varepsilon/2$ and Lemma 3.11,

$$S_1 \ll_\varepsilon q^{u+r_0/2+r\varepsilon/3} \le q^{n-r\varepsilon/2+n\varepsilon/3}.$$

Either way, these provide sufficient power savings.

Finally to bound S_2, we can directly apply Theorem 2.1 which completes the proof.

5.2 Proof of Theorem 2.3

This proof is quite similar to the proof of Theorem 2.2 and just requires slightly more attention to detail to obtain more explicit bounds. This expands upon some ideas from [15, 6]. Again, recall the notation a_0, r_0, and F_0 from (5.1). We may assume $n > 3r/4$ since otherwise, the result is trivial.

First, suppose that $r_0 < 7n/16$. Then Lemma 5.1 implies

$$\left| \sum_{\substack{\deg x < n \\ (x,F)=1}} \mu(x) e_F(a\overline{x}) \right| \ll_\varepsilon q^{n(15n/16+\varepsilon)},$$

as desired, and thus, we may now assume $r_0 \ge 7n/16$.

By letting $U = r_0/3$, we need to bound S_1 and S_2 as in Lemma 5.2. First, we deal with S_1 and split the argument up into cases depending on the sizes of u and $n - u$.

Case 1: $u \le 2r_0/3$ and $r_0 \le n - u$. Here, we apply Lemma 3.12 to the inner sum over y to obtain

$$\begin{aligned} S_1 &\ll_\varepsilon q^{n-u-r_0+\varepsilon n/2} \sum_{\substack{\deg x \le u \\ (x,F)=1}} \left| \sum_{\substack{\deg y < r_0 \\ (y,F)=1}} e_{F_0}(a_0 \overline{xy}) \right| \\ &\ll_\varepsilon q^{n-r_0/2+\varepsilon n} \le q^{25n/32+\varepsilon n} \end{aligned}$$

since $r_0 \ge 7n/16$.

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Case 2: $u \le r_0/3$ and $r_0/3 \le n-u \le r_0$. Using Equation (4.3) together with Lemma 3.14 yields

$$S_1 \ll_\varepsilon q^{3(n-u)/4+r_0/4+u/2+\varepsilon n} = q^{3n/4+r_0/4-u/4+\varepsilon n}.$$

Also, separately applying Lemma 3.12 to S_1 (to the inner sum over y) implies

$$S_1 \ll_\varepsilon q^{u+r_0/2+\varepsilon n}.$$

By combining these two estimates, we have

$$S_1 \ll_\varepsilon q^{3n/5+3r_0/10+\varepsilon n} \le q^{2n/3+r/4+\varepsilon n}$$

since $n > 3r/4$.

Case 3: $r_0/3 \le u \le /3$ and $r_0/3 \le n-u \le r_0$. Here, we use Lemma 4.1 with $k_1 = k_2 = 2$ together with Lemma 3.14, giving

$$S_1 \ll_\varepsilon q^{n+\varepsilon n+\frac{1}{8}(7u/2-r_0/2+7(n-u)/2-r_0/2+r_0-4t_1-4(n-u))} = q^{15n/16+\varepsilon n}.$$

For the remaining cases bounding S_1, we may assume that $n-u \le r_0/3$, which implies $u \ge n - r_0/3$. Note that this also implies $n \le r_0$ since $u \le 2r_0/3$.

Case 4: $n - r_0/3 \le u \le 2n/3$ and $n-u \le r_0/3$. Here, by again applying Equation (4.3) with Lemma 3.14, we have

$$S_1 \ll_\varepsilon q^{3u/4+r_0/4+(n-u)/2+\varepsilon n} \le q^{2n/3+r/4+\varepsilon n}.$$

Case 5: $2n/3 \le u \le 2r_0/3$ and $n-u \le r_0/3$. Applying the Cauchy-Schwarz inequality directly to S_1 shows

$$S_1^2 \ll_\varepsilon q^{u+\varepsilon n} \sum_{\substack{y_1,y_2 \\ \deg y_i < n-u \\ (y_i,F)=1}} \sum_{\substack{\deg x<u \\ (x,F)=1}} e_F(a\bar{x}(\overline{y_1}+\overline{y_2})).$$

Isolating the case $y_1 = -y_2$ and then applying Lemma 3.11 to the sum over x implies

$$S_1^2 \ll_\varepsilon q^{n+u+\varepsilon n} + q^{u+r/2+\varepsilon n} T, \tag{5.2}$$

where

$$T = \sum_{\substack{y_1,y_2 \\ \deg y_i < n-u \\ (y_i,F)=1}} q^{\deg(F,\overline{y_1}+\overline{y_2})/2}.$$

We can rearrange and write

$$T = \sum_{\substack{d|F \\ d \text{ monic}}} q^{\deg d/2} \sum_{\substack{0 \le \deg a < r \\ (a,F)=d}} I_{F,a}(n-u)$$

with $I_{F,a}(n-u)$ as in (3.1). Now mimicking the argument after Equation (3.4) identically shows

$$T \le \sum_{\substack{d|F \\ d \text{ monic}}} q^{\deg d/2} q^{2n-2u-\deg d} \ll_\varepsilon q^{2n-2u+\varepsilon n}.$$

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Substituting back into (5.2) yields

$$S_1 \ll_\varepsilon q^{n/2+u/2+\varepsilon n} + q^{r/4+n-u/2+\varepsilon n} \ll q^{2n/3+r/4+\varepsilon n},$$

where we have again used $n > 3r/4$.

Combining all 5 cases above yields a suitable bound for S_1. We now focus on bounding S_2 and similarly consider a number of cases depending on the size of v and $n-v$. Without loss of generality, we may assume that $v \le n-v$.

Case 1: $r_0/3 \le v \le r_0$ and $r_0/3 \le n-v \le r_0$. Here, we may apply bounds identically to Case 3 above when bounding S_1.

The last two cases both use Equation (4.3) with Lemma 3.14.

Case 2: $r_0/3 \le v \le r_0$ and $r_0 \le n-v$. Here,

$$S_2 \ll_\varepsilon q^{n-v+7v/8-r_0/8+\varepsilon n} \le q^{15n/16+\varepsilon n}$$

since $r_0 \ge 7n/16$.

Case 3: $r_0 \le v$ and $r_0 \le n-v$. In this case,

$$S_2 \ll_\varepsilon q^{v+n-v-r_0+\varepsilon n} \le q^{n-r_0/4+\varepsilon n} \le q^{15n/16+\varepsilon n}$$

since $r_0 \ge 7n/16$.

Combining these cases yields a suitable bound for S_2, which now completes the proof.

5.3 Proof of Theorem 2.4

Again, this proof is similar to the other proofs previously in this section. Recall that we are wanting to bound

$$S = \sum_{\deg F=r} \max_{a\in\mathbb{F}_q[T]} \left| \sum_{\substack{\deg x<n \\ (x,F)=1}} \mu(x) e_F(a\bar{x}) \right|.$$

For each F, let a_F denote the value of a for which the maximum on the inner sum is achieved. Then we can say

$$\begin{aligned}
S &= \sum_{\substack{\deg d\le r \\ d \text{ monic}}} \sum_{\substack{\deg F=r \\ (a_F,F)=d}} \left| \sum_{\substack{\deg x<n \\ (x,F)=1}} \mu(x) e_{F/d}((a_F/d)\bar{x}) \right| \\
&\le \sum_{\substack{\deg d\le r \\ d \text{ monic}}} \sum_{\deg F=r-\deg d} \max_{(a,F)=1} \left| \sum_{\substack{\deg x<n \\ (x,F)=1}} \mu(x) e_F(a\bar{x}) \right| \\
&= \sum_{j=1}^{r} q^j \sum_{\deg F=r-j} \max_{(a,F)=1} \left| \sum_{\substack{\deg x<n \\ (x,F)=1}} \mu(x) e_F(a\bar{x}) \right| \\
&\ll_\varepsilon q^{\varepsilon n+j} \sum_{\deg F=r-j} \max_{(a,F)=1} \left| \sum_{\substack{\deg x<n \\ (x,F)=1}} \mu(x) e_F(a\bar{x}) \right| \qquad (5.3)
\end{aligned}$$

for some integer $1 \le j \le r$.

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First, suppose $j > r - 2n/5$. Then applying Lemma 5.1 to the inner-sum implies

$$q^{\varepsilon n+j} \sum_{\deg F=r-j} \max_{a\in\mathbb{F}_q[T]} \left| \sum_{\substack{\deg x<n \\ (x,F)=1}} \mu(x) e_F(a\bar{x}) \right| \ll_\varepsilon q^{j+r-j+r-j+n/2+n\varepsilon}$$

$$\ll q^{r+9n/10+n\varepsilon},$$

so we may assume that $j \le r - 2n/5$.

We let $U = \min\{n/3, 5n/8 - r/4\}$. By Lemma 5.2 and Equation (5.3), the problem reduceS to bounding

$$T_1 = q^{\varepsilon n+j} \sum_{\deg F=r-j} \max_{(a,F)=1} S_1, \quad T_2 = q^{\varepsilon n+j} \sum_{\deg F=r-j} \max_{(a,F)=1} S_2$$

with S_1 and S_2 as in Lemma 5.2. In Lemma 5.2, the condition on u is given as $u \le 2U$. But since $2U \le n - U$ here, the case of $U \le u \le 2U$ is covered when dealing with S_2 (since all of our methods for bounding S_2 also apply to S_1). So when bounding S_1, we may assume $u \le U$.

First, we deal with T_1. We may apply Lemma 3.11 to the inner sum over y. If $n - u \le r - j$, then

$$T_1 \ll_\varepsilon q^{n\varepsilon+j+r-j+u+(r-j)/2} \ll q^{3r/2+u+n\varepsilon} \ll q^{5n/8+5r/4+n\varepsilon},$$

or if $n - u \ge r - j$, then

$$T_1 \ll_\varepsilon q^{n\varepsilon+j+r-j+u+n-u-(r-j)+(r-j)/2} \ll q^{r+4n/5+n\varepsilon},$$

where we have used $j \le r - 2n/5$.

Next, we deal with T_2. By Lemma 4.3 and Lemma 3.15, we have that for any positive integer k,

$$T_2 \ll_\varepsilon q^{\varepsilon n+j} q^{(v+r-j)\frac{2k-1}{2k}+\max\{v,r-j\}\frac{1}{2k}} \left(q^{(r-j)/2k+n/2-v/2} + q^{n-v}\right). \tag{5.4}$$

We consider two cases depending on the size of v and $n - v$. Since we have treated the inner sum S_2 in T_2 as a bilinear Kloosterman sum with arbitrary weights, and the ranges on v and $n - v$ are equal, we may also interchange v and $n - v$. Thus, considering the range $2n/5 \le v \le n - U$ is enough, since if $v \le 2n/5$, then $n - v \ge 3n/5$, so we may swap v and $n - v$ to get back into the range $2n/5 \le v \le n - U$.

Case 1: $2n/5 \le v \le n/2$ and $\max\{v, r - j\} = r - j$. Here, we use (5.4) with $k = 2$,

$$T_2 \ll_\varepsilon q^{r+n\varepsilon}(q^{n-v/4} + q^{n/2+r/4-j/4+v/4}) \ll_\varepsilon q^{r+n\varepsilon}(q^{9n/10} + q^{r/4+5n/8}).$$

Case 2: $3n/5 \le v \le n - U$ and $\max\{v, r - j\} = r - j$. Here, we use (5.4) with $k = 3$,

$$\begin{aligned} T_2 &\ll_\varepsilon q^{r+n\varepsilon}(q^{r/6-j/6+n/2+v/3} + q^{n-v/6}) \\ &\ll q^{r+n\varepsilon}(q^{r/6+5n/6-U/3} + q^{9n/10}) \\ &\ll q^{r+n\varepsilon}(q^{13n/18+r/6} + q^{5n/8+r/4} + q^{9n/10}), \end{aligned}$$

where we have used $U = \min\{n/3, 5n/8 - r/4\}$.

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Case 3: $2n/5 \le \nu \le n - U$ and $\max\{\nu, r - j\} = \nu$. Here, we use (5.4) with $k = 2$,

$$\begin{aligned} T_2 &\ll_\varepsilon q^{r+n\varepsilon}(q^{n/2+\nu/2} + q^{n-r/4+j/4}) \\ &\ll q^{r+n\varepsilon}(q^{5n/6} + q^{11n/16+r/8} + q^{9n/10}), \end{aligned}$$

where we have used $j \le r - 2n/5$ and $U = \min\{n/3, 5n/8 - r/4\}$.

Combining all of our estimates for T_1 and T_2 yields the desired result.

5.4 Proof of Theorem 2.5

This result follows from substituting Theorem 2.3 instead of [25, Theorem 1.8] into the proof of [25, Theorem 1.9], but we sketch the details here. We let $d = n - r$, and thus, the condition that $n\omega \ge r$ for some $\omega < 1/2 + 1/62$ can be rewritten as

$$d \ge r\frac{1-\omega}{\omega} = r(1-\omega')$$

for some $\omega' < 1/16$. We let $\theta > 0$ (which will be taken to be sufficiently small as needed). Also, we let

$$\varepsilon = \frac{16}{15}\left(\frac{1}{16} - \omega' - 2\theta\right).$$

By [25, equation (5.9)], it suffices to bound

$$S = \sum_{k=1}^{d+r} k \sum_{\substack{x \in \mathcal{M}_k \\ (x,F)=1}} \mu(x) \sum_{\substack{y \in \mathcal{M}_{r+d-k} \\ xy \equiv a \pmod F}} 1.$$

As in [25], if $k \le d$, we can apply Lemma 3.8 to contribute the main term.

We denote the remaining sum over $k > d$ by S_0 and note that

$$S_0 \le rk \left| \sum_{\substack{x \in \mathcal{M}_{k_0} \\ (x,F)=1}} \mu(x) \sum_{\substack{y \in \mathcal{M}_{r+d-k_0} \\ xy \equiv a \pmod F}} 1 \right|$$

for some k satisfying $d \le k \le d + r$. If $k \le r(1+\varepsilon)$. Then using [25, equation (5.10)], applying Theorem 2.3, and using $k \le r(1+\varepsilon)$ yields

$$\begin{aligned} S_0 &\le rkq^{d-k} \sum_{\deg h < k-d} \left| \sum_{\substack{x \in \mathcal{M}_k \\ (x,F)=1}} \mu(x) e_F(ah\bar{x}) \right| \\ &\ll_\theta rkq^{\theta r}(q^{15k/16} + q^{2k/3+r/4}) \\ &\ll_\theta q^{15r/16+15r\varepsilon/16+r\theta}. \end{aligned} \tag{5.5}$$

We now use $\varepsilon = (1/16 - \omega' - 2\theta)16/15$ and then $r \le d + r\omega'$ to conclude

$$S_0 \ll_\theta q^{r - r\omega - r\theta} \le q^{d - r\theta},$$

which is sufficient. So we may now assume that $k > r(1+\varepsilon)$. We also let $\beta > 0$. Rearranging S_0, we arrive at [25, equation (5.13)],

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$$S_0 \le rk \sum_{y \in \mathcal{M}_{r+d-k_0}} \left| \sum_{\substack{x \in \mathcal{M}_k \\ xy \equiv a \pmod{F}}} \mu(x) \right|.$$

Thus, we may apply Lemma 3.7 to yield

$$S_0 \ll_{\theta,\beta} rkq^{r+d-k} q^{(k-r)(1-\beta/p)} \ll_\beta q^{d-r\beta/p\varepsilon},$$

which again, is sufficient. This holds as long as

$$q > \left(pe\frac{\varepsilon+2}{\varepsilon}\right)^{\frac{2}{1-2\beta}} = \left(pe\left(1+\frac{30}{1-16\delta'-32\theta}\right)\right)^{\frac{2}{1-2\beta}}.$$

But since we fix p and q, we may choose θ and β sufficiently small so that we only require

$$q > p^2e^2\left(1+\frac{30}{1-16\omega'}\right)^2. \tag{5.6}$$

By substituting ω and rearranging, we obtain the desired result.

5.5 Proof of Theorem 2.6

This proof uses essentially the same ideas as the proof of Theorem 2.5, although it is slightly more technical. We may assume that $n = O(r)$, since for small r, this is implied by other results (for example, by Theorem 2.5). We let $d = n - R$, and thus, the condition that $R \le n\omega$ for some $\omega < 1/2 + 1/38$ can be rewritten as $d \ge R(1-\omega')$ for some $\omega' < 1/10$. We let $\theta > 0$ (which will be taken to be sufficiently small as needed) Also, we let

$$\varepsilon = \frac{10}{9}\left(\frac{1}{10} - w' - 2\theta\right).$$

We rewrite the sum in question as

$$S = \sum_{r=1}^{R-1} \sum_{\deg F = r} \max_{(a,F)=1} \left| \sum_{\substack{x \in \mathcal{M}_n \\ x \equiv a \pmod{F}}} \Lambda(x) - \frac{q^n}{\phi(F)} \right|.$$

For each r in this sum, let $d_r = n - r$. Expanding this identically as in [25, equation (5.9)], we can say $S \ll S_1 + S_2 + S_3$, where

$$S_1 = \sum_{r=1}^{R-1} \sum_{\deg F = r} \left| -q^{d_r} \sum_{k=1}^{d_r} kq^{-k} \sum_{\substack{x \in \mathcal{M}_k \\ (x,F)=1}} \mu(x) - \frac{q^n}{\phi(F)} \right|,$$

$$S_2 = \sum_{r=1}^{R-1} \sum_{\deg F = r} \max_{(a,F)=1} \sum_{\substack{d_r < k < d_r + r \\ k \le r(1+\varepsilon)}} k \left| \sum_{\substack{x \in \mathcal{M}_k \\ (x,F)=1}} \mu(x) \sum_{\substack{y \in \mathcal{M}_{r+d_r-k} \\ xy \equiv a \pmod{F}}} 1 \right|,$$

$$S_3 = \sum_{r=1}^{R-1} \sum_{\deg F = r} \max_{(a,F)=1} \sum_{\substack{d_r < k < d_r + r \\ k > r(1+\varepsilon)}} k \left| \sum_{\substack{x \in \mathcal{M}_k \\ (x,F)=1}} \mu(x) \sum_{\substack{y \in \mathcal{M}_{r+d_r-k} \\ xy \equiv a \pmod{F}}} 1 \right|,$$

and it suffices to show each $S_i \ll_\omega q^{n-R\delta}$ for some $\delta > 0$.

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To bound the contribution from S_1, we apply Lemma 3.8 directly to see

$$S_1 = \sum_{r=1}^{R-1} \sum_{\deg F = r} \left| q^{d_r} \left(-\frac{q^r}{\phi(F)} + q^{o(n)-d_r} \right) + \frac{q^n}{\phi(F)} \right|$$
$$\ll_\theta q^{R+\theta n} < q^{n-R\delta}$$

for some $\delta > 0$, since we can choose θ and δ sufficiently small and $R < n$.

Next, we consider S_2. Identically as in Equation (5.5), we can use Lemma 3.9 to say

$$S_2 \ll \sum_{r=1}^{R-1} \sum_{\substack{d_r < k < d_r + r \\ k \le r(1+\varepsilon)}} k q^{d_r - k} \sum_{\deg h < k - d_r} \sum_{\deg F = r} \max_{(a,F)=1} \left| \sum_{\substack{x \in \mathcal{M}_k \\ (x,F)=1}} \mu(x) e_F(ah\bar{x}) \right|.$$

Then applying Theorem 2.4 and using $k \le r(1+\varepsilon)$ and $r \le R$ yields

$$S_2 \ll_\theta \sum_{r=1}^{R-1} \sum_{\substack{d_r < k < d_r + r \\ k \le r(1+\varepsilon)}} k \left(q^{5r/4 + 5k/8} + q^{r + 9k/10} + q^{7r/6 + 13k/18} \right) q^{\theta r/2}$$
$$\ll_\theta q^{R + 9R/10 + 9R\varepsilon/10 + \theta R},$$

where here, we have used $kR \ll_\theta q^{R\theta/2}$. Using $\varepsilon = (1/10 - \omega' - 2\theta)10/9$ and $R \le d + R\omega' = n - R + R\omega'$ means

$$S_2 \ll_\theta q^{n - R\theta},$$

as desired.

Finally, to bound S_3, we let $\beta, \beta' > 0$ (which we will take to be sufficiently small as needed), and we can apply Lemma 3.7. Note that working identically to Equation (5.6), this will only hold for

$$q > p^2 e^2 \left(1 + \frac{18}{1 - 10\omega'} \right)^2.$$

Regardless, regarranging S_3 and then applying Lemma 3.7 means

$$S_3 = \sum_{r=1}^{R-1} \sum_{\deg F = r} \max_{(a,F)=1} \sum_{\substack{d_r < k < d_r + r \\ k > r(1+\varepsilon)}} k \sum_{\substack{y \in \mathcal{M}_{r + d_r - k} \\ (y,F)=1}} \left| \sum_{\substack{x \in \mathcal{M}_k \\ x \equiv \bar{y} a \pmod{F}}} \mu(x) \right|$$
$$\ll_{\beta, \beta', \theta} \sum_{r=1}^{R-1} \sum_{\substack{d_r < k < d_r + r \\ k > r(1+\varepsilon)}} q^{n - \beta/p(k-r) + n\beta'},$$

where we have used $k \ll_{\beta'} q^{n\beta'}$. We now deal with two parts of this sum separately. For $r < R/3$ (which means $r < n/3$), we make the substitution $k > d_r = n - r$ to give

$$\sum_{r=1}^{R/3} \sum_{\substack{d_r < k < d_r + r \\ k > r(1+\varepsilon)}} q^{n - \beta/p(k-r) + n\beta'} \ll_{\beta'} \sum_{r=1}^{R/3} q^{n - \beta/p(n-2r) + 2n\beta'}$$
$$\ll_{\beta'} q^{n - n(\beta/(3p) - 3\beta')}$$
$$\ll q^{n - R(\beta/(3p) - 3\beta')},$$

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which is admissable for β and β' chosen suitably. Finally, for $r \geq R/3$, we make the substitutions $k > r(1+\varepsilon)$, $\varepsilon = (1/10 - \omega' - 2\theta)10/9$ and $d_r = n - r$ to give

$$\begin{aligned}\sum_{r=R/3}^{R-1} \sum_{\substack{d_r<k<d_r+r \\ k>r(1+\varepsilon)}} q^{n-\beta/p(k-r)+n\beta'} &\ll_{\beta'} \sum_{r=R/3}^{R-1} q^{n-r\varepsilon\beta/p+2n\beta'} \\ &\ll_{\beta'} q^{n-R\varepsilon\beta/(3p)+3n\beta'},\end{aligned}$$

which is admissible, since we have assumed that $n = O(r)$, and we may choose β, β' suitably.

Combining our estimates for S_1, S_2, and each part of S_3 gives the result.

References

[1] C. Bagshaw, *Square-free smooth polynomials in residue classes and generators of irreducible polynomials*. Proc. Amer. Math. Soc. **151**(2023), no. 03, 1017–1029.

[2] C. Bagshaw, *Bilinear forms with Kloosterman and Gauss sums in function fields*. Finite Fields Appl. **2024**(2024), no. 94, 102356.

[3] C. Bagshaw and B. Kerr, Lattices in function fields and applications.Preprint, 2023, https://arxiv.org/abs/2304.05009.

[4] C. Bagshaw and I. Shparlinski, *Energy bounds, bilinear forms and their applications in function fields*. Finite Fields Appl. **82**(2022), 102048.

[5] R. Baker, *Kloosterman sums with prime variable*. Acta Arith. **156**(2012), no. 4, 351–372.

[6] W. Banks, A. Harcharras, and I. Shparlinski, *Short Kloosterman sums for polynomials over finite fields*. Canad. J. Math. **55**(2003), no. 2, 225–246.

[7] A. Bhowmick, T. H. Lê, and Y. R. Liu, *A note on character sums in finite fields*. Finite Fields Appl. **46**(2017), 247–254.

[8] J. Bourgain, *More on the sum-product phenomenon in prime fields and its applications*. Int. J. Number Theory **1**(2005), no. 01, 1–32.

[9] J. Bourgain and M. Z. Garaev, *Kloosterman sums in residue rings*. Acta Arith. **164**(2014), no. 1, 43–64.

[10] J. Bourgain and M. Z. Garaev, *Sumsets of reciprocals in prime fields and multilinear Kloosterman sums*. Izv. Math. **78**(2014), 656.

[11] J. Cilleruelo and I. Shparlinski, *Concentration of points on curves in finite fields*. Monatsh. Math. **171**(2013), no. 3, 315–327.

[12] É. Fouvry and P. Michel, *Sur certaines sommes d'exponentielles sur les nombres premiers*. Ann. Sci. Ec. Norm. Supér. **31**(1998), no. 1, 93–130.

[13] É. Fouvry and I. Shparlinski, *On a ternary quadratic form over primes*. Acta Arith. **150**(2011), no. 3, 285–314.

[14] J. Friedlander and H. Iwaniec, *The Brun-Titchmarsh theorem*, London Math. Soc. Lecture Note Ser., Vol **247**, 1997.

[15] M. Z. Garaev, *Estimation of Kloosterman sums with primes and its application*. Math. Notes **88**(2010), 330–337.

[16] D. Han, *A note on character sums in function fields*. Finite Fields Appl. **68**(2020), 101734.

[17] D. Hayes, *The expression of a polynomial as a sum of three irreducibles*. Acta. Arith. **11**(1966), 461–481.

[18] A. Irving, *Average bounds for Kloosterman sums over primes*. Funct. Approx. Comment. Math. **51**(2014), no. 2, 221–235.

[19] H. Iwaniec and E. Kowalski, *Analytic number theory*, American Mathematical Soc., Vol. **53**, 2004.

[20] A. Karatsuba, *Fractional parts of functions of a special form*. Izv. Math. **59**(1995), 721–740.

[21] M. A. Korolev and M. E. Changa, *New estimate for Kloosterman sums with primes*. Math. Notes **108**(2020), no. 1–2, 87–93.

[22] W. Luo, *Bounds for incomplete hyper-Kloosterman sums*. J. Number Theory **75**(1999), no. 1, 41–46.

[23] M. Rosen, *Number theory in function fields, Vol. 210, Springer Science & Business* Media, 2013.

[24] W. Sawin, *Square-root cancellation for sums of factorization functions over square-free progressions in Fq[t]*. Acta Math. (to appear), 2023.

[25] W. Sawin and M. Shusterman, *On the Chowla and twin primes conjectures over Fq[T]*. Ann. Math. **196**(2022), 457–506.
[26] W. Sawin and M. Shusterman, *Möbius cancellation on polynomial sequences and the quadratic Bateman—Horn conjecture over function fields*. Invent. Math. **229**(2022), no. 2, 751–927.
[27] I. Shparlinski and A. Zumalacárregui, *Sums of inverses in thin sets of finite fields*. Proc. Amer. Math. Soc. **146**(2018), 1377–1388.
[28] I. E. Shparlinski, *Bounds of incomplete multiple Kloosterman sums*. J. Number Theory **126**(2007), no. 1, 68–73.

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