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# ON THE GROUPS OF BRITTON'S THEOREM A

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In his simple proof of the unsolvability of the word problem for groups [**1**], J. L. Britton proved a normal form theorem (Britton's Lemma) for groups obtained by the *HNN* construction. In the appendix to that paper he described a generalization of the *HNN* construction and sketched the proof of a generalization of Britton's Lemma for this new construction, Britton's Theorem A. In this note we demonstrate that all groups obtained by means of this generalized construction are in fact *HNN* groups; and it will follow that Theorem A is simply a restatement of Britton's Lemma. This argument makes it clear that while Theorem A can be (and has been) useful in various group theoretic situations, in practice, every application of this generalized construction can be replaced by a straightforward application of the *HNN* construction. In the light of this paper, Miller's proof of Theorem A [**3**] can be reduced to a translation of a standard proof of Britton's Lemma into the context of Theorem A.

**1. The HNN and Britton constructions.** The following group theoretic construction is due to Higman, Neumann, and Neumann [**2**]. Let $G$ be a group with a presentation $(S;D)$. Let $\{A_{ij}\}$ and $\{B_{ij}\}$ for $j \in J_i$ and $i \in I$ be sets of words on the generators $S$ such that for each $i \in I$, the map $f_i : A_{ij} \rightarrow B_{ij}$ for all $j$ in $J_i$ extends to an isomorphism from the subgroup $(A_{ij}, j \in J_i)$ generated by the indicated elements onto the subgroup $(B_{ij}, j \in J_i)$. Let $t_i$, $i \in I$ be distinct new letters different from all the letters in $S$. Let $G^*$ be presented by $(S \cup \{t_i, i \in I\}; D \cup \{t_i^{-1}A_{ij}t_i = B_{ij}, j \in J_i, i \in I\})$. $G^*$ is called an *HNN extension of $G$ with stable letters $t_i$, $i \in I$.* The subgroups $(A_{ij}, j \in J_i)$ and $(B_{ij}, j \in J_i)$ are called *stable subgroups*. Higman, Neumann, and Neumann proved that the group $G$ is embedded into $G^*$ via the identity isomorphism.

BRITTON'S LEMMA. *Let $G^*$ be an HNN extension of $G$ with stable letters $t_i$, $i \in I$, and let $W$ be a word on the generators of $G^*$. Then*

(i) *$W$ is equal in $G^*$ to a word $W'$ which contains no subwords of form $t_i^{-1}At_i$ where $A$ is a word on $S$ and $A \in (A_{ij}, j \in J_i)$, nor any subwords of form $t_iBt_i^{-1}$ where $B$ is a word on $S$ and $B \in (B_{ij}, j \in J_i)$.*

(ii) *If $W'$ and $W''$ be any two representations of $W$ in the form described in* (i), *then the sequences of $t_i^{\pm 1}$ in $W'$ and $W''$ are identical.*

Britton's generalization of the *HNN* construction is as follows. Again let $G$

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be a group presented by $(S;D)$. Let $P$ be a set of distinct letters, different from all the letters in $S$, and let $P$ be indexed by $V$. Let $p(v)$ be the element of $P$ corresponding to $v \in V$. Let $\{A_i\}$ and $\{B_i\}$ be sets of elements of $G$ (given as words on $S$) indexed by some set $I$, and let $\{x_i\}$ and $\{y_i\}$ be sets of elements of $V$ also indexed by $I$. Let $G^*$ be the group presented by

$$(S \cup P; D \cup \{p(x_i)^{-1}A_i p(y_i) = B_i, i \in I\}).$$

We introduce an equivalence relation $E$ on $P$ by $pEp'$ if and only if $p = p'$ in the free group $(P; p(x_i) = p(y_i), i \in I)$. For each $v \in V$, let $A(v)$ be the subgroup of $G$ $(A_i, p(x_i)Ep(v))$ generated by the indicated words, and let $B(v)$ be the subgroup $(B_i, p(x_i)Ep(v))$. If for each $v \in V$, the map $A_i \rightarrow B_i$, for those $A_i$ in $A(v)$, extends to an isomorphism $f(v)$ of $A(v)$ onto $B(v)$, then $G^*$ is called a *Theorem A extension of G with stable letters P*.

Let $G^*$ be a Theorem A extension of $G$ as above, and let $T$ and $U$ be words on the letters of $S$. Following Britton, we say that $Tp(v)$ *produces* $p(w)U$, if $Tp(v)$ can be transformed to $p(w)U$ by a finite sequence of operations of the form

$$XA_i p(x_i)Y \rightarrow Xp(y_i)B_i Y, \quad \text{or}$$
$$XA_i^{-1}p(w_i)Y \rightarrow Xp(v_i)B_i^{-1}Y$$

BRITTON'S THEOREM A. *Let $G^*$ be a Theorem A extension of $G$, as above. Then:*

(i) *$G$ is embedded into $G^*$ via the identity map.*

(ii) *If $W$ involves one of the letters $p(v)$ and $W = 1$ in $G^*$, then $W$ contains a subword $p(w)^{-1}Ap(v)$ where $p(w)Ep(v)$, and $A$ is a word on $S$ equal in $G$ to an element $W(A_i)$ of $A(v)$, or else $W$ contains a subword $p(w)Bp(v)^{-1}$, where $p(w)Ep(v)$, and $B$ is a word on $S$ equal in $G$ to an element $W(B_i)$ of $B(v)$. In either case $W(A_i)p(v)$ produces $p(w)W(B_i)$.*

The main result of this paper is the following

MAIN THEOREM. *If $G^*$ is a Theorem A extension of $G$ with stable letters $P$, then $G^*$ is also an HNN extension of $G$ with a subset of $P$ as stable letters, and subgroups of the groups $A(v)$ and $B(v)$ as stable subgroups.*

Subsequently, we will show the equivalence of Theorem A and Britton's Lemma.

**2. Proof of the main theorem.** Let $G^*$ be a Theorem $A$ extension of $G$ with stable letters $P$. Well order $P$, and let $\alpha$ be the order type of this ordering. By transfinite recursion on the ordinals $\beta \leqq \alpha$ we define a sequence of presentations $\pi_\beta = (S_\beta;D_\beta)$, two sequences $A(v, \beta)$ and $B(v, \beta)$ of sets of subgroups of $G$ corresponding to elements $v \in V$, and a sequence $f(v, \beta)$ of sets of isomorphisms where $f(v, \beta)$ maps $A(v, \beta)$ onto $B(v, \beta)$.

Let $\pi_0$ be the given presentation for $G^*$, and for each $v \in V$, let $A(v, 0)$, $B(v, 0)$, and $f(v, 0)$ be $A(v)$, $B(v)$ and $f(v)$ respectively.

Suppose that $\pi_\beta = (S_\beta;D_\beta)$, $A(v, \beta)$ $B(v, \beta)$, and $f(v, \beta)$ have all been defined. Consider $p_{\beta+1}$, the $(\beta + 1)$st element of $P$ under the ordering. If $p_{\beta+1}$ is not equivalent (under $E$) to any $p_\gamma$ for $\gamma \leqq \beta$ let $\pi_{\beta+1}$ be $\pi_\beta$, and for each $v \in V$, let $A(v, \beta + 1)$, $B(v, \beta + 1)$, and $f(v, \beta + 1)$ be $A(v, \beta)$, $B(v, \beta)$ and $f(v, \beta)$, respectively.

Now suppose that $p_{\beta+1}Ep_\gamma$ for some $\gamma \leqq \beta$. Suppose that, in fact $\gamma$ is the least ordinal such that $p_{\beta+1}Ep_\gamma$. Choose a sequence of relations

$$p(u_1)^{-1}A_1p(v_1) = B_1$$
$$p(u_2)^{-1}A_2p(v_2) = B_2$$
$$\vdots$$
$$p(u_n)^{-1}A_np(v_n) = B_n$$

satisfying all of the following:

(i) All of the $p$'s which appear are distinct but equivalent under $E$.

(ii) One of $\{p(u_1), p(v_1)\}$ is $p_\gamma$.

(iii) One of $\{p(u_n), p(v_n)\}$ is $p_{\gamma+1}$.

(iv) For each $1 < i < n$, one of $\{p(u_i), p(v_i)\}$ is in $\{p(u_{i-1}), p(v_{i-1})\}$ and the other is in $\{p(u_{i+1}), p(v_{i+1})\}$.

Such a sequence must exist since the relation $E$ is defined to be equality in a free quotient of $G^*$. A sequence as we have just constructed enables us to express $p_{\beta+1}$ in terms of $p_\gamma$ and two elements of $G$, by solving the last equation for $p_{\beta+1}$, in terms of the remaining $p$-symbol $p'$, solving the preceding equation for $p'$ in terms of the remaining $p$-symbol $p''$, and so on. For example, suppose that our sequence is

$$p_\gamma^{-1}A_1p^{(3)} = B_1$$
$$p''^{-1}A_2p^{(3)} = B_2$$
$$p''^{-1}A_3p' = B_3$$
$$p'^{-1}A_4p_{\beta+1} = B_4$$

Then we have $p_{\beta+1} = A_4^{-1}p'B_4$, $p' = A_3^{-1}p''B_3$, $p'' = A_2p^{(3)}B_2^{-1}$, and $p^{(3)} = A_1^{-1}p_\gamma B_1$. Thus

$$p_{\beta+1} = A_4^{-1}A_3^{-1}A_2A_1^{-1}p_\gamma B_1B_2^{-1}B_3B_4.$$

By induction on the length $n$ of the chosen sequence, one can show that the resulting equation will be of form $p_{\beta+1} = \bar{A}^{-1}p_\gamma\bar{B}$, where $\bar{A} \in A(v)$ and $\bar{B}$ is the corresponding element of $B(v)$ for some $v \in V$ such that $p(v)Ep_\gamma$.

To obtain $D_{\beta+1}$ from $D_\beta$, replace each relation of form $p_{\beta+1}^{-1}Ap' = B$ by $p_\gamma^{-1}\bar{A}Ap' = \bar{B}B$, and each relation of form $p'^{-1}Ap_{\beta+1} = B$ by $p'^{-1}A\bar{A}^{-1}p_\gamma = B\bar{B}^{-1}$; then delete all trivial relations. $S_{\beta+1} = S_\beta - \{p_{\beta+1}\}$. For each $v \in V$, the group $A(v, \beta + 1)$ is the subgroup of $A(v, \beta)$ generated by the elements

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$A \in G$ such that $D_{\beta+1}$ contains a relation of form $p^{-1}Ap' = B$ for $pEp'$ and $p'Ep(v)$. Similarly, $B(v, \beta + 1)$ is the subgroup of $B(v, \beta)$ generated by the set of $B$ for which $D_{\beta+1}$ contains a relation $p^{-1}Ap' = B$, where again $pEp'$ and $p'Ep(v)$. By induction on the length of the chosen sequence of relations, one can demonstrate that $f(v, \beta)[A(v, \beta + 1)] = B(v, \beta + 1)$. Let $f(v, \beta + 1)$ be the restriction of $f(v, \beta)$ to $A(v, \beta + 1)$.

If $\beta$ is a limit ordinal, then $S_\beta = \bigcap_{\gamma<\beta} S_\gamma$. To define $D_\beta$, note that each relation $R$ in $D_0$ is altered only finitely many times at stages below $\beta$ in the construction; consequently, $\lim_{\gamma\to\beta} R$ is well defined. Let

$$D_\beta = \{\lim_{\gamma\to\beta} R | R \in D_0\}.$$

Let $A(v, \beta) = \bigcap_{\gamma<\beta} A(v, \gamma)$, let $B(v, \beta) = \bigcap_{\gamma<\beta} B(v, \gamma)$, and let $f(v, \beta)$ be the restriction of $f(v, 0)$ to $A(v, \beta)$.

LEMMA 1. *For each $\beta < \alpha$, $\pi_\beta$ is a presentation for $G^*$.*

*Proof.* By transfinite induction on $\beta$, one can show that each $\pi_\beta$ results from $\pi_0$ by a sequence of Tietze transformations.

LEMMA 2. *Each presentation $\pi_\beta$ gives $G^*$ explicitly as a Theorem A extension of $G$.*

*Proof.* The stable letters are the stable letters of $\pi_0$, which either come after $p_\beta$ in the ordering of $P$, or else are minimal (in that ordering) in their equivalence class under $E$. The isomorphism condition is verified by means of the maps $f(v, \beta)$.

LEMMA 3. *If $p_\beta$ is not minimal (in the ordering) in its equivalence class under $E$, then $p_\beta$ does not appear in any $S_\delta$ for $\delta > \beta$.*

*Proof.* $p_\beta$ is eliminated at the $\beta$th state in the construction.

COROLLARY 1. *$S_\alpha$ contains only the minimal elements (in the ordering of $P$) of each equivalence class under $E$.*

For each $v \in V$, let $\bar{v}$ be the element of $V$ such that $p(\bar{v})$ is minimal in the ordering of $P$ among the elements equivalent to $p(v)$ under $E$.

COROLLARY 2. *The presentation $\pi_\alpha = (S_\alpha; D_\alpha)$ gives $G^*$ explicitly as an HNN extension of $G$ with stable letters $\{p(\bar{v})\}$ which conjugate the subgroups $A(\bar{v}, \alpha)$ onto their isomorphic images $B(\bar{v}, \alpha)$.*

This completes the proof of the main theorem.

**3. Britton's Theorem A.** Let $W$ be a word on the generators $S_0$ which involves some $p(v)$ such that $W = 1$ in $G^*$. Apply the sequence of Tietze transformations, specified in the construction of $\pi_\alpha$ for $\pi_0$, to $W$, and call the resulting word $W^*$. Since $W = W^* = 1$ in $G^*$, $W^*$ must contain as a consecutive subword some $p(\bar{v})^{-1}Ap(\bar{v})$ or $p(\bar{v})Bp(\bar{v})^{-1}$, where $A \in A(\bar{v}, \alpha)$ and

$B \in B(\bar{v}, \alpha)$ by Britton's Lemma. The two occurrences of $p(\bar{v})$ arose in $W^*$, from the occurrence of two letters $p(u)$ and $p(w)$ in $W$, where $p(\bar{v})Ep(u)$ and $p(u)Ep(w)$. Since only finitely many Tietze transformations altered $W$, we may as well perform these finitely many in reverse order to $W^*$ to obtain $p(u)^{-1}A'p(w)$ from $p(\bar{v})^{-1}Ap(\bar{v})$, or $p(u)B'p(w)$ from $p(\bar{v})Bp(\bar{v})^{-1}$.

To see that $A'p(w)$ produces $p(u)B'$, let $p(u) = A_1^{-1}p(\bar{v})B_1$ and $p(w) = A_2^{-1}p(\bar{v})B_2$. Then

$$p(u)^{-1}A'p(w) = B_1^{-1}p(\bar{v})^{-1}A_1A'A_2^{-1}p(\bar{v})B_2 = B_1^{-1}(B_1B'B_2^{-1})B_2 = B'.$$

Thus $A'p(w)$ produces $p(u)B'$, since the $A_i$'s and the $B_i$'s were obtained from the sequence of relations of § 2. An analogous argument applies in the case of $p(u)B'p(w)$.

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