ON SIEGEL MODULAR FORMS
PART II

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1. Introduction

In this paper we compute dimension formulas for rings of Siegel modular forms of genus $g = 3$. Let denote $\Gamma_g(2)$ the main congruence subgroup of level two, $\Gamma_{g,0}(2)$ the Hecke subgroup of level two and Γ_g the full modular group. We give the dimension formulas for genus $g = 3$ for the above mentioned groups Γ and determine the graded ring $A(\Gamma_3(2))$ of modular forms with respect to $\Gamma_3(2)$.

The dimension formula in the case $\Gamma = \Gamma_3$ was first given by Tsuyumine in [T1]. Tsuyumine, following a method of Igusa, used the sequence

$$0 \to \chi_{18} A(\Gamma_3) \to A(\Gamma_3) \to S(2,8)$$

where χ_{18} is a cusp form of weight 18 defining the closure of the hyperelliptic locus and $S(2,8)$ is the graded ring of invariants of binary 8-forms. Tsuyumine uses the structure of $S(2,8)$, given by Shioda [Sh], and restriction of a bigger ring $A'(\Gamma_3)$ with respect to a second divisor.

For our generalization of Tsuyumine's result we use a more direct approach. In [R] we computed the ring of modular forms for $\Gamma_3(2,4)$ (the Igusa subgroup of level two). Principally this allows to compute all rings of modular forms for subgroups Γ with $\Gamma_3(2,4) \subset \Gamma \subset \Gamma_3$. However, this involves subtle computations of rings of invariants with respect to the finite group $\Gamma/\Gamma_3(2,4)$. It turns out that the computation is simplified by constructing a certain central extension H_g of $\Gamma_g/\Gamma_g(2,4)$. This group seems to be of independent interest because of its importance in coding theory. The main ingredient is a decomposition of Bruhat type for the group H_g. This decomposition is closely connected with the theory of partial Fourier transformation.

Finally, in the last chapter we give a characterization of decomposable points in the Satake compactification, which gives another method for computing $A(\Gamma_3(2))$.

2.   A decomposition of Bruhat type

We give two descriptions of the group H_g. For the first description consider

$$V = \{f : \mathbf{F}_2^g \to \mathbf{C}\} = L^2(\mathbf{F}_2^g)$$

together with the standard scalar product $\langle, \rangle : \mathbf{F}_2^g \times \mathbf{F}_2^g \to \mathbf{F}_2$. Let f_a for $a \in \mathbf{F}_2^g$ be the standard basis of V. We regard Fourier transformation on V, i.e.

$$\hat{f}(a) = 2^{-g} \sum_{b \in \mathbf{F}_2^g} f(b)(-1)^{\langle a,b \rangle}$$

and the convolution

$$f * g(a) = 2^{-g} \sum_{b \in \mathbf{F}_2^g} f(b)g(a - b).$$

We have the obvious properties $\hat{\hat{f}}(a) = 2^{-g} f(-a)$ and $\widehat{f * g} = \hat{f}\hat{g}$. Furthermore Fourier transformation commutes with complex conjugation. Secondly we have the Gauß functions

$$f_S(a) = i^{S[a]}$$

for any symmetric matrix $S \in \mathrm{Mat}_{g,g}(\mathbf{Z})$ (here a is regarded as a vector with entries 0, 1 and $S[a] = {}^t aSa$ is the standard quadratic form). The function f_S is a character of second degree in the sense of Weil [W]. The functions $f_S^2 = f_{2S}$ are characters in the usual sense. We refer for general facts to [W]. Let denote D_g the abelian group generated by all Gauß functions and let $D_{g,g} = D_g/\langle f^2$ for $f \in D_g \rangle$ be the factor group regarded as functions f_S with S containing only 0 and 1. We call these functions f_S *pure*. We include the proof of the following lemma, although elementary, because it is essential for the decomposition.

LEMMA 2.1.   *If $\hat{f}_S(a) \in \{0, \pm 1, \pm i\}$ for all $a \in \mathbf{F}_2^g$ and $\hat{f}_S(a)_{a \in \mathbf{F}_2^g}$, regarded as vector in $\mathbf{C}^{2^g}$, has length 1, then $S \equiv 0 \pmod 2$, hence f_S is a character on $\mathbf{F}_2^g$ with values in (± 1).*

Proof.   We first compute for $f = f_S$

$$f * \bar{f}(a) = 2^{-g} \sum_{b \in \mathbf{F}_2^g} f(b)\bar{f}(a - b) = 2^{-g} \sum_{b \in \mathbf{F}_2^g} i^{S[b]}(-i)^{S[a-b]}$$

$$= (-i)^{S[a]} 2^{-g} \sum_{b \in \mathbf{F}_2^g} (-1)^{\langle Sa,b \rangle}$$

$$= \begin{cases} (-i)^{S[a]} & \text{if } Sa \equiv 0 \bmod 2 \\ 0 & \text{else.} \end{cases}$$

Hence

$$\widehat{f\bar{f}}(a) = \widehat{f * \bar{f}}(a) = 2^{-g} \sum_{b \in \mathbf{F}_2^g \text{ with } Sb \equiv 0(2)} (-1)^{\langle b, Sb/2 + a \rangle}$$

$$\leq 2^{d-g}$$

where $d = \dim(\ker(S))$ regarded as matrix in $\mathrm{Mat}_{g,g}(\mathbf{F}_2)$. The assumption implies that $d = g$ hence the result. $\qquad\square$

Now we give a matrix description. The Fourier transformation corresponds (up to some scalar factor) to

$$T_g = \left(\frac{1 + i}{2}\right)^g ((-1)^{\langle a,b \rangle})_{ab \in \mathbf{F}_2^g}$$

and the Gauß functions correspond to

$$D_S = \mathrm{diag}(i^{S[a]} \text{ for } a \in \mathbf{F}_2^g)$$

for any symmetric $g \times g$-matrix S with integer coefficients.

Let H_g be the subgroup of $Gl(V)$ generated by T_g and the D_S, where S runs over the set of symmetric $g \times g$-matrices.

However, for our purpose we need a more combinatorical approach. This leads to a second description of the group H_g. We regard the affine linear group

$$AGl(g, \mathbf{F}_2) := \mathbf{F}_2^g \rtimes Gl(g, \mathbf{F}_2)$$

together with the standard action $(x, M)(a) = Ma + x$ on $\mathbf{F}_2^g$. The group $AGl(g, \mathbf{F}_2)$ will be considered as a subgroup in $Gl(V) \cong Gl(2^g, \mathbf{C})$ just by permuting the f_a.

We recall the following facts [R]:
We defined $N_g = \langle D_S^2, T_g^{-1} D_S^2 T_g$ for all $S \rangle$, the abelian subgroup $D_g = \langle D_S$ for all $S \rangle$ and the subgroup $H_{g,4} = D_g * AGl(g, \mathbf{F}_2)$. The notation $H_{g,4}$ is motivated by the fact, that $H_{g,4}$ is acting on the fourth powers f_a^4 only by permutations. It is a monomial group, regarded as subgroup in $U\left(2^g, \mathbf{Z}\left[\frac{1 + i}{2}\right]\right) \subset Gl(2^g, \mathbf{C})$ (see

[R], 2.6). In particular, the entries of a matrix in $H_{g,4}$ are integral over $\mathbf{Z}$ and of absolute value 1 or 0, hence contained in $\{0, \pm 1, \pm i\}$. The group N_g is a normal subgroup in H_g and $N_g \subset H_{g,4}$. One has an exact diagram

$$
\begin{array}{ccccccccc}
0 & \to & N_g & \to & H_g & \to & Sp(2g, \mathbf{F}_2) & \to & 0 \\
 & & \cup & & \cup & & \cup & & \\
0 & \to & \mathbf{F}_2^g & \to & AGl(g) & \to & Gl(g, \mathbf{F}_2) & \to & 0
\end{array}
$$

and the index of $H_{g,4}$ in H_g is $1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2^g + 1)$.

We have to consider partial Fourier transformation M with respect to the standard flag of subvector spaces in $\mathbf{F}_2^g$. In matrix notation with respect to the first standard vector (one digit, if one considers $\mathbf{F}_2^g$ as binary numbers)

$$M = \frac{1+i}{2} \begin{pmatrix} 1 & 1 & & & & \\ 1 & -1 & & & 0 & \\ & & 1 & 1 & & \\ & & 1 & -1 & & \cdots \\ & 0 & & & & \\ & & & \vdots & & \ddots \end{pmatrix}$$

is the matrix with 2^{g-1} blocks $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ along the diagonal. The typical Gauß function corresponds to the diagonal matrix (with 2^g entries)

$$E = (1, i, 1, i, \ldots).$$

Then $P = iME^2M = (0,1) \ldots (2^g - 2, 2^g - 1)$ (in cycle notation) is a permutation in $AGl(g, \mathbf{F}_2)$. We have the relations: $MEM = EMEP$, $(MEM)^2 = -P$, $ME^2M = iP$, $M^2 = i$, $(ME)^3 = -1$, $ME^2 = PM$, $E^2M = MP$, $P^2 = 1 = E^4 = (PM)^4$. This is easily reduced to a $Gl(2, \mathbf{C})$-computation.

More generally we have to regard d-dimensional partial Fourier transformation $M^{(d)}$ with respect to the first d standard basis vestors. In matrix notation $M^{(d)}$ for $d \leq g$ is given by the matrix, which consists of 2^{g-d} blocks T_d along the diagonal (for example in case $d = g = 2$

$$M^{(2)} = \frac{i}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}).$$

Let denote by $\Delta_{d,g} : H_d \rightarrow H_g$ the standard diagonal embedding, which maps a matrix W to the matrix with 2^{g-d} blocks W along the diagonal. Then $M^{(d)} = \Delta_{d,g}(T_d)$, $M = \Delta_{1,g}(T_1)$. Let denote by $D_{d,g}$ the set of diagonal matrices in D_g, given by elements D_S with S of the form $\begin{pmatrix} S' & 0 \\ 0 & 0 \end{pmatrix}$ for a symmetric $d \times d$-block S' with entries 0 and 1. The cardinality of this set (of pure Gauß functions belonging to the first d standard vectors) is obviously $2^{\binom{d+1}{2}}$.

More general, it is possible to define partial Fourier transformation for any linear subspace $U \subset \mathbf{F}_2^g$. The group $Gl(g, \mathbf{F}_2)$ acts by conjugation on the sets $D_{d,g}$

and the elements $M^{(d)}$. We get for any d-dimensional $U \subset \mathbf{F}_2^g$ an element M^U and a set D_g^U. The elements M^U are not unique (depending on a choice of a basis), but nevertheless we get the following result of Bruhat type

THEOREM 2.2. *There is a decomposition*

$$H_g = \biguplus_{U \subset \mathbf{F}_2^g} D_g^U * M^U * D_g * AIl(g, \mathbf{F}_2).$$

Proof. Remember $H_{g,4} = D_g * AGl(g, \mathbf{F}_2)$. Furthermore

$$\#\{d\text{-dim subspaces U in } \mathbf{F}_2^g\} = \frac{(2^g - 1)(2^{g-1} - 1) \ldots (2^{g-d+1} - 1)}{(2^d - 1)(2^{d-1} - 1) \ldots 1},$$

$$\#\{\text{pure Gauß functions in } D^U\} = 2^{\binom{d+1}{2}} \text{ for } d \geq 1.$$

Induction shows

$$[H_g : H_{g,4}] = 1 + \sum_{b=1,\ldots,g} \#\{d\text{-dim subspaces } U \text{ in } \mathbf{F}_2^g\} 2^{\binom{d+1}{2}}.$$

Hence it only remains to show the disjointness of the decomposition. The element M^U is clearly unique, it corresponds to a block decomposition of the matrix. Hence we may assume that $M^U = M^{(d)}$ (the standard flag). We regard two product representations $D_1 M^{(d)} B_1 = D_2 M^{(d)} B_2$ with $D_i \in D_{d,g}$ and $B_i \in H_{g,4}$. Obviously is again $D_1 D_2^{-1} \in D_{d,g}$ (up to a character) a pure Gauß function and nontrivial for $D_1 \neq D_2$. Using the maps $\Delta_{d,g}$ it remains to show the following lemma. □

LEMMA 2.3. *It holds* $[D_S, T_g] \in H_{g,4}$ *iff* $S \equiv 0 \bmod 2$.

Proof. It is obvious that $[D_S^2, T_g] \in H_{g,4}$. On the other hand we have for the commutator

$$[D_S, T_g](f_a) = \sum_{c \in \mathbf{F}_2^g} K_{a,c} f_c = \sum_{c \in \mathbf{F}_2^g} (-1)^{\langle a+c,c \rangle} \bar{\bar{f}}_S(a + c + S_c) f_c.$$

If the commutator (i.e. $(K_{a,c})$) is a monomial matrix, it follows from Lemma 2.1 that $S \equiv \bmod 2$ hence $D_S \in N_g$. □

COROLLARY 2.4. *We have the following equality of groups*

$$H_g = \langle Gl(g), P, E, M \rangle \text{ and } H_{g,4} = \langle Gl(g), P, E \rangle.$$

$H_{g,4}$ *is the monomial subgroup of* H_g, *i.e. the group of all matrices in* $Gl(2^g, \mathbf{C}) \cap H_g$ *which have exactly one nonzero entry in every row and column.*

The dimension d of the cells in the decomposition are characterized by the property, that the number of nonzero entries in the matrices is just $2^{g-d}4^d$. Dividing by N_g gives the Bruhat decomposition with respect to the parabolic subgroup $Gl(g, \mathbf{F}_2)$ (corresponding to the set of shorter roots in the Dynkin diagram C_g).

3.   The computation of rings of modular forms

We recall from [R] the group homomorphism

$$\phi : \Gamma_g \to H_g/(\pm 1).$$

For any congruence subgroup Γ, such that there exists a subgroup $G \subset H_g$ with $\Gamma_g^*(2, 4) \subset \Gamma \subset \Gamma_g$ and $\phi(\Gamma) = G/(\pm 1)$, we can compute the ring of Siegel modular forms of level Γ and even weight as

$$A(\Gamma)_{(2)} = \bigoplus_{2|s} [\Gamma, s] = (\mathbf{C}[f_a \text{ for } a \in \mathbf{F}_2^g]^G / \{\text{relations}\})^N,$$

where $\mathbf{N}$ denotes the normalization in its field of fractions and "relations" are the theta relations. In genus one and two, there are no relations and in genus three the relations form a principal ideal generated by a polynomial ψ of degree 16 with 471 terms (see Chapter 5). For $g \leq 3$ there is no normalization necessary [loc. cit.]. We get for the invariant rings $R = \mathbf{C}[f_a]^G$ the dimension formula:

$$\Phi_G(\lambda) = \sum_{l \geq 0} \dim_{\mathbf{C}} R_{(l)} \times \lambda^l = \frac{1}{\# G} \sum_{\sigma \in G} \frac{1}{\det(1 - \lambda\sigma)}$$

(We remark that the degree of a polynomial in the invariant ring is always two times the weight as a modular form.)

Before giving generators for the invariant rings we make some general remarks (see also Stanley [S1], [S2]). All subgroups G of $H_{g,4}$ are monomial groups (in the matrix representation with the basis f_a there is one nonzero entry in every row and column). We have a homomorphism $\pi : G \to S_{2^g}$ (the group of permutation matrices in $Gl(2^g, \mathbf{C})$), the group $\ker(\pi)$ is an abelian group consisting of diagonal matrices and G is the semi direct product of $\ker(\pi)$ and $\pi(G)$. All invariants are given as symmetrization (over $\mathrm{im}(\pi)$ in S_{2^g}) of admissible monomials (i.e. invariant under the group $\ker(\pi)$). We fix the notation

$$(b_0, \ldots, b_{2^g-1}) = \frac{1}{\# \operatorname{Stab}(b_0, \ldots, b_{2^g-1})} \sum_{\sigma \in \pi(G)} (\prod_{a \in \mathbf{F}_2^g} f_a^{b_a})^\sigma$$

where $(b_0, \ldots, b_{2^g-1})$ is $\ker(\pi)$-invariant and the stabilizer is taken inside G. It

will always be clear from the context, if $(b_0, \ldots, b_{2^g-1})$ denotes a (usually very large) homogeneous polynomial in $\mathbf{C}[f_a]$ or the monomial $\Pi_{a \in \mathbf{F}_2^g} f_a^{b_a}$.

The full modular group Γ_g corresponds to H_g, which is generated by the monomial group $H_{g,4}$ and one more element (e.g. M). In this case invariants will always be given as linear combinations of invariants under $H_{g,4}$.

For groups generated by pseudo reflections the invariant ring is a polynomial ring ([S2], 4.1). In this case it is enough to find 2^g algebraic independent polynomials in degrees given by the Poincaré-series.

For arbitrary groups it is more difficult to compute the invariant ring. The first step is to find 2^g algebraic independent invariants in degrees given by the denominator of *some* representation of the Poincaré-series in the following form.

$$\Phi_G(\lambda) = \frac{1 + \sum_{i=2^g+1 \ldots r} \lambda^{d_i}}{\Pi_{t=1 \ldots 2^g}(1 - \lambda^{d_i})}$$

This representation is not unique, but if one can find polynomials $g_1, \ldots, g_{2^g}$ of degree $d_1, \ldots, d_{2^g}$ with only finitely many common zeros, there also exist $g_{2^g+1}, \ldots, g_r$ which generate the invariant ring as a free module over the ring $\mathbf{C}[g_1, \ldots, g_{2^g}]$(see [S2], Chapter 3). The invariant ring is always Cohen–Macaulay (loc. cit. 3.2.).

To find a minimal set of generators as a ring one has to check up to the degree of the numerator that all invariants are polynomials in the chosen set of invariants.

If the invariant ring is a complete intersection

$$R = \mathbf{C}[g_1, \ldots, g_{2^g+c}] / (h_1, \ldots, h_c)$$

(a hypersurface for $c = 1$), the Poincaré-series has the form

$$\Phi_G(\lambda) = \frac{\Pi_{i=1 \ldots c}(1 - \lambda^{e_i})}{\Pi_{i=1 \ldots 2^g+c}(1 - \lambda^{d_i})}$$

with g_i and h_i polynomials of degree d_i and e_i respectively (c is the codimension, see [S1], 3.3). In particular the Poincaré-series is a quotient of products of cyclotomic polynomials.

4.   The Poincaré-series

The computation of the Poincaré-series is done in two steps. The first step is to compute the Poincaré-series for the action of H_3 on the polynomial ring, the second is to multiply this series with $(1 - \lambda^{16})$ or $(1 - \lambda^8)$ according to degree or

weight-grading.

As described above we have to compute many determinants. For genus three there are 371589120. The first simplification we use comes from computing modulo the center of the group.

The main simplification comes from the decomposition theorem 2.2. Obviously

$$\det(1 - \lambda\sigma_1\sigma_2) = \det(1 - \lambda\sigma_2\sigma_1)$$

$$\det(1 - \lambda\sigma) = \det(\sigma^{-1} - \lambda)\det(\sigma).$$

For $g \geq 3$ the group H_g is a subgroup of $SU(2^g)$, moreover $(M^{(d)})^2 = i^d$. Let denote $a_{0,3} = 1$, $a_{1,3} = 14$, $a_{2,3} = 56$, $a_{3,3} = 64$. All together we get

$$\Phi_{H_g}(\lambda) = \sum_{l \geq 0} \dim_{\mathbf{C}} R_{(l)} \times \lambda^l = \frac{1}{\# H_g} \sum_{\sigma \in H_g} \frac{1}{\det(1 - \lambda\sigma)}$$

$$= \frac{1}{\# H_g} \sum_{d = 0, \ldots, g} a_{d,g} \sum_{\tau \in H_{g,4}} \frac{1}{\det(M^{(d)}\tau - \lambda)}$$

It is interesting to note, that we do not need the full knowledge of the conjugacy-classes of H_g.

This restricts the computation considerably. Only 2752512 determinants of size 8×8 remain. A tedious computation gives 145 types of characteristic polynomials. In an appendix we give a complete list.

Because H_3 is a subgroup of $SU(8)$ all the characteristic polynomials P fulfill the equation

$$\overline{P(X)} = X^8 * P(1/X).$$

Moreover by the finiteness of the group H_3 all the coefficients of a characteristic polynomial are elements in $\mathbf{Z}[i]$, hence all characteristic polynomials are determined by 7 integers.

For any characteristic polynomial P also $P(iX)$, $P(-X)$ and $P(-iX)$ occur with the same cardinality. The symmetrization over the center

$$S(P) = 1/P(X) + 1/P(iX) + 1/P(-X) + 1/P(-iX)$$

yields 33 types of rational function, where the denominator is a product of cyclotomic polynomials C_j of degree at most 16 and j the order of an element of H_g hence a divisor of $371589120 = 2^{17} * 3^4 * 5 * 7$.

The set of possible characteristic polynomials is more or less described by this observations. The difficult point is to count how often the 145 types of polynomials occur.

Before giving the main result for the full modular group let us discuss the easier cases of level 2 and Hecke case. (The Hecke subgroup $\Gamma_{g,0}(2)$ is for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$ defined by the condition $C \equiv 0 \bmod(2)$ and corresponds to $H_{g,4}$.)

The Poincaré-series are as follows:

$$\Phi_{N_3}(\lambda^{1/2})$$
$$= \frac{1 + 7\lambda^2 + 43\lambda^4 + 154\lambda^6 + 43\lambda^8 + 7\lambda^{10} + \lambda^{12}}{(1 - \lambda^2)^8}$$
$$= 1 + 15\lambda^2 + 135\lambda^4 + 870\lambda^6 + 3993\lambda^8 +$$
$$+ 14157\lambda^{10} + 41535\lambda^{12} + 105740\lambda^{14} + \ldots$$

As a consequence one has to compute the invariants up to weight 12, i.e. with polynomials of degree 24 in the 8 $f_0, \ldots, f_7$. The invariant ring is not a complete intersection (the numerator is irreducible in $\mathbf{Q}[\lambda]$ and not cyclotomic).

Multiplication with $(1 - \lambda^8)$ yields:

$$\Phi_{\Gamma_3(2)}(\lambda)$$
$$= \frac{(1 + 7\lambda^2 + 43\lambda^4 + 154\lambda^6 + 43\lambda^8 + 7\lambda^{10} + \lambda^{12})(1 + \lambda^2)(1 + \lambda^4)}{(1 - \lambda^2)^7}$$
$$= 1 + 15\lambda^2 + 135\lambda^4 + 870\lambda^6 + 3992\lambda^8 + 14142\lambda^{10} +$$
$$+ 41400\lambda^{12} + 104870\lambda^{14} + \ldots$$

For the Hecke subgroup the result is as follows:

$$\Phi_{H_{3,4}}(\lambda^{1/2}) = \frac{N_{\text{Hecke}}(\lambda)(1 + \lambda^2)}{(1 - \lambda^2)(1 - \lambda^4)^3(1 - \lambda^6)(1 - \lambda^8)(1 - \lambda^{12})(1 - \lambda^{14})}$$
$$= 1 + \lambda^2 + 4\lambda^4 + 6\lambda^6 + 15\lambda^8 + 24\lambda^{10} + 49\lambda^{12} + 78\lambda^{14} + \ldots$$

The numerator is given by:

$$N_{\text{Hecke}}(\lambda) = 1 - \lambda^2 + \lambda^4 + 2\lambda^8 + \lambda^{10} + 2\lambda^{12} + \lambda^{14} + 5\lambda^{16} + \lambda^{18} + 6\lambda^{20} + 2\lambda^{22}$$
$$+ 6\lambda^{24} + 2\lambda^{26} + 6\lambda^{28} + \lambda^{30} + 5\lambda^{32} + \lambda^{34} + 2\lambda^{36} + \lambda^{38} + 2\lambda^{40} + \lambda^{44} - \lambda^{46} + \lambda^{48}$$

As a consequence one has to compute the invariants up to weight 50, i.e. with polynomials of degree 100 in the 8 $f_0, \ldots, f_7$. The invariant ring is not a complete intersection (N_{Hecke} is irreducible in $\mathbf{Q}[\lambda]$ and not cyclotomic).

Multiplication with $(1 - \lambda^8)$ yields:

$$\Phi_{\Gamma_{3,0}(2)}(\lambda) = \frac{N_{Hecke}(\lambda)(1 + \lambda^2)}{(1 - \lambda^2)(1 - \lambda^4)^3(1 - \lambda^6)(1 - \lambda^{12})(1 - \lambda^{14})}$$

$$= 1 + \lambda^2 + 4\lambda^4 + 6\lambda^6 + 14\lambda^8 + 23\lambda^{10} + 45\lambda^{12} + 72\lambda^{14} + \ldots$$

For the full modular group (or H_3) the result is as follows:

$$\Phi_{H_3}(\lambda^{1/2}) = \frac{N_{\mathrm{mod}}(\lambda)(1 + \lambda^2)}{(1 - \lambda^4)(1 - \lambda^8)(1 - \lambda^{12})^2(1 - \lambda^{14})(1 - \lambda^{18})(1 - \lambda^{20})(1 - \lambda^{30})}$$

$$= 1 + \lambda^4 + \lambda^6 + 2\lambda^8 + 2\lambda^{10} + 5\lambda^{12} + 4\lambda^{14} + \ldots$$

The numerator is given by:

$$N_{\mathrm{mod}}(\lambda) = 1 - \lambda^2 + \lambda^4 + \lambda^{10} + 3\lambda^{16} - \lambda^{18} + 3\lambda^{20} + 2\lambda^{22} + 2\lambda^{24} + 3\lambda^{26} + 4\lambda^{28}$$
$$+ 2\lambda^{30} + 7\lambda^{32} + 3\lambda^{34} + 7\lambda^{36} + 5\lambda^{38} + 9\lambda^{40} + 6\lambda^{42} + 10\lambda^{44} + 8\lambda^{46} + 10\lambda^{48} +$$
$$9\lambda^{50} + 12\lambda^{52} + 7\lambda^{54} + 14\lambda^{56} + 7\lambda^{58} + 12\lambda^{60} + 9\lambda^{62} + 10\lambda^{64} + 8\lambda^{66} + 10\lambda^{68} +$$
$$6\lambda^{70} + 9\lambda^{72} + 5\lambda^{74} + 7\lambda^{76} + 3\lambda^{78} + 7\lambda^{80} + 2\lambda^{82} + 4\lambda^{84} + 3\lambda^{86} + 2\lambda^{88} + 2\lambda^{90} +$$
$$3\lambda^{92} - \lambda^{94} + 3\lambda^{96} + \lambda^{102} + \lambda^{108} - \lambda^{110} + \lambda^{112}$$

To get a numerator with positive coefficients one has to multiply $N_{\mathrm{mod}}(\lambda)$ with $(1 + \lambda^2)$. As a consequence one has to compute the invariants at least up to weight 114, i.e. with polynomials of degree 228 in the 8 $f_0, \ldots, f_7$. The invariant ring is not a complete intersection (N_{mod} is irreducible in $\mathbf{Q}[\lambda]$ and not cyclotomic).

Multiplication with $(1 - \lambda^8)$ yields:

$$\Phi_{\Gamma_3}(\lambda) = \frac{N_{\mathrm{mod}}(\lambda)(1 + \lambda^2)}{(1 - \lambda^4)(1 - \lambda^{12})^2(1 - \lambda^{14})(1 - \lambda^{18})(1 - \lambda^{20})(1 - \lambda^{30})}$$

$$= 1 + \lambda^4 + \lambda^6 + \lambda^8 + 2\lambda^{10} + 4\lambda^{12} + 3\lambda^{14} + \ldots$$

(One may check that $N_{\mathrm{mod}} * (1 + \lambda_2)$ is the same numerator given in Tsuyumine's paper [T1]. The statement in [T1], [T3] and [T4], that the ring is not Cohen–Macaulay is not correct.)

Remark. All the Poincaré-series given in this paper have the property

$$(-\lambda)^d \Phi_G(1/\lambda) = \lambda^\tau \Phi_G(\lambda)$$

for some $r \in \mathbf{Z}$ which is equivalent to the Gorenstein-property. It turns out that r is the number of pseudo reflections in G. For example for $G \subset Sl(d, \mathbf{C})$ the invariant rings are Gorenstein-rings [S1], 4.4. and 5.5. or [S2], 8.1). We remark,

that this is *not* equivalent to the Gorenstein-property of the Satake compactification as a variety.

5.   The case $\Gamma_3(2)$

In the notation with the f_a the relation is a polynomial of degree 16 with 471 terms. We have (recall from part I [R])

$$
\begin{aligned}
\phi = {} & 2\,(9,1,1,1,1,1,1,1) - (8,0,0,0,2,2,2,2) \\
& + (6,2,2,2,4,0,0,0) - 4\,(5,5,1,1,1,1,1,1) \\
& - (4,4,4,0,4,0,0,0) - 2\,(4,4,0,0,2,2,2,2) \\
& + 16\,(3,3,3,3,1,1,1,1) - 72\,(2,2,2,2,2,2,2,2),
\end{aligned}
$$

where the writing is given with respect to the monomial group $H_{3,4}$. For every monomial one has to take the orbit under the $AGl\,(3, \mathbf{F}_2)$-action. For $\Gamma_3(2)$—which corresponds to the case N_3—the permutation part is only an abelian group of order 8, hence one has to collect the 471 terms to 79 orbits in this case to get a writing in accordance with the convention described in chapter 3. We have to define the notion of an admissible monomial (i.e. invariance under the kernel of the canonical map $N_3 \rightarrow S_8$).

LEMMA 5.1.   *A monomial* $(b_0,\ b_1,\ b_2,\ b_3,\ b_4,\ b_5,\ b_6,\ b_7)$ *is admissible*$\Leftrightarrow$

$$
\begin{aligned}
& \sum_{i=0..7} b_i = n = 2\,(\text{weight}) \equiv 0 \bmod (4) \\
& b_0 + b_1 + b_2 + b_3 \equiv 0 \bmod (2) \\
& b_0 + b_1 + b_4 + b_5 \equiv 0 \bmod (2) \\
& b_0 + b_2 + b_4 + b_6 \equiv 0 \bmod (2).
\end{aligned}
$$

Proof.   The point is to find a set of conditions, which uniquely defines a monomial modulo the group action. The group of diagonal matrices in this case is generated by i and the following diagonal matrices:

$$
\begin{aligned}
& (-1, -1, -1, -1, 1, 1, 1, 1) \\
& (-1, -1, 1, 1, -1, -1, 1, 1) \\
& (-1, 1, -1, 1, -1, 1, -1, 1)
\end{aligned}
$$

$\square$

The computation of all modular forms for a certain weight consists in dividing the set of admissible monomials in orbits under the action of the group $\mathbf{F}_2^3$ in

the natural decomposition of $AGl(3, \mathbf{F}_2)$. We will give the invariants by some choice with respect to the permutation orbit under $AGl(3, \mathbf{F}_2)$. $\mathbf{F}_2^3$ is a normal subgroup of $AGl(3, \mathbf{F}_2)$ and very orbit under $AGl(3, \mathbf{F}_2)$ decomposes in orbits under $\mathbf{F}_2^3$. Zeros are omitted if possible.

weight 2 (degree 4).

In the following tables we choose in every orbit under $AGl(3, \mathbf{F}_2)$ a representative (first column) and give the cardinality of the orbit under $AGl(3, \mathbf{F}_2)$ (second column) and the number of linear independent invariants of that type for $\mathbf{F}_2^3$ (third column).

(4)	8	1
(2,2)	28	7
(1,1,1,1)	14	7

weight 4 (degree 8).

(8)	8	1	(3,1,1,1,2)	224	28
(6,2)	56	7	(2,2,2,2)	14	7
(5,1,1,1)	56	7	(1,1,1,1,2,2)	84	21
(4,4)	28	7	(2,2,2,0,2,0,0,0)	56	7
(4,2,2)	168	21	(1,1,1,1,4)	56	7
(3,3,1,1)	84	21	(1,1,1,1,1,1,1,1)	1	1

It follows from the Poincaré-series that the tables aboves give all the 15 and 135 modular forms, which generate the space of modular forms in weight 2 and 4. To get generators for the ring of modular forms, we take all the 15 forms in weight 2 and the last 15 in weight 4.

THEOREM 5.2. $A(\Gamma_3(2))$ *is minimally generated by the* 30 *modular forms*

(4,0,0,0,0,0,0,0), (2,2,0,0,0,0,0,0), (2,0,2,0,0,0,0,0), (2,0,0,2,0,0,0,0),
(2,0,0,0,2,0,0,0), (2,0,0,0,0,2,0,0), (2,0,0,0,0,0,2,0), (2,0,0,0,0,0,0,2),
(1,1,1,1,0,0,0,0), (1,1,0,0,1,1,0,0), (1,1,0,0,0,0,1,1), (1,0,0,1,1,0,0,1),
(1,0,0,1,0,1,1,0), (1,0,1,0,1,0,1,0), (1,0,1,0,0,1,0,1), (2,2,2,0,2,0,0,0),
(2,2,2,0,0,2,0,0), (2,2,2,0,0,0,2,0), (2,2,2,0,0,0,0,2), (2,2,0,0,2,0,2,0),
(2,2,0,0,2,0,0,2), (2,0,0,2,0,2,0,2), (1,1,1,1,4,0,0,0), (1,1,4,0,1,1,0,0),
(1,1,4,0,0,0,1,1), (1,4,0,1,1,0,0,1), (1,4,0,1,0,1,1,0), (1,4,1,0,1,0,1,0),
(1,4,1,0,0,1,0,1), (1,1,1,1,1,1,1,1).

We start with a lemma. Let denote

$$F_b = \sum_{a \in \mathbf{F}_2^g} f_a f_{a+b} \text{ for } b \in \mathbf{F}_2^g.$$

If one substitutes the f_a by its squares one gets the first eight polynomials of the list. The quadrics F_b may simultaneously be brought into normal form by a linear change of variable as described in the proof of the following lemma.

LEMMA 5.3. *The F_b have only 0 as a common zero.*

Proof. We have

$$T_g(F_b) = \left(\frac{i}{2}\right)^g \sum_{a \in \mathbf{F}_2^g} \left(\sum_{c \in \mathbf{F}_2^g} (-1)^{\langle a,c \rangle} f_c\right) \left(\sum_{d \in \mathbf{F}_2^g} (-1)^{\langle a+b,d \rangle} f_d\right)$$

$$= \left(\frac{i}{2}\right)^g \sum_{a,c,d \in \mathbf{F}_2^g} (-1)^{\langle a,c+d \rangle + \langle b,d \rangle} f_c f_d$$

$$= i^g \sum_{c \in \mathbf{F}_2^g} (-1)^{\langle b,c \rangle} f_c^2.$$

The matrix $((-1)^{\langle b,c \rangle})_{b,c}$ is regular, hence the result. □

Proof. (of the theorem) It is enough to show that the polynomials generate the (polynomial) invariant ring. The first eight polynomials have no common zero except $(0, \ldots, 0)$ by Lemma 5.3. Hence there exists a set of generators containing the first eight polynomials of the list with degrees given by the Poincaré-series, especially by the numerator in the above form.

One can easily see that all basis invariants in degree 8, different from the last fifteen generators, are linear combinations in the last fifteen and mixed products of first fifteen generators. The last generator $F = \Pi_{a \in \mathbf{F}_2} f_a$ plays a special role. F^3 corresponds to the term λ^{12} in the numerator and $F^2 *$ any modular forms of degree 8 (weight 4) (and degree 4 (weight 2)) correspond to the term $7\lambda^{10}$ (and $43\lambda^8$) in the numerator of the Poincaré-series. Hence the following problem of linear algebra remains: Is it possible to write all of the 870 basis invariants in degree 12 as a linear combination in the $135 * 15 = 2025$ products of basis invariants of degree 8 and 4? With a computer or with tedious computations one can check that the answer is yes. □

6. The characterization of decomposable points

In this chapter we want to derive some consequences for the set of decomposable points from the theory developed in part I, in particular Propositions 2.9, 2.10, and 3.15.

As a point set the Satake compactification is given by $\mathrm{Proj}(\mathbf{C}[f_a])$. For every $r + s = g$ one has a commutative diagram

$$
\begin{array}{ccc}
\mathscr{A}_r(2,4) \times \mathscr{A}_s(2,4) & \to & \mathscr{A}_g(2,4) \\
\downarrow & & \downarrow \\
\mathbf{P}^{2^r-1} \times \mathbf{P}^{2^s-1} & \to & \mathbf{P}^{2^g-1}.
\end{array}
$$

The map in the upper row is usually denoted by Ψ. The map in the lower row is the Segre map. It is well known that the Segre map is an embedding (see [Ha], Ex. I, 2.14).

One may check quite easily that the diagram is equivariant under the various groups N_r, N_s and N_g. Hence we get the following

THEOREM 6.1.   *There is a fibre product diagram of injective maps for every $r + s = g$ as follows*

$$
\begin{array}{ccc}
\mathscr{A}_r(2) \times \mathscr{A}_s(2) & \to & \mathscr{A}_g(2) \\
\downarrow & & \downarrow \\
\mathbf{P}^{2^r-1}/N_r \times \mathbf{P}^{2^s-1}/N_s & \to & \mathbf{P}^{2^g-1}/N_g.
\end{array}
$$

For $r = s = 1$ the Segre embedding is given by the principal ideal $(f_0 f_3 - f_1 f_2)$, therefore it is easy to compute the ring of modular forms in genus two by restriction to the diagonal as it is done in Freitag [F](III, 1.4).

For higher genus there are different types of decomposable points given by different types of Segre embeddings. Let $\mathfrak{a}_S$ denote the ideal of a Segre embedding S.

COROLLARY 6.2.

$$
Th(\mathscr{A}_g(2)^{\text{decomposabie}}) = \bigcup_{S \text{ Segre map}} Z(\mathfrak{a}_S^{N_g})
$$

where the index set are the various Segre maps.

EXAMPLE 6.3. In genus 3 the ideal of one Segre embedding is given by $\mathfrak{a}_S =$
$(f_0 f_5 - f_1 f_4, \; f_2 f_7 - f_3 f_6, \; f_0 f_6 - f_2 f_4, \; f_1 f_7 - f_3 f_5, \; f_0 f_7 - f_3 f_4, \; f_1 f_6 - f_2 f_5)$ and the
group N_3 is acting on the generators by transpositions and multiplication with
± 1. By lifting the generators of $A_2(2) \times A_1(2)$ to $A_3(2)$ and computing a basis of
$\mathfrak{a}_S^{N_3}$ one has another possibility for computing $A_3(2)$.

The cartesian product of two graded rings R_* and S_* is given by $(R \times S)_i$
$= R_i \otimes S_i$ (see [Ha], II. Ex. 5.11) and is a direct summand of the tensor product
(with half of the grading). For the groups N_r, N_s and N_g the group action may be
extended to the tensor product of the graded rings.

Remark 6.4. In [SM1], [SM2] Salvati Manni gives a characterization of decomposable points in terms of theta constants (of the first kind).

Table

We give a list of all characteristic polynomials of elements in H_3 and count
their occurrence. The polynomials have the type

$$P(X) = 1 + a_1 X + a_2 X^2 + a_3 X^3 + a_4 X^4 + a_5 X^5 + a_6 X^6 + a_7 X^7 + X^8$$

with $\overline{a_1} = a_7$, $\overline{a_2} = a_6$, $\overline{a_3} = a_5$ and $a_4 \in \mathbf{Z}$. This table is included to allow comparison with other approaches.

In the table, after the coefficients, we give the number of types under the action of the center (a row may belong to at most four characteristic polynomials) and the numbers corresponding to the d-dimensional $M^{(d)}$, the number of cases is the sum $\#\{0\text{-dim cases}\} + 14 * \#\{1\text{-dim cases}\} + 56 * \#\{2\text{-dim cases}\} + 64 * \#\{3\text{-dim cases}\}$. The last column gives the product decomposition into cyclotomic polynomials of the denominator of the symmetrization $S(P)$. There are different symmetrizations with equal cyclotomic type. By summation over the 0–dimensional cases one gets the Poincaré-series for $H_{3,4}$.

a_1	a_2	a_3	a_4	#	0-dim	1-dim	2-dim	3-dim	cyclotomic type
i	0	0	0	4	196608	49152	122880	86016	$28,14,7,4^2,2^2,1^2$
0	0	0	2	1	193536	62976	44544	34944	8^2
0	$2i$	0	-2	2	16128	13824	4608	4032	8^2
$1+i$	0	$1-i$	2	4	21504	39936	16896	21504	8^2
0	0	0	-2	1	246456	34848	56640	29568	$4^2,2^2,1^2$
0	2	0	2	2	32256	9216	13824	8064	$4^2,2^2,1^2$
$1+i$	$2i$	$-1+i$	-2	4	21504	3072	26112	21504	$4^2,2^2,1^2$
2	2	2	2	4	16128	4608	6912	4032	$4^2,2^2,1^2$
2	4	6	6	4	1344	384	192	672	$4^2,2^2,1^2$

$2-2i$	$-4i$	$-2-2i$	-2	4	1568	512	128	224	$4^2,2^2,1^2$
0	1	0	0	2	172032	36864	55296	21504	$12,6,4^2,3,2^2,1^2$
0	i	0	0	2	114688	90112	61440	86016	$24,8^2$
0	0	0	0	1	172032	172032	172032	172032	16
$1+i$	i	$1-i$	2	4	28672	4096	11264	8960	$12,6,4^2,3,2^2,1^2$
$1+i$	i	$-1+i$	-2	4	28672	40960	19456	34048	$12,6,3$
0	0	0	1	1	0	179712	40704	91392	$12,6,3$
1	1	0	0	4	0	61440	36864	75264	$12,6,3$
$1+i$	$2i$	$-2+2i$	-3	4	0	9216	13824	16128	$12,6,3$
0	$4i$	0	-6	2	3584	1280	256	448	8^4
0	4	0	6	2	5670	384	480	0	$4^4,2^4,1^4$
2	0	-2	-2	4	14784	384	2112	672	$4^4,2^4,1^4$
2	1	2	4	4	14336	0	2560	896	$12^2,6^2,4^4,3^2,2^4,1^4$
2	1	-2	-4	4	14336	6144	6656	2688	$12^2,6^2,4^2,3^2,2^2,1^2$
$2+2i$	$2i$	$2-2i$	6	4	1344	1920	192	336	$8^2,4^2,2^2,1^2$
$2+2i$	$6i$	$-6+6i$	-10	4	1344	384	576	336	$8^2,4^2,2^2,1^2$
4	8	12	14	4	420	48	48	0	$4^4,2^4,1^4$
4	4	-4	-10	4	364	0	16	0	$4^6,2^6,1^6$
$4+4i$	$16i$	$-20+20i$	-34	4	14	8	0	0	$4^4,2^4,1^4$
8	28	56	70	4	1	0	0	0	$4^8,2^8,1^8$
0	2	0	3	2	0	12672	9408	5376	$12^2,6^2,3^2$
0	0	0	-1	1	0	79872	89088	86016	24
0	i	0	-1	2	0	73728	73728	64512	40
$1+i$	i	0	1	4	0	36864	36864	32256	$20,10,5$
0	1	0	1	2	0	36864	36864	32256	$20,10,5$
1	0	-1	-1	4	0	49152	49152	43008	$60,30,15$
0	$2i$	0	-3	2	0	15360	3584	3584	24^2
$1+i$	0	0	-1	4	0	33792	7680	16128	$24,12,6,3$
2	2	0	-1	4	0	5376	4224	2688	$12^2,6^2,3^2$
1	1	1	1	4	0	49152	73728	86016	$36,18,12,9,6,3$
1	1	-2	-2	4	0	2048	0	1792	$12^3,6^3,4^2,3^3,2^2,1^2$
2	3	4	5	4	0	6144	6144	5376	$20^2,10^2,5^2$
1	1	2	2	4	0	2048	12288	8960	$12^2,6^2,4^2,3^2,2^2,1^2$
2	4	4	5	4	0	1792	384	1792	$12^3,6^3,3^3$
$2+2i$	$4i$	$-4+4i$	-7	4	0	1920	448	448	$12^2,6^2,3^2$
4	10	16	19	4	0	64	32	0	$12^4,6^4,3^4$

Appendix

As an appendix we give the Poincaré-series and the graded rings of modular forms for genus one and two. In this cases our method reduces to a very simple one. A second method (in level 2) would be just to apply the (surjective) Φ-operator from genus three to get the results in genus one and two. In the following tables we give the congruence subgroup, the order of the corresponding finite subgroup in H_g, the Poincaré-series and the ring of modular forms for that congruence subgroup. Although the results are known, it may be useful for the applications in coding theory, to give the invariant rings in terms of theta constants of second order (and in genus two the exceptional $\Theta = \Pi_{\text{even } m}\theta[m]$).

$$\Gamma_1(4) \quad 2 \quad \frac{1 + \lambda}{(1 - \lambda)^2} \qquad \mathbf{C}[(2,0),(0,2),(1,1)]$$

$$\Gamma_1(2) \quad 16 \quad \frac{1}{(1 - \lambda^2)^2} \qquad \mathbf{C}[(4,0),(2,2)]$$

$$\Gamma_{1,0}(2) \quad 32 \quad \frac{1}{(1 - \lambda^2)(1 - \lambda^4)} \quad \mathbf{C}[(4,0),(4,4)]$$

$$\Gamma_1 \quad 96 \quad \frac{1}{(1 - \lambda^4)(1 - \lambda^6)} \quad \mathbf{C}[(8,0) + 14(4,4),(12,0) - 33(8,4)]$$

That corresponds to the fact that the three group N_1, $N_{1,4}$ and H_1 are generated by pseudo reflections, whereas $\Gamma_1(4)$ corresponds to ± 1.

In the genus two case we will always distinguish between the case of even weight and the general case. In the case of even weight we get:

$$\Gamma_2(2) \quad 32 \quad \frac{1 - \lambda^8}{(1 - \lambda^2)^5} \quad \mathbf{C}[(4,0,0,0),(1,1,1,1),(2,2,0,0),(2,0,2,0),(2,0,0,2)]$$

$$\Gamma_{2,0}(2) \quad 3072 \quad \frac{1}{(1 - \lambda^2)(1 - \lambda^4)^2(1 - \lambda^6)}$$

$$\mathbf{C}[(4,0,0,0),(4,4,0,0),(2,2,2,2),(12,0,0,0)]$$

In the case of general weight we get (following [R], 4.2):

$$\Gamma_2(2) \quad 32 \quad \frac{(1 - \lambda^8)(1 + \lambda^5)}{(1 - \lambda^2)^5}$$

$$\mathbf{C}[(4,0,0,0),(1,1,1,1),(2,2,0,0),(2,0,2,0),(2,0,0,2), \Theta]$$

$$\Gamma_{2,0}(2) \quad 3072 \quad \frac{(1 + \lambda^{19})}{(1 - \lambda^2)(1 - \lambda^4)^2(1 - \lambda^6)}$$

$$\mathbf{C}[(4,0,0,0),(4,4,0,0),(2,2,2,2),(12,0,0,0), \Theta * P_{28}]$$

If one denotes

$$A = (4,0,0,0),\ B = (1,1,1,1),\ C = (2,2,0,0),\ D = (2,0,2,0),\ E = (2,0,0,2),$$

then there is a relation of degree 16 between the generators in level 2 as follows.

$$16B^4 + A^2B^2 + C^2D^2 + D^2E^2 + C^2E^2 = ACDE + 4B^2(C^2 + D^2 + E^2)$$

Hence $\mathscr{A}_2(2)$ is a quartic hypersurface in $\mathbf{P}^4$ (see also Igusa [11], [12]). In level $\Gamma_{2,0}(2)$ the computation is easy because the group $H_{2,4}$ is a group generated by pseudo reflections of order 3072. Hence $\mathscr{A}_{2,0}(2)$ is a weighted $\mathbf{P}^3$ and hence rational (see also Ibukiyama [Ib1, 2]). The polynomial P_{28} is given as follows:

$$P_{28} = f_0 f_1 f_2 f_3 * (f_0^4 - f_1^4)(f_0^4 - f_2^4)(f_0^4 - f_3^4)(f_1^4 - f_2^4)(f_1^4 - f_3^4)(f_2^4 - f_3^4)$$

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