INTEGRAL OPERATOR METHODS FOR GENERALIZED AXIALLY SYMMETRIC POTENTIALS IN $(n+1)$ VARIABLES *

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(Received 16 October 1964)

1. Introduction

In this paper we shall use the integral operator method of Bergman, B[1−6], to investigate solutions of the partial differential equation

$$\frac{\partial^2 u}{\partial x_1^2}+\cdots+\frac{\partial^2 u}{\partial x_n^2}+\frac{\partial^2 u}{\partial \rho^2}+\frac{s}{\rho}\frac{\partial u}{\partial \rho}=0, \tag{1.1}$$

where $s > -1$. In particular, information concerning the growth, and location of singularities, of solutions of (1.1) will be obtained. Equations of the form (1.1) with $s = 1, 2, \cdots$ arise from the $(n+k+1)$-dimensional Laplace equation $\Delta_{n+k+1}u = 0$ in the "axially symmetric" coordinates $x_1, \cdots x_n, \rho$ where the relationship between cartesian and "axially symmetric" coordinates is given by

$$\begin{aligned} X_1 &\to x_1 \\ &\vdots \\ X_n &\to x_n \\ X_{n+1}^2+\cdots+X_{n+k+1}^2 &\to \rho^2 \end{aligned}$$

We shall refer to equation (1.1) as the generalized axially symmetric potential equation of nth order (GASPN), as is done in W[1]. The special case $n = 1$ has been extensively studied. Reference is made to Gilbert, G[1−4], Erdélyi, E[1], Henrici, H[2−4], and Gilbert and Howard, GH[1−2].

2. Integral operators generating solutions to the GASPN

We first obtain a set of particular solutions to the GASPN. The motivation and notation is mainly that of Appel and Kampé de Fériet, A-de F[1],

* This research was supported at the University of Maryland in part by the Air Force Office of Scientific Research under Grant AFOSR 400−64 and by the National Science Foundation under Grants NSFGP−2067 and GP−3937.

to which the reader is referred for details.

We define $W_{m_1\cdots m_n}(x_1,\cdots,x_n,\rho)$ by means of the generating function $W\equiv[(x_1-a_1)^2+\cdots+(x_n-a_n)^2+\rho^2]^{-\frac{1}{2}(n+s-1)}$ as follows:

$$W=\sum_{m_1\cdots m_n=0}^{\infty} a_1^{m_1}\cdots a_n^{m_n}W^{(s)}_{m_1\cdots m_n}. \tag{2.1}$$

From the expression for W it is clear that

$$W^{(s)}_{m_1\cdots m_n}=\frac{(-1)^{\mu}}{(m_1)!\cdots(m_n)!}\frac{\partial^{\mu}}{\partial x_1^{m_1}\cdots\partial x_n^{m_n}}\left(\frac{1}{r^{n+s-1}}\right) \tag{2.2}$$

where $\mu=m_1+\cdots+m_n$ and $r^2=x_1^2+\cdots+x_n^2+\rho^2$.

We now show that the $W^{(s)}_{m_1\cdots m_n}$ are solutions of the GASPN. First one notes, by an elementary computation, that

$$\rho\frac{\partial}{\partial x_j}\left(\frac{1}{r^{n+s-1}}\right)-x_j\frac{\partial}{\partial\rho}\left(\frac{1}{r^{n+s-1}}\right)=0 \tag{2.3}$$

for $j=1,2,\cdots,n$. From this we get

$$\begin{aligned}\rho\frac{\partial^{m_1}}{\partial x_1^{m_1}}\cdots\frac{\partial^{m_{j-1}}}{\partial x_{j-1}^{m_{j-1}}}\frac{\partial^{m_j+2}}{\partial x_j^{m_j+2}}\frac{\partial^{m_{j+1}}}{\partial x_{j+1}^{m_{j+1}}}\cdots\frac{\partial^{m_n}}{\partial x_n^{m_n}}\left(\frac{1}{r^{n+s-1}}\right)\\ =\frac{\partial^{m_j+1}}{\partial x_j^{m_j+1}}\left\{x_j\frac{\partial}{\partial\rho}\frac{\partial^{m_1}}{\partial x_1^{m_1}}\cdots\frac{\partial^{m_{j-1}}}{\partial x_{j-1}^{m_{j-1}}}\frac{\partial^{m_{j+1}}}{\partial x_{j+1}^{m_{j+1}}}\cdots\frac{\partial^{m_n}}{\partial x_n^{m_n}}\left(\frac{1}{r^{n+s-1}}\right)\right\}.\end{aligned} \tag{2.4}$$

Using the Leibniz product rule on the right hand side of (2.4) and recalling (2.2) we see that (2.4) can be written as

$$\rho\frac{\partial^2}{\partial x_j^2}W^{(s)}_{m_1\cdots m_n}=\frac{\partial^2}{\partial\rho\,\partial x_j}W^{(s)}_{m_1\cdots m_n}+(m_j+1)\frac{\partial}{\partial\rho}W^{(s)}_{m_1\cdots m_n} \tag{2.5}$$

$j=1,2,\cdots,n$. Adding these equations for $j=1,2,\cdots,n$ we get

(2.6)

$$\begin{aligned}&\rho\left[\frac{\partial^2}{\partial x_1^2}W^{(s)}_{m_1\cdots m_n}+\cdots+\frac{\partial^2}{\partial x_n^2}W^{(s)}_{m_1\cdots m_n}\right]\\ &\quad=\frac{\partial}{\partial\rho}\left[x_1\frac{\partial}{\partial x_1}W^{(s)}_{m_1\cdots m_n}+\cdots+x_n\frac{\partial}{\partial x_n}W^{(s)}_{m_1\cdots m_n}\right]+(\mu+n)\frac{\partial}{\partial\rho}W^{(s)}_{m_1\cdots m_n}.\end{aligned}$$

But one readily shows from (2.1) and the definition of W that $W^{(s)}_{m_1\cdots m_n}$ is a homogeneous function in $x_1,\cdots,x_n,\rho$ of degree $-(\mu+n+s-1)$, so

$$\begin{aligned}x_1\frac{\partial}{\partial x_1}W^{(s)}_{m_1\cdots m_n}+\cdots+x_n\frac{\partial}{\partial x_n}W^{(s)}_{m_1\cdots m_n}&=-(\mu+n+s-1)W^{(s)}_{m_1\cdots m_n}\\ &\quad-\rho\frac{\partial}{\partial\rho}W^{(s)}_{m_1\cdots m_n}.\end{aligned}$$

Substituting this in (2.6) gives the equation

$$\Delta_n W^{(s)}_{m_1\cdots m_n} + \frac{[\mu+n+s-1-\mu-n]}{\rho}\frac{\partial}{\partial\rho} W^{(s)}_{m_1\cdots m_n}$$

$$(2.7) \qquad + \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho} W^{(s)}_{m_1\cdots m_n}\right)$$

$$\equiv \Delta_n W^{(s)}_{m_1\cdots m_n} + \frac{\partial^2}{\partial\rho^2} W^{(s)}_{m_1\cdots m_n} + \frac{s}{\rho}\frac{\partial}{\partial\rho} W^{(s)}_{m_1\cdots m_n} = 0,$$

which shows the GASPN is satisfied by $W^{(s)}_{m_1\cdots m_n}$. One notes that up to this point the variables (x_i, ρ) and the parameter s may be taken to be arbitrary complex quantities if desired. The reason for the restriction on s will become apparent later.

We shall now obtain an integral operator generating solutions to the GASPN. Defining first the polynomials $V^{(s)}_{m_1\cdots m_n}(\xi_1, \cdots, \xi_n)$ by the relation

$$(2.8) \qquad (1-2\sum_1^n a_i\xi_i + \sum_1^n a_i^2)^{-\frac{1}{2}(n+s-1)} = \sum_{m_1,\cdots,m_n=0}^{\infty} a_1^{m_1}\cdots a_n^{m_n} V^{(s)}_{m_1\cdots m_n}(\xi_i\cdots\xi_n)$$

we obtain a connection between $V^{(s)}_{m_1\cdots m_n}$ and $W^{(s)}_{m_1\cdots m_n}$ by noting that

$$\sum_{m_1,\cdots,m_n=0}^{\infty} a_1^{m_1}\cdots a_n^{m_n} W^{(s)}_{m_1\cdots m_n} = [x_1^2+\cdots+x_n^2+\rho^2-2\sum_1^n a_i x_i + \sum_1^n a_i^2]^{-\frac{1}{2}(n+s-1)}$$

$$(2.9) \qquad = (r^2)^{-\frac{1}{2}(n+s-1)}\left[1-2\sum_1^n \frac{a_i}{r}\left(\frac{x_i}{r}\right) + \sum_1^n \frac{a_i^2}{r^2}\right]^{-\frac{1}{2}(n+s-1)}$$

$$= r^{-(n+s-1)} \sum_{m_1,\cdots,m_n=0}^{\infty}\left(\frac{a_1}{r}\right)^{m_1}\cdots\left(\frac{a_n}{r}\right)^{m_n} V^{(s)}_{m_1\cdots m_n}\left(\frac{x_1}{r},\cdots,\frac{x_n}{r}\right).$$

Thus $W_{m_1,\cdots,m_n}(x_1,\cdots,x_n,\rho) \equiv r^{-(\mu+n+s-1)} V^{(s)}_{m_1\cdots m_n}(\xi_1,\cdots,\xi_n)$, where $\xi_i = x_i/r$ and $\mu = m_1+\cdots+m_n$.

For suitably chosen contours $\mathfrak{C}_k$, $(k = 1, 2, \cdots, n)$ (homologous to zero), and such that $\prod_{k=1}^n \mathfrak{C}_k$ lies outside a sufficiently large polycylinder[1], we have by Cauchy's formula for several complex variable that

$$\left(\frac{1}{2\pi i}\right)^n \iint_{\mathfrak{C}_1\cdots\mathfrak{C}_n} \frac{(\zeta_1^{-k_1}\cdots\zeta_n^{-k_n})\dfrac{d\zeta_1\cdots d\zeta_n}{\zeta_1\cdots\zeta_n}}{(1-2\sum_{i=1}^n \zeta_i\xi_i + \sum_{i=1}^n \zeta_i^2)^{\frac{1}{2}(n+s-1)}}$$

$$(2.10) \qquad = \left(\frac{1}{2\pi i}\right)^n \int_{\mathfrak{C}_1}\cdots\int_{\mathfrak{C}_n}(\zeta_1^{-k_1}\cdots\zeta_n^{-k_n})\frac{d\zeta_1\cdots d\zeta_n}{\zeta_1\cdots\zeta_n}$$

$$\times\left\{\sum_{m_1,\cdots,m_n=0}^{\infty}\zeta_1^{m_1}\cdots\zeta_n^{m_n} V^{(s)}_{m_1\cdots m_n}(\xi_1,\cdots,\xi_n)\right\}$$

$$= V^{(s)}_{k_1,\cdots,k_n}(\xi_1,\cdots,\xi_n).$$

[1] We might restrict these further, by demanding that each $\mathfrak{C}_k$ lies outside a suitably large disk.

Multiplying both sides by $r^{-(k_1+\cdots+k_n)-(n+s-1)}$ gives

$$W^{(s)}_{k_1,\cdots,k_n}(x_1,\cdots,x_n,\rho) = \frac{r^{-(n+s-1)}}{(2\pi i)^n}\int_{\mathfrak{C}_1}\cdots\int_{\mathfrak{C}_n}[(r\zeta_1)^{-k_1}\cdots(r\zeta_n)^{-k_n}] \times\left\{(1-2\sum_{i=1}^n\zeta_i\xi_i+\sum_{i=1}^n\zeta_i^2)^{-\frac{1}{2}(n+s-1)}\right\}\frac{d\zeta_1\cdots d\zeta_n}{\zeta_1\cdots\zeta_n}. \tag{2.11}$$

If the function,

$$g(\zeta r) \equiv \sum_{k_1,\cdots,k_n=0}^{\infty} a_{k_1\cdots k_n}(\zeta_1 r)^{-k_1}\cdots(\zeta_n r)^{-k_n}, \tag{2.12}$$

converges uniformly on the compact set $(\zeta r)\,\varepsilon\prod_{k=1}^n\mathfrak{C}_k$, we may interchange the orders of summation and integration, and thereby obtain an integral representation for solutions of GASPN in the form,

$$\begin{aligned} W(X) &\equiv \sum_{k_1,\cdots,k_n}^{\infty} a_{k_1\cdots k_n}W^{(s)}_{k_1\cdots k_n}(x_1,\cdots,x_n,\rho) = \mathfrak{A}^{(s)}_n[g(\zeta r)],\\ \mathfrak{A}^{(s)}_n[g] &= \frac{r^{-(n+s-1)}}{(2\pi i)^n}\int_{\mathfrak{D}}\frac{g(\zeta r)d\zeta/\zeta}{[1-2(\zeta,\xi)+||\zeta||^2]^{\frac{1}{2}(n+s-1)}}, \end{aligned} \tag{2.13}$$

where

$$\mathfrak{D}\equiv\prod_{k=1}^n\mathfrak{C}_k,\quad (\zeta,x)=\sum_{i=1}^n\zeta_i\xi_i,\quad ||\zeta||^2=(\zeta,\zeta),$$

and

$$d\zeta/\zeta\equiv\frac{d\zeta_1\cdots d\zeta_n}{\zeta_1\cdots\zeta_n}.$$

We remark that this representation for $W(X)$, $[X\equiv(x_1,\cdots,x_n,\rho)]$ is valid in the small; it is valid in a sufficiently small neighborhood of an initial *point of definition* X^0 for the function element represented by the series development $\sum_k a_k W^{(s)}_k(X)$, $(k\equiv k_1,\cdots,k_n)$.

In the future we shall refer to the operator $\mathfrak{A}[g]\equiv\mathfrak{A}^{(s)}_n[g]$ which maps holomorphic functions of n complex variables (2.12) onto solutions of the GASPN equation with the form (2.13), as the GASPN operator of the first kind.

We shall now show it is possible to find an integral operator $\mathfrak{A}^{-1}[W]$, which plays the role of an inverse operator for $\mathfrak{A}[g]$. We recall first that the polynomials $U^{(s)}_{m_1,\cdots,m_n}(\xi_1,\cdots,\xi_n)$, A. de F. [1], defined by the generating function

$$\begin{aligned}[((\zeta,\xi)-1)^2+||\zeta||^2(1-||\xi||^2)]^{-s/2} &= \sum_{M=0}^{\infty}\zeta^M U^{(s)}_M(\xi)\\ &\equiv\sum_{m_1,\cdots,m_n=0}^{\infty}\zeta_1^{m_1}\cdots\zeta_n^{m_n}U^{(s)}_{m_1\cdots m_n}(\xi_1,\cdots,\xi_n),\end{aligned} \tag{2.14}$$

for a biorthogonal system with the $V_M^{(s)}(\xi)$. Indeed, Erdélyi, E[2] (pg. 277, Vol. 2) gives the result that

$$(2.15)\qquad \int_{0\leq\|\xi\|\leq 1}(1-\|\xi\|^2)^{\frac{1}{2}(s-1)}V_M^{(s)}(\xi)U_L^{(s)}(\xi)\,d^n\xi$$

vanishes, except when $M\equiv L$, i.e. $m_i = l_i$ $(i = 1, 2, \cdots, n)$, and has the value

$$\frac{2\pi^{\frac{1}{2}n}}{(2m+n+s-1)}\,\frac{\Gamma(\frac{1}{2}s+1)}{\Gamma(\frac{1}{2}[n+s-1])}\,\frac{\Gamma(s+m)}{\Gamma(s)m_1!\cdots m_n!},$$

when $M\equiv L$, where $m = m_1+\cdots+m_n$. Using this result we see by formal computation that

$$(2.16)\qquad \begin{aligned} g(\zeta r) &\equiv \sum_{K=0}^{\infty} a_K(\zeta_1 r)^{-k_1}\cdots(\zeta_n r)^{-k_n}\\ &= \int_{\|x\|\leq 1}\left\{\sum_{K=0}^{\infty} a_K r^{k_1+\cdots+k_n}V_K^{(s)}(\xi)\right\}\\ &\quad\times\left\{\frac{\Gamma(s)\Gamma(\frac{1}{2}[n+s-1])}{2\pi^{\frac{1}{2}n}\Gamma(\frac{1}{2}s+1)}\sum_{K=0}^{\infty}\left[\frac{(k_1)!\cdots(k_n)!(2k+n+s-1)}{\Gamma(k+s)(\zeta_1 r)^{k_1}\cdots(\zeta_n r)^{k_n}}U_K^{(s)}(\xi)\right]\right.\\ &\quad\left.\times\,(1-\|\xi\|^2)^{\frac{1}{2}(s-1)}\right\}d^n\xi, \end{aligned}$$

where $K\equiv k_1, k_2, \cdots, k_n$, and $k\equiv k_1+k_2+\cdots+k_n$. The restriction $s>-1$ is clearly necessary to ensure the convergence of the integral in (2.16).

In order to show that the *kernel*,

$$\mathfrak{K}(\xi;\zeta r)\equiv(1-\|\xi\|^2)^{\frac{1}{2}(s-1)}\left\{\sum_{K=0}\frac{k_1!\cdots k_n!(2k+n+s-1)}{\Gamma(k+s)(\zeta_1 r)^{k_1}\cdots(\zeta_n r)^{k_n}}U_K^{(s)}(\xi)\right\}$$

converges to a holomorphic function of the variables[2] $(\xi;\zeta r)$ in a domain $\mathfrak{B}^{(2n)}\subset C^{2n}$, such that there exists a domain

$$\mathfrak{B}^{(n)}\subset\bigcup_{\xi\in\Omega}\mathfrak{B}^{(2n)}\cap\{\xi=\xi^0\}\equiv\mathfrak{B}^{(2n)}\cap\Omega,\text{ where }\Omega\equiv\{0\leqq\|\xi\|\leqq 1;\ \xi_i=\text{real}\}$$

we compare the general term in $\mathfrak{K}$ with the general term of the generating function for the $U_k^{(s)}$. The purpose for this is to show the existence of a germ, or function element $g(\zeta r)$ which exists in $\mathfrak{B}^{(n)}$.

The associated radii of convergence $\tilde{r}_1, \cdots, \tilde{r}_n$ of a power series $\sum a_K\zeta_1^{k_1}\cdots\zeta_n^{k_n}$ satisfy the relation [F.1] (pg. 48)

$$\lim_{k\to\infty}(|a_K|\tilde{r}^k)^{1/k}=1$$

where $k = k_1+\cdots+k_n$, $\tilde{r}^k = \tilde{r}_1^{k_1}\cdots\tilde{r}_n^{k_n}$, etc. Hence for each fixed value

[2] The variables ξ_i $(i = 1, \cdots, n)$ are complex unless otherwise noted.

of ξ, the associated radii of convergence of $\sum_k U_K^{(s)}(\xi)\zeta^k$ satisfy,

$$\frac{1}{\tilde{r}_1 \cdots \tilde{r}_n} = \lim_{k\to\infty} |U_K^{(s)}(\xi)|^{1/k};$$

that is, for $|\zeta_k| < \tilde{r}_k$ $(k = 1, \cdots, n)$ the series representation of the generating function converges. From (2.14) it is clear that for $\xi \in \Delta \equiv \prod_{i=1}^n \{|\xi_i| \leqq \varepsilon\}$, where $\varepsilon > 0$ is sufficiently small, we may compute finite associated radii $(\tilde{r}_1, \cdots, \tilde{r}_n)$. Likewise, one may compute the associated radii of convergence for the series in $\mathfrak{K}(\xi; \zeta r)$, for $\xi \in \Delta$, to be $(\tilde{r}_1, \tilde{r}_2, \cdots, \tilde{r}_n)$ also since

$$\frac{1}{\tilde{r}_1 \cdots \tilde{r}_n} = \lim_{k\to\infty} \left(\frac{k_1! \cdots k_n!(2k+n+s-1)}{\Gamma(k+s)}\, U_K^{(s)}(\xi)\right)^{1/k}$$
$$= \lim_{k\to\infty} (U_K^{(s)}(\xi))^{1/k}.$$

By the Weierstrass comparison theorem plus Hartogs' theorem [B.M.1] pg. 141 this establishes $\mathfrak{K}(\xi; r\zeta)$ as a holomorphic function of $(2n)$ complex variables in $\Delta\times\Delta^*(\Delta^* \equiv \prod_{i=1}^n \{|r\zeta_i| > \tilde{r}_i^{-1}\})$. In fact, it is clear that the series for $\mathfrak{K}(\xi; r\zeta)$ converges in each polycylinder $\Delta\times\Delta^*$, which does not meet $\mathfrak{S} \equiv \{((\zeta, \xi)-1)^2+||\zeta||^2(1-||\xi||^2) = 0\}$.

If $\xi \in \Omega \equiv \{0 \leqq ||\xi|| \leqq 1;\ \xi_i = \text{real}\}$, then for $\zeta = iz^0 \equiv i(z_1^0, \cdots, z_n^0)$ we have $(\xi, \zeta) = i(\xi, z^0) \neq$ a real, and $||\zeta||^2 = (iz^0, iz^0) = -||z^0||^2$, where $z_i =$ real. Also, we have that $(i(\xi, z^0)-1/-||z^0||^2) \neq (||\xi||^2-1)$ for all $\xi \in \Omega^n$. We conclude that there exists a complex neighborhood of $\zeta = iz^0$, $\mathfrak{N}(iz^0)$, such that $\mathfrak{N}(iz^0) \cap \mathfrak{S} = \phi$. We conclude $\mathfrak{K}(\xi; r\zeta)$ is a holomorphic function of the n complex variables $(r\zeta_1, \cdots, r\zeta_n)$ for $(r\zeta) \in \mathfrak{N}(iz^0)$ for each fixed $\xi \in \Omega^n$. This shows that the domain $\mathfrak{B}^{(n)} \neq \phi$. Consequently we realize that our integral representation,

$$\mathfrak{A}^{-1}[w] = \frac{\Gamma(s)\Gamma(n+s-1)}{2\pi^{\frac{1}{2}n}\Gamma(\frac{1}{2}s+1)} \int_{||\xi|| \leq 1} w(\xi)\mathfrak{K}(\xi; \zeta r) d^n\xi,$$

constitutes an inverse integral operator for GASPN in the following sense; given a function $w(X)$ with a representation (2.13), then there exists a holomorphic function element $g(\zeta r)$ with the local representation $g(\zeta r) = \mathfrak{A}^{-1}[w]$ given above.

In the following sections we will investigate further the analytic properties of these operators.

3. Singularities of functions of several complex variables represented by integrals

In this section we prove several results concerning holomorphic functions represented by integrals whose kernels are singular on analytic sets.

These results are generalizations of those appearing in [G.3,4,]. The results, for the most part are more general than we actually need here; we prove them in this form for future reference.

LEMMA 3.1. *Let* $K(z;\zeta) \equiv K(z_1, \cdots, z_n; \zeta_1, \cdots, \zeta_n)$ *be an analytic function of* $(m+n)$ *complex variables, and regular analytic in the product domain* $(\mathfrak{D}^{(n)} \times \mathfrak{B}^{(m)})$, *where* $\mathfrak{B}^{(m)} \equiv \prod_{\mu=1}^{m} \mathfrak{B}_{\mu}^{(1)}$. *Furthermore, let the singularities of* $K(z;\zeta)$ *lie on the analytic set* $\mathfrak{S}^{(n+m-1)} \equiv \{S(z;\zeta) = 0\}$. *Then the analytic function* $F(z)$ *represented in* $\mathfrak{D}^{(n)}$ *by*

$$F(z) = \left(\frac{1}{2\pi i}\right)^m \int_{\mathfrak{L}_1} \cdots \int_{\mathfrak{L}_m} K(z;\zeta) d\zeta$$

has a continuation which is regular at all points (z) *that may be connected to* $\mathfrak{D}^{(n)}$ *by an arc* $\mathfrak{C}_z^{(1)}$ *such that no point of* $\mathfrak{C}_z^{(1)}$ *lies on the set* $\bigcap_{\mu=0}^{m} \mathfrak{S}_{\mu}^{(n+m-1)}$, $\mathfrak{S}_0^{(n+m-1)} \equiv \mathfrak{S}^{(n+m-1)}$, *and* $\mathfrak{S}_{\mu}^{(n+m-1)} \equiv \{S_{\zeta_\mu}(z;\zeta) = 0\}$, $(\mu \geqq 1)$.

(Remark: The superscripts in $\mathfrak{D}^{(n)}$ indicate the complex dimensionality of the domain.)

PROOF. The representation for $F(z)$ is clearly valid for all points (z) which may be reached by analytic continuation along a contour $\mathfrak{C}_{(z)}^{(1)}$ starting from some point in $\mathfrak{D}^{(n)}$ providing no point of $\mathfrak{C}_{(z)}^{(1)}$ lies in $\mathfrak{S}^{(n+m-1)}$ for $(\zeta) \in \Lambda^{(m)} \equiv \prod_{\mu=1}^{m} \mathfrak{L}_{\mu}$. These points (z) are contained in what we shall refer to as the initial domain of definition of $F(z)$. It is clear that it may be possible to extend $F(z)$ by deforming its domain of integration, provided that in so doing we do not pass over a singularity of the integrand. We ask, when is it no longer possible to avoid an "encroaching" singularity and continue $F(z)$ past a point (z_1)? This may be answered in the negative sense by realizing when it is always possible to deform $\Lambda^{(m)}$ about a point $(\zeta) = (\alpha)$ for which $S(z_1;\alpha) \equiv 0$. If not *all* of the terms $\partial S(z_1;\alpha)/\partial\zeta_\mu$ vanish [3], then we may approximate $S(z_1;\zeta)$ in a suitably small poly-disk $\Delta^{(m)}(\varepsilon) \equiv \prod_{\mu=1}^{m}\{|\zeta_\mu - \alpha_\mu| < \varepsilon_\mu\}$ by

$$S(z_1;\zeta) = (\zeta_1 - \alpha_1)\frac{\partial S}{\partial \zeta_1}(z_1;\alpha) + \cdots + (\zeta_m - \alpha_m)\frac{\partial S}{\partial \zeta_m}(z_1;\alpha),$$

where we assume without loss of generality that $\partial S/\partial\zeta_1(z_1,\alpha) \neq 0$. It is clear that in this case we may choose arbitrarily $(m-1)$ of the contours, that is we choose $\zeta_\mu = \alpha_\mu + \varepsilon_\mu/2\, e^{i\psi_\mu}$ for $\mu = 2, 3, \cdots, m$. The last contour we choose to be $\zeta_1 = \alpha_1 + \rho e^{i\psi}$ where $\rho < \varepsilon_1$ is such that

$$\rho \neq \left| \left(\frac{\partial S(z_1,\alpha)}{\partial \zeta_1}\right)^{-1} \left(\sum_{\mu=2}^{m} \tfrac{1}{2}\varepsilon_\mu \frac{\partial S(z_1;\alpha)}{\partial \zeta_\mu}\right)\right|.$$

[3] *We are answering the question by finding when it is possible to avoid a singularity, not when it may not be possible to avoid it.*

https://doi.org/10.1017/S1446788700027762 Published online by Cambridge University Press

In this case $S(z_1; \zeta) \neq 0$ for (ζ) in the set $\lambda^{(m)} \equiv \{|\zeta_1 - \alpha_1| = \rho\} \times \prod_{\mu=2}^{m} \{|\zeta_\mu - \alpha_\mu| = \varepsilon_\mu/2\}$, and we may avoid the singularity $(\zeta) = (\alpha)$ by deforming $\Lambda^{(m)}$ to follow a portion of $\lambda^{(m)}$ about this point. This concludes our proof.

LEMMA 3.2. *Let $f_\nu(z; \zeta) \equiv f_\nu(z_1, \cdots, z_n; \zeta_1, \cdots, \zeta_m)$ $(\nu = 1, 2, \cdots)$ be analytic functions of $(n+m)$ complex variables, and regular-analytic in the product domain $(\mathfrak{D}^{(n)} \times \prod_{\mu=1}^{m} \mathfrak{B}_\mu^{(1)})$. Furthermore, let the singularities of $f_\nu(z; \zeta)$ lie on the analytic set $\mathfrak{S}_{\nu,0}^{(n+m-1)} \equiv \{S_\nu(z; \zeta) = 0\}$. Then the continuation of the function element given in $\mathfrak{D}^{(n)}$ by $F(z) = (1/2\pi i)^m \int_{\mathfrak{L}_1} \cdots \int_{\mathfrak{L}_m} f_1(z; \zeta) f_2(z, \zeta) d\zeta$, where $\mathfrak{L}_\mu$ $(\mu = 1, 2, \cdots, m)$ are rectifiable curves, is regular at all points not contained in the set*

$$\left(\bigcup_{\nu=1}^{2} \bigcap_{\mu=0}^{m} \mathfrak{S}_{\nu,\mu}^{(n+m-1)}\right) \left[\left(\bigcap_{\mu=2}^{m} \mathfrak{H}_{1,\mu}^{(n+m-1)}\right) \cap \mathfrak{S}_{1,0}^{(n+m-1)} \cap \mathfrak{S}_{2,0}^{(n+m-1)}\right],$$

where

$$\mathfrak{H}_{\nu,\mu}^{(n+m-1)} \equiv \left\{\frac{\partial(S_1, S_2)}{\partial(\zeta_\nu, \zeta_\mu)} = 0\right\} \quad \textit{and} \quad \mathfrak{S}_{\nu,\mu}^{(n+m-1)} \equiv \left\{\frac{\partial S_\nu(z, \zeta)}{\partial \zeta_\mu} = 0\right\}.$$

PROOF. Applying the envelope method to each $f_\nu(z; \zeta)$ separately yields $\bigcap_{\mu=0}^{m} \mathfrak{S}_{\nu,\mu}^{(n+m-1)}$. Applying the *Hadamard method* [G.1] and [H.1] with respect to the variable ζ_1, assuming at least one of the terms $\partial S_\nu/\partial \zeta_1 \neq 0$, yields[4] $\mathfrak{H}_{1,\mu}^{(n+m-1)} \cap \mathfrak{S}_{1,0}^{(n+m-1)} \cap \mathfrak{S}_{2,0}^{(n+m-1)}$ for each $\mu \geqq 2$. The condition for a point $(z, \zeta) \in \mathfrak{H}_{1,\mu}^{(n+m-1)}$ is that $(\partial S_1/\partial \zeta_1)/(\partial S_2/\partial \zeta_1) = (\partial S_1/\partial \zeta_\mu)/(\partial S_2/\partial \zeta_\mu)$ for $\mu \geqq 2$, if the terms $\partial S_2/\partial \zeta_\mu (\mu \geqq 1)$ do not vanish. This is clearly the same as the condition $(z, \zeta) \in \mathfrak{H}_{\nu,\mu}$. (If terms $\partial S_2/\partial \zeta_\mu = 0$ the condition is trivially satisfied.) We notice therefore, that it does not matter which one of the ζ_μ-variables is first eliminated by the Hadamard method, i.e.

$$\mathfrak{H}_{1,\mu}^{(n+m-1)} = \mathfrak{H}_{\nu,\mu}^{(n+m-1)}, \qquad (\nu = 1, 2, \cdots, m).$$

This concludes our proof.

A somewhat less sharp estimate of the set of possible singularities may be obtained by applying the envelope method to the entire integrand. If

$$\mathfrak{A}_0^{(n+m-1)} \equiv \{S_1(z; \zeta) S_2(z; \zeta) = 0\},$$

$$\mathfrak{A}_\mu^{(n+m-1)} \equiv \left\{\frac{\partial S_1}{\partial \zeta_\mu} S_2 + S_1 \frac{\partial S_2}{\partial \zeta_\mu} = 0\right\}, \quad (\mu = 1, 2, \cdots, m),$$

then the candidates for singularities must lie on the set $\bigcap_{\mu=0}^{m} \mathfrak{A}_\mu^{(n+m-1)}$.

[4] If $\partial S_1/\partial \zeta_\nu = \partial S_2/\partial \zeta_\nu = 0$ for all $\nu = 1, 2, \ldots, n$, then we have the set $\bigcup_{\mu=1}^{2} \bigcap_{\nu=0}^{n} \mathfrak{S}_{\mu,\nu}$, which we have already included in the set of possible singularities.

4. Singularities of generalized axially symmetric potentials

As an interesting application of the preceding lemmas we consider those solutions of GASPN, which are represented by the operator $\mathfrak{A}[g(\zeta r)]$, where $g(\zeta r)$ is singular on the analytic set $\mathfrak{S}_{1,0} \equiv \{\psi(\zeta r) = 0\}$. We denote the singularities of the kernel by

$$(4.1) \qquad \mathfrak{S}_{2,0} \equiv \{1-2(\xi, \zeta)+||\zeta||^2 = 0\},$$

then

$$(4.2) \qquad \mathfrak{S}_{2,\mu} \equiv \{\zeta_\mu = \xi_\mu\}$$

and

$$(4.3) \qquad \bigcap_{\mu=0}^{n} \mathfrak{S}_{2,\mu} \equiv \{1-2(\xi, \xi)+(\xi, \xi) = 0\} \equiv \{||\xi|| = 1\}.$$

These are apparent singularities which arise from the kernel, and are *not related* to the function $g(\zeta r)$.

The set

$$\mathfrak{H}_{1,\mu}^{(2n-1)} \equiv \left\{(\zeta_\mu - \xi_\mu)\frac{\partial\psi}{\partial\zeta_1} - (\zeta_1-\xi_1)\frac{\partial\psi}{\partial\zeta_\mu} = 0\right\}$$

occurs when we use the Hadamard method to eliminate ζ_1 between $\mathfrak{S}_{1,0}$ and $\mathfrak{S}_{2,0}$ provided $\zeta_1 \neq \xi_1$. Considering the points $(\xi, \zeta) \in \mathfrak{H}_{1,\mu}^{(2n-1)}$ we realize they are contained in the set,

$$\left\{(\xi, \zeta) \,\middle|\, \frac{\partial\psi(\eta)}{\partial\eta_\nu} = (\eta_\nu - x_\nu)\Omega(\eta; x_1)\right\},$$

where $\Omega(\eta; x_1) \equiv (\eta_1 - x_1)^{-1}\partial\psi/\partial\eta_1$, and $x_\nu = \xi_\nu r$, $\eta_\nu = \zeta_\nu r$, $(\nu = 1, 2, \cdots, n)$. Since $\psi(\eta)$ is a function of the η-variables alone, i.e. not a function of the (x)-variables, and furthermore since $\psi(\eta)$ is a holomorphic-regular function of the (η)-variables, the above relation can hold only when either (i) $\eta_\nu = x_\nu$ is a zero of $\partial\psi/\partial\eta_\nu$, or (ii) a zero of $\partial\psi/\partial\eta_\nu$ with a zero of $\partial\psi/\partial\eta_1$; or (iii) ψ is independent of η_1 and η_ν.[5] In the case where $\eta_\nu = x_\nu$ is a zero of $\partial\psi/\partial\eta_\nu$ we have

$$(4.4) \qquad \mathfrak{H}_{1,\mu}^{(2n-1)} \equiv \{(x, \eta)|x_\nu = \eta_\nu;\ x_\mu \in C^1,\ \eta_\mu \in C^1, \text{ for } \mu \neq \nu\}.$$

If we have case (ii), then $\{\partial\psi(\eta)/\partial\eta_\nu \equiv \alpha(\eta) = 0\} \cap \{\partial\psi/\partial\eta_1 \equiv \beta(\eta) = 0\}$ is in general an $(n-2)$-complex dimensional analytic set, and we have

$$\mathfrak{H}_{1,\mu}^{(2n-1)} \equiv \{(x, \eta)|x \in C^1;\ \alpha(\eta) = \beta(\eta) = 0\}$$

[5] We rule this last case out in virtue of the fact that the singularities of $\psi(\eta)$ must lie on an analytic set.

reducing to a $(2n-2)$ dimensional set. In this latter case we have actually eliminated two η_μ-variables in the sense that the (η) variables now span a $(n-2)$-dimensional space. In this case the system of equations in the (ξ)-variable associated with the analytic sets $\mathfrak{H}_{1,\mu}^{(2n-1)}$ $(\mu = 2, 3, \cdots, n)$ an either $\mathfrak{S}_{1,0}$, or $\mathfrak{S}_{2,0}$ is over determined and the intersection (4.4) is of complex dimension $\leqq (n-2)$. Using analytic completion methods (see Bochner and Martin pg. 70, [B.M.1]) we can show that these singularities are removable, hence the only possible singularities must occur for case (i) on the previous page. We then have for case (i)

$$\bigcap_{\nu=2}^{n} \{(x, \eta) | x_\nu = \eta_\nu ; x_\mu \in C^1, \eta_\mu \in C^1, \mu \neq \nu\} \cap \mathfrak{S}_{2,0} \equiv \{||x|| = r\} \tag{4.5}$$

$$\bigcap_{\nu=2}^{n} \{(x, \eta) | x_\nu = \eta_\nu ; x_\mu \in C^1, \eta_\mu \in C^1, \mu \neq \nu\} \cap \mathfrak{S}_{1,0} \equiv \{\psi(x) = 0\}, \tag{4.6}$$

and

$$\left(\bigcap_{\nu=2}^{n} \mathfrak{H}_{1,\nu}^{(2n-1)}\right) \cap \mathfrak{S}_{1,0} \cap \mathfrak{S}_{2,0} \equiv \{(x) | \psi(x) = 0, \text{ and } ||x|| = r\}, \tag{4.7}$$

that is the real singularities lie on the intersection of the surface $\psi = 0$ with the $\rho = 0$ plane.

Finally, we consider the points contained in

$$P^0(r) \equiv \bigcap_{\mu=0}^{n} \mathfrak{S}_{1,\mu} \equiv \bigcap_{\mu=1}^{n} \left\{ r \;\middle|\; \frac{\partial \psi(\eta)}{\partial \eta_\mu} = 0 ; \eta_\mu = \zeta_\mu r, r > 0 \right\} \cap \mathfrak{S}_{1,0};$$

the set $P^0(r)$ is in general a set of real dimension zero. We now introduce the set,

$$\Lambda_2^{(n-1)} \equiv \{X | \, ||X|| = r ; r \in P^0(r)\},$$

which is a set upon which the integral may be singular.

We summarize the above discussion with the following theorem.

THEOREM 4.1. *Let $W(X) = \mathfrak{A}_n^{(s)}[g(\zeta r)]$ be a solution of the GASPN equation, where $g(\zeta r)$ is a holomorphic function of the n-complex variables $(\zeta r) \equiv (\zeta_1 r, \cdots, \zeta_n r)$. Furthermore, let the only singularities of $g(\zeta r)$ lie on the analytic set $\mathfrak{S} \equiv \{\psi(\zeta r) = 0\}$. Then $W(X)$ is regular for all points $X \equiv (x_1, \cdots, x_n, \rho)$ which are not contained on the following set $\Lambda_1^{(n-2)} \cup \Lambda_2^{(n-1)}$, where*

$$\Lambda_1^{(n-2)} \equiv \{\psi(x) = 0\} \cap \{\rho = 0\}$$

and

$$\Lambda_2^{(n-1)} \equiv \{X | \, ||X|| = r ; r \in P^0(r)\},$$

where $P^0(r)$ is defined above.

We next add a further result concerning the singularities of holomorphic functions represented by integrals, which will be of use in studying the properties of the operator $\mathfrak{A}^{-1}$.

LEMMA 4.2. *Let* $K(z;\zeta) \equiv K(z_1,\cdots,z_n;\zeta_1,\cdots,\zeta_m)$ *be an analytic function of* $(m+n)$*-complex variables, and regular-analytic in the product domain* $(\mathfrak{D}^{(n)}\times\mathfrak{B}^{(m)})$. *Let* Γ *be topologically an* m*-dimensional cycle whose boundary* $\partial\Gamma$ *remains fixed. Furthermore, let the singularities of* $K(z;\zeta)$ *lie on the analytic set* $\mathfrak{S}_0^{(n+m-1)} \equiv \{S(z;\zeta) = 0\}$. *Then the analytic function* $F(z)$ *represented in* $\mathfrak{D}^{(n)}$ *by*

$$F(z) = \int_\Gamma K(z;\zeta)\,d\zeta_1 \Lambda \cdots \Lambda d\zeta_n$$

has a continuation which is regular at all points (z) *that do not lie on the set*

$$\left\{\bigcap_{\mu=0}^{m} \mathfrak{S}_\mu^{(n+m-1)}\right\} \cup \left\{\bigcup_{\zeta\in\partial\Gamma} \mathfrak{S}_0^{(n+m-1)}\right\}.$$

PROOF. By the Cauchy-Poincaré theorem, Fuks [F.1; pg. 264] we know that

$$\int_\Gamma K(z;\zeta)d\zeta_1\Lambda\cdots\Lambda d\zeta_n = \int_{\Gamma'} K(z;\zeta)d\zeta_1\Lambda\cdots\Lambda d\zeta_n,$$

where Γ' is obtained by continuously deforming Γ, leaving $\partial\Gamma$ fixed, providing that in the domain bounded by $(\Gamma-\Gamma')\,K(z;\zeta)$ is for fixed (z) regular analytic in ζ. The proof then follows along the same lines as lemma 3.1 by showing the set $\bigcap_{\mu=0}^m \mathfrak{S}_\mu^{(n+m-1)}$ corresponds to those points we cannot avoid by letting Γ' follow an n-dimensional polycircle about.

The points lying on $\partial\Gamma$ have a special status since we cannot vary them, and must be therefore considered as candidates for singularities.

We conclude therefore that the points (z) which correspond to points of $\mathfrak{S}_0^{(n+m-1)} \equiv \{S(z;\zeta) = 0\}$ when (ζ) lies on the boundary, $(\zeta)\in\partial\Gamma$, may be singularities of $F(z)$. This proves our result.

We next consider this result applied to the case of the operator $\mathfrak{A}^{-1}[W]$. Let us suppose that the GASPN function $W(X)$ is singular on the analytic set $\mathfrak{S}_{1,0} \equiv \{\Phi(X) = 0\}$. The kernel may be seen to be singular on the set [G. H.1,2],

$$\mathfrak{S}_{2,0}^{(2n-1)} \equiv \{[(\zeta,\xi)-1]^2 + ||\zeta||^2(1-||\xi||^2) = 0\}.$$

The sets $\mathfrak{S}_{2,\mu}$ are given by

$$\mathfrak{S}_{2,\mu}^{(2n-1)} \equiv \{\zeta_\mu[(\zeta,\xi)-1] - \xi_\mu||\zeta||^2 = 0\},$$

and

$$\mathfrak{H}_{1,\mu}^{(2n-1)} \equiv \left\{(\xi,\zeta)\,|\,(\zeta_\mu[(\zeta,\xi)-1]-\xi_\mu||\zeta||^2)\frac{\partial\Phi}{\partial\xi_1} - (\zeta_1[(\zeta,\xi)-1]-\xi_1||\zeta||^2)\frac{\partial\Phi}{\partial\xi_\mu} = 0\right\}.$$

We compute $\bigcap_{\mu=0}^{n}\mathfrak{S}_{2,\mu}^{(2n-1)} \equiv \{||\zeta|| = 0\}$, since summing $\zeta_\mu\{\zeta_\mu[(\zeta, \xi)-1] -\xi_\mu||\zeta||^2\}$ over μ yields $||\zeta||^2(\zeta, \xi)-||\zeta||^2 = (\zeta, \xi)||\zeta||^2$. One may also obtain this result by direct elimination of the ξ_μ.

For the inverse operator the points on the set $\mathfrak{H}_{1,\mu}^{(2n-1)}$ ($\mu = 2, 3, \cdots, n$) do not simplify as for the operator $\mathfrak{A}[g]$. We have

$$\mathfrak{H}_{1,\mu}^{(2n-1)} \equiv \left\{(\xi, \zeta) | (\zeta_\mu[(\zeta, \xi)-1]-\xi_\mu||\zeta||^2) \frac{\partial \Phi}{\partial \xi_1} \right.$$
$$\left. -(\zeta_1[(\zeta, \xi)-1]-\xi_1||\zeta||^2) \frac{\partial \Phi}{\partial \xi_\mu} = 0\right\}.$$

As before the Hadamard [H.1], [G.1] plus envelope method [G.3] yields potential singularities on $\bigcap_{\mu=2}^{m}\mathfrak{H}_{1,\mu}^{(2n-1)} \cap \mathfrak{C}_{1,0} \cap \mathfrak{S}_{2,0}$; however, here we are no longer able to use the methods of separating independent variables as in the earlier case.

The integrand also has singularities on $\bigcap_{\mu=0}^{n}\mathfrak{S}_{1,\mu}$, i.e. for all $(\xi) \in P^{(0)} \equiv \bigcap_{\mu=1}^{n}\{(\xi)|\partial\Phi/\partial\xi_\mu = 0\} \cap \{(\xi)|\Phi(X) = 0\}$; however, since these points are not connected with the ζ-variables through $\Phi(X) = 0$ they do not lead to singularities of the integral in the usual manner. It is clear that the integral operator may not be well-defined if the domain of integration contains points of the set $P^{(0)}$. However, if it is defined then the points $(\zeta) \in \mathfrak{S}_{2,0} \cap P^{(0)}$ may be possible singularities of the associate function defined by the integral operator.

We mention lastly, that depending on the value of s, singularities may occur because of the coincidence of points on the boundary, $\{(\xi)|\, ||\xi|| = 1\}$, with $\mathfrak{S}_{2,0}$. These points are contained in the set $\bigcup_{||\xi||=1}\{(\zeta)|(\zeta, \xi) = 1\}$ ($\xi_i = \text{real}; i = 1, \cdots, n$).

5. Integral operators for special cases of GASPN

In this section we shall give another pair of operators associated with the GASPN, valid when the parameter s in Equation (1.1) is a positive integer, Indeed we have the following representation for $V_{m_1,\cdots,m_n}^{(s)}(\xi_1, \cdots, \xi_n) \equiv V_M^{(s)}(\xi)$, [A de F.1] (pg. 256)

$$\frac{\pi^{\frac{1}{2}n}\Gamma(s/2)}{\Gamma(\frac{1}{2}[n+s])} V_M^{(s)}(\xi) = \frac{\Gamma(n+s-1+\mu)(i)^\mu}{\Gamma(n+s-1)m_1!\cdots m_n!}$$
$$\times \iint_{0\leq||t||\leq 1} \frac{t_1^{m_1}\cdots t_n^{m_n}[1-||t||^2]^{\frac{1}{2}s-1}dt_1\cdots dt_n}{(\sqrt{1-||\xi||^2}+i(t, \xi))^{n+s-1+\mu}} \tag{5.1}$$

where the usual notation for the scalar product of two vectors has been used.

As in section II we generate a solution to the GASPN by multiplying both sides of (5.1) by

$$a_M r^{-(\mu+n+s-1)} (a_M \equiv a_{m_1, \cdots, m_n})$$

and summing. It is assumed that the requisite interchanges of integration and summation are justified by uniform convergence. One has as a representation for a solution $W(X) \equiv W(x_1, \cdots, x_n, \rho)$ as

$$(5.2)\quad \begin{aligned} W(X) &\equiv \sum_{M=0}^{\infty} a_M W_M^{(s)}(X) \\ &= \frac{\Gamma(\frac{1}{2}[n+s])r^{-(n+s-1)}}{\pi^{\frac{1}{2}n}\Gamma(\frac{1}{2}s)\Gamma(n+s-1)} \iint_{0 \leq ||t|| \leq 1} \left\{ \sum_{M=0}^{\infty} \frac{a_M (i)^\mu \Gamma(n+s-1+\mu)}{m_1! \cdots m_n!} \right. \\ &\quad \left. \frac{r^{-\mu} t_1^{m_1} \cdots t_n^{m_n}}{(\sqrt{1-||\xi||^2}+i(t, \xi))^\mu} \right\} \frac{(1-||t||^2)^{\frac{1}{2}s-1} dt_1, \cdots, dt_n}{(\sqrt{1-||\xi||^2}+i(t, \xi))^{n+s-1}}, \end{aligned}$$

where $\mu = m_1 + \cdots + m_n$ with m_i non-negative integers. This may be written more compactly as

$$(5.3)\quad W(X) = A[f] \equiv \alpha_{n,s} r^{-(n+s-1)} \iint_{0 \leq ||t|| \leq 1} f\left(\frac{t}{r\sigma}\right) \frac{(1-||t||^2)^{\frac{1}{2}s-1}}{\sigma^{n+s-1}} d^n t$$

where $\alpha_{n,s}$ is the constant multiplying the integrand in (5.2) and $\sigma \equiv \sqrt{1-||\xi||^2}+i(t, \xi)$. *This representation is valid in the exterior of a hypersphere.* Kelvin's transformation may be used to continue it to the interior of the hypersphere.

By means of the biorthogonality of the sets $\{V_M^{(s)}(\xi)\}$ and $\{U_M^{(s)}(\xi)\}$ (see section II) we can generate an operator inverse to $A[f]$. Indeed, from [A. de F.1] (pg. 263), we have

$$(5.4)\quad \begin{aligned} &\iint_{0 \leq ||t|| \leq 1} (1-||\xi||^2)^{\frac{1}{2}(s-1)} V_M^{(s)}(\xi) V_K^{(s)}(\xi) d^n \xi \\ &\qquad = \delta_K^M \frac{2\pi^{\frac{1}{2}n}}{(2\mu+n+s-1)} \frac{\Gamma(\frac{1}{2}[s+1])\Gamma(\mu+s)}{\Gamma(\frac{1}{2}[n+s-1])\Gamma(s) m_1! \cdots m_n!}. \end{aligned}$$

By use of this we have that

$$(5.5)\quad \begin{aligned} A^{-1}[W] &\equiv \int \cdots \int_{0 \leq ||t|| \leq 1} W(X) K(t/\sigma; r, \xi) d^n \xi \\ &= f\left(\frac{t}{r\sigma}\right) \equiv f\left(\frac{t_1}{r\sigma}, \cdots, \frac{t_n}{r\sigma}\right), \end{aligned}$$

where the kernel $K(t/\sigma; r, \xi)$ is given by

$$K(t/\sigma; r, \xi) \equiv \frac{r^{n+s-1}\Gamma(s)\Gamma(\frac{1}{2}[n+s-1])}{\Gamma(\frac{1}{2}[s+1])2\pi^{\frac{1}{2}n}} \sum_{\mu=0}^{\infty} \frac{(2\mu+n+s-1)m_1!\cdots m_n!}{\Gamma(\mu+s)}$$

$$\times \frac{t_1^{m_1}\cdots t_n^{m_n}}{\sigma^{\mu}} U^{(s)}_{m_1\cdots m_n}(\xi_1, \cdots, \xi_n) \tag{5.6}$$

$$\equiv \frac{r^{n+s-1}\Gamma(s)\Gamma(\frac{1}{2}[n+s-1])}{\Gamma(\frac{1}{2}[s+1])2\pi^{\frac{1}{2}n}} \sum_{M=0}^{\infty} (2\mu+n+s-1) \frac{m!t^m}{\sigma^{\mu}} U^{(s)}_M(\xi),$$

that is, (5.5) is an inverse operator for the operator $W(X) = A[f]$. We remark that we may extend these definitions of $A[f]$ and its inverse by introducing integration over n-dimensional chains, Γ where we hold the boundary $\partial\Gamma \equiv \{t| \,||t|| = 1\}$ fixed. Hence

$$W(X) = A[f] \equiv \alpha_{n,s} r^{-(n+s-1)} \int_{\Gamma} \frac{f(t/r\sigma)}{\sigma^{n+s-1}} (1-||t||^2)^{\frac{1}{2}s-1} d^n t, \tag{5.7}$$

where $\partial\Gamma \equiv \{t| \,||t|| = 1\}$, and

$$f(t/r\sigma) = A^{-1}[W] \equiv \int_G W(X)K(t/\sigma; r, \xi)d^n\xi \tag{5.8}$$

where $\partial G \equiv \{\xi| \,||\xi|| = 1\}$. We may then investigate the singularities of these functions using the methods introduced in section 3.

6. Inequality for GASPN functions defined by integral operators

In this section we shall find bounds for GASPN functions in terms of their associates. (See Bergman [B.1] for results of this kind in the case of harmonic functions of three variables.) We introduce first the notation

$$D_R \equiv \left\{(z) \,\middle|\, \left(\frac{z}{R}\right) \in D \subset C^n\right\}, \tag{6.1}$$

where D is an arbitrary, n-circular domain, and

$$M[g; D_R] = \sup_{z\in D_R} |g(z)|, \tag{6.2}$$

where $g(z) \equiv \sum_{K=0}^{\infty} a_{k_1,\cdots,k_n} z_1^{k_1}, \cdots, z_n^{k_n}$. We now consider the GASPN functions defined by the operators $\mathfrak{A}[f]$ and $A[g]$, where $f \equiv f(1/\zeta r)$, and $g \equiv g(t/r\sigma)$ respectively.

To be specific in the case of the operator $\mathfrak{A}[f]$ let us choose for our n-circular domain the unit hypersphere in the maximum norm, i.e. $\Delta \equiv \{(z)| \,||z||_m < 1\}$, where $||z||_m = \max_{1\leq k\leq n}|z_k|$. We notice that if $(1/\zeta r) \in \Delta_R$ then $\max_{1\leq k\leq n} 1/\zeta_k < Rr$ or $(\zeta) \notin \Delta_{1/rR}$ which follows from $\max_k|rR\zeta_k| \geqq \min_k|rR\zeta_k| > 1$.

Let us now introduce

$$\tilde{M}[f; \Delta_{R^{-1}}] = \sup_{(\zeta r) \notin \Delta_{R^{-1}}} |f(1/\zeta r)|, \tag{6.3}$$

and take $R = 1/r$. We assume that r, and hence R, is fixed. Then for $||\zeta||_m \geqq 1$ and taking as our domain of integration the set $\prod_{K=1}^{n}\{\zeta_K = e^{i\theta_K}\}$ we have

$$|W(X)| \leqq \frac{r^{-(n+s-1)}}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{|f(1/\zeta r)| d\theta_1, \cdots, d\theta_n}{|1-2(\zeta, \xi)+||\zeta||^2|^{\frac{1}{2}(n+s-1)}}. \tag{6.4}$$

We now proceed to obtain a lower estimate for $I \equiv |1-2(\zeta, \xi)+||\zeta||^2|$ by considering its extremum as $\zeta_K = e^{i\theta_k}$ varies. Since the quantity $I \geqq 0$ we have a minimum for $\zeta_K = \xi_K$ $(K = 1, \cdots, n)$ and this yields,

$$|1-2(\zeta, \xi)+||\zeta||^2| \geqq |1-||\xi||^2|;$$

hence,

$$\begin{aligned} |W(X)| &\leqq \frac{r^{-(n+s-1)}}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \tilde{M}[f; \Delta_{1/R}] \frac{d\theta_1, \cdots, d\theta_n}{(1-||\xi||^2)^{\frac{1}{2}(n+s-1)}} \\ &\leqq \frac{\tilde{M}[f; \Delta_{1/R}]}{\rho^{n+s-1}}. \end{aligned} \tag{6.5}$$

Thus we have a bound for those x's and ρ such that $x_1^2+\cdots+x_n^2+\rho^2 = r^2$. But $W(X)$ satisfies a maximum principle and we therefore have

$$W(X) \leqq \frac{\tilde{M}[f; \Delta_{1/R}]}{\rho^{n+s-1}}$$

for all $x_1^2+x_2^2+\cdots+x_n^2+\rho^2 \geqq 1/R^2$. We list this result as a theorem.

THEOREM 6.1. *Let $W(X)$ be a GASPN function whose $\mathfrak{A}$-associate $f(1/\zeta r)$ is holomorphic regular for (ζr) in the exterior of a (maximum-norm) hypersphere, $||z||_m \leqq R$. Furthermore, if $\tilde{M}[f; \Delta_R] = \sup_{(\zeta r)/\Delta_R}|f(1/\zeta r)|$, and $||X||_e^2 \equiv x_1^2+\cdots+x_n^2+\rho^2$, then*

$$|W(X)| \leqq \frac{\tilde{M}[f; \Delta_{1/R}]}{\rho^{n+s-1}}, \quad \text{for} \quad ||X||_e \geqq \frac{1}{R} > 0.$$

We next consider the GASPN functions generated by the operator $A[g(t/r\sigma)]$. Again let us use as a reference domain the (euclidean) hypersphere $||z||_e \leqq R$. We notice, that

$$|\sigma|^2 = |\sqrt{1-||\xi||^2}+i(\xi, t)|^2 \geqq |1-||\xi||^2|$$

for ξ, t *real*, and hence that

$$\frac{|t_k|}{r|\sigma|} - \frac{|t_k|}{r|1-||\xi||^2|^{\frac{1}{2}}} \leqq \frac{|t_k|}{\rho} \qquad (k = 1, \cdots, n).$$

We conclude that if $(t/\rho) \in \Delta_R$, i.e. $(t) \in \Delta_{\rho R}$, that $(t/r\sigma) \in \Delta_R$. Since the domain of integration is taken to be $||t||_e \leqq 1$ we must have $\rho R \geqq 1$ if $(t/r\sigma) \in \Delta_R$ *for all* t *in the domain of integration.* We proceed as before to obtain the bound,

$$(6.6)\quad \begin{aligned} |W(X)| &\leqq \frac{\Gamma(\frac{1}{2}[n+s])r^{-(n+s-1)}}{\pi^{\frac{1}{2}n}\Gamma(\frac{1}{2}s)\Gamma(n+s-1)} \iint_{||t||\leq 1} \frac{|g(t/r\sigma)|(1-||t||^2)^{\frac{1}{2}s-1}|d^n t|}{|\sigma|^{n+s-1}} \\ &\leqq \frac{\Gamma(\frac{1}{2}[n+s])}{\pi^{\frac{1}{2}n}\Gamma(\frac{1}{2}s)\Gamma(n+s-1)} \frac{M[g;\Delta_R]}{\rho^{n+s-1}} \iint_{||t||\leq 1} (1-||t||^2)^{\frac{1}{2}s-1}|d^n t| \end{aligned}$$

for $||X||_e \geqq \rho \geqq 1/R$. Since this bound is essentially of the same type as given in Theorem 6.1, we will not list this result as a separate theorem.

For a conclusion we shall list some extensions of results obtained by Gol'dberg ([F.1] pg. 339) concerning the growth of entire functions in C^n to the case of GASPN functions. We recall the results of Gol'dberg below. Let

$$(6.7)\qquad \lambda[D] = \overline{\lim_{R\to\infty}} \frac{\ln \ln M[f, D_R]}{\ln R},$$

and

$$(6.8)\qquad \sigma[D] = \overline{\lim_{R\to\infty}} \frac{\ln M[f, D_R]}{R^{\lambda[D]}},$$

be associated with the entire function $f(z) = \sum_{M=0}^{\infty} a_M z^M$; then one has

$$(6.9)\qquad \lambda[D] \equiv \overline{\lim_{||K||_a\to\infty}} \frac{||K||_a \ln ||K||_a}{-\ln |a_K|},$$

and

$$(6.10)\qquad (e\lambda\sigma[D])^{1/\lambda} = \overline{\lim_{||K||_a}} \{||K||_a^{1/\lambda}[|a_K|d_K[D]]^{1/||K||_a}\},$$

where $||K||_a = k_1 + \cdots + k_n$ is the "addition" norm, $d_K[D] = \sup_{(z)\in D}|z^K|$, and $z^K \equiv z_1^{k_1}, \cdots, z_n^{k_n}$. We list the following extension of Gol'dberg's results.

THEOREM 6.2. *Let* $W(X)$ *be a GASPN function whose* $\mathfrak{A}$*-associate* $f(1/\zeta r)$ *is an entire function of the variables* $(1/\zeta r) \in C^n$*, with the following expansion about* $(1/\zeta r) = (0)$,

$$f\left(\frac{1}{\zeta r}\right) = \sum_{K=0}^{\infty} a_{k_1,\cdots,k_n}(\zeta_1 r)^{-k_1} \cdots (\zeta_n r)^{-k_n}.$$

Then we have the bounds

$$(6.11)\qquad |W(X)| \leqq \frac{e^{R\lambda+\varepsilon}}{\rho^{n+s-1}}, \quad \textit{for} \quad ||X||_e \geqq \frac{1}{R} > 0,$$

for arbitrary $\varepsilon > 0$, *and*

(6.12) $$|W(X)| \leqq e^{(\sigma[\Delta_R]+\varepsilon')R^\lambda}, \quad \textit{for} \quad ||X||_e \geqq \frac{1}{R} > 0,$$

arbitrary $\varepsilon' > 0$, *and where*

$$\sigma[\Delta_R] = e^{-1}R^{\lambda-1} \overline{\lim_{||K||_a \to \infty}} \{||K||_a \cdot |a_K|^{\lambda/||K||_a}\}.$$

PROOF. These results are direct transplantations of the results of Gol'dberg if one computes in our case $d_K[\Delta_R] = \sup_{(z)\in\Delta_R}|z^K| = R^{k_1+\cdots+k_n} = R^{||K||_a}$.

We remark in closing that it is possible to obtain numerous other growth properties using different norms to specify n-circular domains, and that it is also possible to obtain lower bounds for the maximum of $|W(X)|$ on the spheres $||X||_e = R_0$ using the methods given in earlier papers by Gilbert [G.3] and Gilbert and Howard [G.H.1,2]. We intend to investigate these other properties in a later paper on GASPN.

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