Canad. Math. Bull. Vol. **27** (2), 1984

ON THE ASSOCIATED LIE RING AND THE ADJOINT GROUP OF A RADICAL RING

BY
HARTMUT LAUE

ABSTRACT. We investigate connections between the associated Lie ring and the adjoint group of a radical ring, studying their upper central chains. Part of a conjecture of S. A. Jennings is proved, and one of our results improves a theorem of his.

It was shown by S. A. Jennings [3] that the associated Lie ring of a radical ring is nilpotent if and only if its adjoint group is nilpotent. He conjectured the nilpotency classes of both structures coincide in this case, and he could prove this for a nilpotent algebra of characteristic 0. Studying the upper central chain of the Lie ring, we sharpen this result, and prove part of Jennings' conjecture (one inequality). Finally we show the first two centers of the adjoint group and the associated Lie ring of a radical ring consist of the same elements, which hints that Jennings' conjecture might be true even in a sharper form.

Let $(R, +, \cdot)$ be an associative ring.[1] For $a, b \in R$ let

$$a \circ b = ab - ba \text{ (Lie product)},$$
$$a * b = a + b + ab \text{ (circle composition)}.$$

Then it is well known that $(R, +, \circ)$ is a Lie ring (called the *associated Lie ring* of $(R, +, \cdot)$, and $(R, *)$ is a semigroup, which is a group (and then called the *adjoint group* of $(R, +, \cdot)$) if and only if $(R, +, \cdot)$ is a radical ring. In general, let $Q(R)$ denote the set of quasi-regular elements of R, i.e. the set of invertible elements of $(R, *)$. Then $(Q(R), *)$ is a group which acts by conjugation on the ring $(R, +, \cdot)$ (hence a fortiori on $(R, +, \circ)$). One easily checks the formula

(1) For all $a \in Q(R), b \in R \qquad \bar{a} * b * a = b + ba + \bar{a}b + \bar{a}ba.$[2]

All characteristic subrings of $(R, +, \cdot)$ (resp. $(R, +, \circ)$) are invariant under the action of $Q(R)$. In particular this is true for the members of the *upper central*

Received by the editors March 22, 1983.
A.M.S. Subject Classification: 16A22.

(1) We do not assume that R contains a unit element.

(2) $\bar{a}$ denotes the inverse of a, with respect to $*$. Clearly $a \circ \bar{a} = 0$.

https://doi.org/10.4153/CMB-1984-031-4 Published online by Cambridge University Press

chain of $(R, +, \circ)$ which are defined as follows:

$$Z_0 = 0,$$

and for each ordinal $\alpha > 0$

$$Z_\alpha = \begin{cases} \{z \mid z \in R, z \circ r \in Z_{\alpha-1} \quad \text{for all} \quad r \in R\} & \text{if } \alpha \text{ is not a limit ordinal.} \\ \bigcup_{\beta<\alpha} Z_\beta & \text{else.} \end{cases}$$

The Lie ring $(R, +, \circ)$ is nilpotent if and only if there is an integer n such that $Z_n = R$, in which case the smallest integer n with this property is the class of $(R, +, \circ)$. We have

$$\text{(2)} \qquad \text{For all} \quad a, b \in R, \qquad a * b = b * a \Leftrightarrow ab = ba \Leftrightarrow a \circ b = 0,$$

whence centralizers with respect to $*$, $\circ$, $\cdot$ are identical. In particular, Z_1 is the center of the semigroup $(R, *)$. We start investigating the upper central chain with

LEMMA 1. *Let* $(R, +, \cdot)$ *be an associative ring. For all ordinals* α *we have*
(i) $(Z_\alpha, +, \cdot)$ *is a subring of* $(R, +, \cdot)$.
(ii) $Q(Z_\alpha) = Z_\alpha \cap Q(R)$.

Proof. We have the "semi-Jacobi identity"

$$\text{(3)} \qquad \text{For all} \quad a, b, c \in R \qquad (ab) \circ c + (ca) \circ b + (bc) \circ a = 0$$

by means of which we prove
(i′) Let $(U, +)$ be a subgroup of $(R, +)$ and

$$Z := \{z \mid z \in R, \qquad z \circ r \in U \quad \text{for all} \quad r \in R\}.$$

Then $(Z, +, \cdot)$ is a subring of $(R, +, \cdot)$.
We have to show Z is closed under the ring multiplication. For $z_1, z_2 \in Z$, $r \in R$, we have by (3)

$$(z_1 z_2) \circ r = z_1 \circ (z_2 r) + z_2 \circ (r z_1) \in U,$$

thus $z_1 z_2 \in Z$. This gives (i′) which implies (i). Next we claim
(ii′) Let $(U, +, \circ)$ be a (Lie) ideal of $(R, +, \circ)$ and

$$Z := \{z \mid z \in R, \qquad z \circ r \in U \quad \text{for all} \quad r \in R\}.$$

Then $Q(Z) = Z \cap Q(R)$.
Easy calculations yield the following preparatory identities:

$$\text{(4)} \qquad \text{For all} \quad a, b, c \in R \qquad a \circ (bc) = (a \circ b)c + b(a \circ c).$$

$$\text{(5)} \qquad \text{For all} \quad a, b, c \in R \qquad (ab) \circ c = a(b \circ c) + (a \circ c)b.$$

$$\text{(6)} \quad \text{For all} \quad a \in Q(R), b \in R \qquad \bar{a} \circ b = b \circ a + \bar{a}(b \circ a) + (b \circ a)\bar{a} + \bar{a}(b \circ a)\bar{a}.$$

Be means of (4), we get in the situation of (ii′)

(7) For all $z \in Z, z', r \in R \qquad z \circ z' = 0 \Rightarrow (z \circ r)z' \in U,$

since $(z \circ r)z' = (z \circ r)z' + r(z \circ z') = z \circ (rz') \in U$. Now we can prove (ii′): By (i′), $(Z, +, \cdot)$ is a subring of $(R, +, \cdot)$, and we have $Q(Z) \subseteq Z \cap Q(R)$. Suppose $z \in Z \cap Q(R)$. Then by (6) and (7), we have for all $r \in R$

$$\begin{aligned} \bar{z} \circ r \equiv \bar{z}(r \circ z)\bar{z} &\equiv ((r \circ z)\bar{z})\bar{z} &&\mod U \\ &= (r \circ z)(\bar{z})^2 \equiv 0 &&\mod U, \end{aligned}$$

by (2) and (7). Therefore $\bar{z} \in Z$, proving (ii′).

Now (ii) follows by an easy induction argument: For $\alpha = 0$, (ii) is trivial. Let $\alpha > 0$ and suppose (ii) is true for all ordinals $< \alpha$. Then if α is not a limit ordinal, we put $U := Z_{\alpha-1}$ and apply (ii′). But if α is a limit ordinal and $z \in Z_\alpha \cap Q(R)$, then there is an ordinal $\beta < \alpha$ such that $z \in Z_\beta \cap Q(R)$, and the induction hypothesis yields $z \in Q(Z_\beta) \subseteq \bigcup_{\gamma < \alpha} Q(Z_\gamma) = Q(Z_\alpha)$. Since we also have $Q(Z_\alpha) \subseteq Z_\alpha \cap Q(R)$, this gives equality, and the proof of (ii) is complete.

We now show that $Q(R)$ not only normalizes the members of the upper central chain but also centralizes the factor groups $(Z_n/Z_{n-1}, +)$, for all natural numbers n:

LEMMA 2. *Let $(R, +, \cdot)$ be an associative ring and n a natural number. Then for all $z \in Z_n$, $a \in Q(R)$ we have $\bar{a} * z * a - z \in Z_{n-1}$.*

Proof. We show by induction on n

(8) For all $z \in Z_n, a, a' \in R \qquad a \circ a' = 0 \Rightarrow a(z \circ a') \in Z_{n-1}.$

For $n = 1$, this is clear. Let $n > 1$, and assume for all $y \in Z_{n-1}$ and for all a, $a' \in R$ such that $a \circ a' = 0$ we have $a(y \circ a') \in Z_{n-2}$. Suppose $z \in Z_n$, a, $a' \in R$ such that $a \circ a' = 0$, and $b \in R$. Then by (4) and (5)

$$\begin{aligned} (z \circ (ba')) \circ a &= ((z \circ b)a' + b(z \circ a')) \circ a \\ &= ((z \circ b) \circ a)a' + (b \circ a)(z \circ a') + b((z \circ a') \circ a) \\ &= a'((z \circ b) \circ a) - (a(z \circ a')) \circ b + (z \circ a') \circ (ba) \\ &\quad - ((z \circ a') \circ b) \circ a + ((z \circ b) \circ a) \circ a'. \end{aligned}$$

Since $z \circ b \in Z_{n-1}$, our induction hypothesis yields $(a(z \circ a')) \circ b \in Z_{n-2}$, whence we conclude $a(z \circ a') \in Z_{n-1}$, i.e. (8).

Now if $z \in Z_n$, $a \in Q(R)$, we have by (1)

$$\begin{aligned} \bar{a} * z * a - z &= za + \bar{a}z + \bar{a}za \\ &= z \circ a + \bar{a}(z \circ a) \\ &\in Z_{n-1}, \end{aligned}$$

due to (8). This proves our lemma.

https://doi.org/10.4153/CMB-1984-031-4 Published online by Cambridge University Press

As a corollary, we have the following theorem which is half of Jennings' conjecture on the nilpotency classes of $(R, +, \circ)$ and $(R, *)$:

THEOREM 1. *Let* $(R, +, \cdot)$ *be a radical ring, and suppose the associated Lie ring* $(R, +, \circ)$ *is nilpotent. Then the group* $(R, *)$ *is nilpotent, and* $\mathrm{cl}(R, *) \leqq \mathrm{cl}(R, +, \circ)$.

Proof. Put $l := \mathrm{cl}(R, +, \circ)$. The group $(R/Z(R), *)$ acts faithfully on $(R, +)$ and, by Lemma 2, stabilizes the chain

$$0 = Z_0 < Z_1 < \cdots < Z_l = R.$$

By a theorem of Kaloujnine [4], this implies $(R/Z(R), *)$ is a nilpotent group, and its class is strictly smaller than the length of the stabilized chain, i.e. $\mathrm{cl}(R/Z(R), *) < l$. Hence $(R, *)$ is nilpotent as well, and $\mathrm{cl}(R, *) \leqq l$. The proof is complete.

Assuming $(R, +, \cdot)$ is a nil algebra over a field of characteristic 0 such that $(R, +, \circ)$ is nilpotent, Jennings [3] proved the equality $\mathrm{cl}(R, *) = \mathrm{cl}(R, +, \circ)$. In fact, under these assumptions a stronger result holds:

THEOREM 2. *Let* $(R, +, \cdot)$ *be a nil algebra over a field* F *of characteristic* p. *Suppose* $p = 0$ *or* $R^p = 0$, *and assume* $(R, +, \circ)$ *is nilpotent. For every nonnegative integer* n, *let* $(Y_n, *)$ *denote the* nth *member of the upper central chain of the nilpotent group* $(R, *)$. *Then* $Y_n = Z_n$.

Proof. For every finite subset T of R and for each integer $d \geq |T|$ we write $T^{(d)}$ for the Lie subalgebra of $(R, +, \circ)$ which is generated by the elements $(\cdots(a_0 \circ a_1) \circ \cdots) \circ a_k$ such that $k + 1 \geq d$ and $T = \{a_0, a_1, \ldots, a_k\}$. Since the factors in such a Lie product run through all of T, we have

$$(9) \qquad T \cap Z_d \neq \varnothing \Rightarrow T^{(d+1)} = 0 \quad \text{for all} \quad d \geq |T|.$$

Let N be the set of all natural numbers if $p = 0$, resp. the set of all natural numbers $< p$ if $p \neq 0$. Let $\sim$ be the unique nontrivial homomorphism of the ring of integers into F, and put

$$\mathfrak{e}: R \to R, a \mapsto \sum_{k \in N} (1/\widetilde{k!}) a^k.$$

(This sum is finite even if $p = 0$, since R is nil.) If $p = 2$, our theorem is trivial. We thus assume $p \neq 2$, in which case the Campbell–Hausdorff formula [1, III, 5.] yields

(10) For all $a, b \in R$ there exists an element $c \in \{a, b\}^{(3)}$ such that $\mathfrak{e}(a) * \mathfrak{e}(b) = \mathfrak{e}(a + b + \frac{1}{2} a \circ b + c)$.

(The reader will readily verify that under the hypotheses of our theorem, the Campbell–Hausdorff formula also applies in the case $p > 0$, the summation

https://doi.org/10.4153/CMB-1984-031-4 Published online by Cambridge University Press

breaking off after $p-1$ summands.) Since $\mathfrak{e}$ is a permutation of R, $\mathfrak{e}$ is invertible by a mapping $\mathfrak{l}$, viz.

$$\mathfrak{l}: R \to R, a \mapsto \sum_{k \in N} ((-1)^{k+1}/\tilde{k})a^k,$$

and from (10) we conclude

(11) For all $a, b \in R$ $\quad \mathfrak{l}(\mathfrak{e}(a) * \mathfrak{e}(b)) \equiv a + b + \frac{1}{2} a \circ b \bmod \{a, b\}^{(3)}$.

Since $a \circ (-a) = 0$, we have $\{a, -a\}^{(3)} = 0$, and by (10) consequently $\overline{\mathfrak{e}(a)} = \mathfrak{e}(-a)$. Thus (11) yields

$$\begin{aligned}\mathfrak{l}(\overline{\mathfrak{e}(a)} * \overline{\mathfrak{e}(b)} * \mathfrak{e}(a) * \mathfrak{e}(b)) &= \mathfrak{l}(\mathfrak{e}(-a) * \mathfrak{e}(-b) * \mathfrak{e}(a) * \mathfrak{e}(b)) \\ &\equiv a \circ b \bmod \{a, b\}^{(3)}.\end{aligned}$$

Inductively, we conclude for $*$-commutators (denoted by brackets) of arbitrary length:

(12) (Jennings [2, 6.1.6]) For all $a_0, \ldots, a_n \in R$

$$\mathfrak{l}([\cdots[a_0, a_1], \ldots, a_n]) \equiv (\cdots(\mathfrak{l}(a_0) \circ \mathfrak{l}(a_1)) \circ \cdots) \circ \mathfrak{l}(a_n) \bmod \{\mathfrak{l}(a_0), \ldots, \mathfrak{l}(a_n)\}^{(n+2)}$$

where we have switched from $\mathfrak{e}(a_i)$ to a_i, from a_i to $\mathfrak{l}(a_i)$.

From Lemma 1(i) we get

(13) For all $a \in R$ $\quad a \in Z_n \Leftrightarrow \mathfrak{l}(a) \in Z_n$.

We can now show the equality $Y_n = Z_n$: Suppose first $a \in Z_n$. Then by (13), $\mathfrak{l}(a) \in Z_n$, thus for all $b_1, \ldots, b_n \in R$, by (9) and (12), $\mathfrak{l}([\cdots[a, b_1], \ldots, b_n]) = 0$, i.e. $[\cdots[a, b_1], \ldots, b_n] = 0$, yielding $a \in Y_n$. Thus $Z_n \subseteq Y_n$. In order to prove the reverse inclusion we show by induction, putting $c := \mathrm{cl}(R, +, \circ)$:

(14) $\quad Y_n \subseteq Z_{c-i}$ for $0 \le i \le c-n$.

This is trivial for $i=0$. Assume $Y_n \subseteq Z_{m+1}$ for an integer $m \ge n$. We have to show $Y_n \subseteq Z_m$. Let $a \in Y_n$. Then by (13) and our induction hypothesis, $\mathfrak{l}(a) \in Z_{m+1}$, and for all $b_1, \ldots, b_m \in R$ we have $\{\mathfrak{l}(a), \mathfrak{l}(b_1), \ldots, \mathfrak{l}(b_m)\}^{(m+2)} = 0$, by (9). Since $m \ge n$, (12) yields $(\cdots(\mathfrak{l}(a) \circ \mathfrak{l}(b_1)) \circ \cdots) \circ \mathfrak{l}(b_m) = 0$, therefore $\mathfrak{l}(a) \in Z_m$, and finally, by (13), $a \in Z_m$. This proves (14) which in the special case $i = c-n$ reduces to the desired inclusion $Y_n \subseteq Z_n$. The proof of our theorem is complete.

This proof does not give any hint how to get rid of the hypotheses of our theorem and prove the equality of the upper centers, say, for arbitrary radical rings, to put it in a most optimistic way. We have noted above that Z_1 and Y_1 are equal. This result being trivial, we found it on the other hand far from easy to treat the second centers and prove equality in the end:

https://doi.org/10.4153/CMB-1984-031-4 Published online by Cambridge University Press

THEOREM 3. *Let* $(R, +, \cdot)$ *be a radical ring and* $(Y_2, *)$ *the second center of the group* $(R, *)$. *Then* $Y_2 = Z_2$.

Proof. We shall need the following simple but powerful rule:

(15) For all $a, b \in R \quad a + ab = 0 \Rightarrow a = 0.$

For $a + ab = 0$ implies $a * b = b$, hence $a = a * b * \bar{b} = b * \bar{b} = 0$. (This shows (15) is true in arbitrary rings R if we suppose $b \in Q(R)$.) Similarly:

(16) For all $a, b \in R \quad a + ba = 0 \Rightarrow a = 0.$

From (1), one easily deduces the following formula for the commutator $[a, b] = \bar{a} * \bar{b} * a * b$:

(17) For all $a, b \in R \quad [a, b] = a \circ b + (\bar{a} * \bar{b})(a \circ b).$

The first main step of our proof is the following statement:

(18) If $z \in Y_2$, then $(z \circ a) \circ a = 0$ for all $a \in R$.

To this end, suppose $z \in Y_2$. Then for all $a \in R$ we have $[z, a] \in Z(R)$, hence by (17) and (5)

$$\begin{aligned} 0 &= [z, a] \circ (a * z) \\ &= (z \circ a + (\bar{z} * \bar{a})(z \circ a)) \circ (a * z) \\ &= (z \circ a) \circ (a * z) + (\bar{z} * \bar{a})((z \circ a) \circ (a * z)), \end{aligned}$$

as $(\bar{z} * \bar{a}) \circ (a * z) = 0$. Now (16) yields

(19) For all $a \in R \quad (z \circ a) \circ (a + z + az) = 0.$

This implies $(z \circ (a + \bar{z})) \circ ((a + \bar{z}) + z + (a + \bar{z})z) = 0$, hence

(20) For all $a \in R \quad (z \circ a) \circ (a + az) = 0.$

Now (19), (20) imply

(21) For all $a \in R \quad (z \circ a) \circ z = 0,$

and (19), (21), (4) yield $0 = (z \circ a) \circ a + (z \circ a) \circ (az) = (z \circ a) \circ a + ((z \circ a) \circ a)z$. Applying (15), we see that (18) holds.

The second step of our proof is

(22) If $z \in R$ and $(z \circ a) \circ a = 0$ for all $a \in R$, then $[z, a] \circ b = (z \circ a) \circ b + (\bar{z} * \bar{a})((z \circ a) \circ b)$ for all $a, b \in R$.

Under the hypothesis of (22), we have $(z \circ (a + b)) \circ (a + b) = 0$ for all $a, b \in R$, whence

(23) For all $a, b \in R \quad (z \circ a) \circ b + (z \circ b) \circ a = 0,$

https://doi.org/10.4153/CMB-1984-031-4 Published online by Cambridge University Press

and in particular

(24) For all $a \in R \qquad (z \circ a) \circ \bar{z} = 0.$

Moreover,

(25) For all $a, b \in R \qquad (b \circ \bar{a})(z \circ a) = 0,$

as (23), (4) and (5) yield

$$\begin{aligned} 0 &= (z \circ (ba)) \circ \bar{a} + (z \circ \bar{a}) \circ (ba) \\ &= ((z \circ b)a + b(z \circ a)) \circ \bar{a} + ((z \circ \bar{a}) \circ b)a + b((z \circ \bar{a}) \circ a) \\ &= ((z \circ b) \circ \bar{a})a + b((z \circ a) \circ \bar{a}) + (b \circ \bar{a})(z \circ a) \\ &\quad + ((z \circ \bar{a}) \circ b)a + b((z \circ \bar{a}) \circ a) \\ &= (b \circ \bar{a})(z \circ a). \end{aligned}$$

In a similar way, but using (24) additionally, we get

(26) For all $a, b \in R (\bar{z} \circ b)(z \circ a) = 0,$

because

$$\begin{aligned} 0 &= (z \circ (\bar{z}a)) \circ b + (z \circ b) \circ (\bar{z}a) \\ &= (\bar{z}(z \circ a)) \circ b + ((z \circ b) \circ \bar{z})a + \bar{z}((z \circ b) \circ a) \\ &= (\bar{z}((z \circ a) \circ b) + (\bar{z} \circ b)(z \circ a) + \bar{z}((z \circ b) \circ a) \\ &= (\bar{z} \circ b)(z \circ a). \end{aligned}$$

Furthermore, (2) and the hypothesis of (22) yield

(27) For all $a \in R \qquad (z \circ a) \circ \bar{a} = 0.$

Applying (5), (25), (26) and (27), we calculate

$$\begin{aligned} ((\bar{z} * \bar{a}) \circ b)(z \circ a) &= (\bar{z} \circ b)(z \circ a) + (\bar{a} \circ b)(z \circ a) + ((\bar{z}\bar{a}) \circ b)(z \circ a) \\ &= \bar{z}(\bar{a} \circ b)(z \circ a) + (\bar{z} \circ b)\bar{a}(z \circ a) \\ &= 0 \end{aligned}$$

for all $a, b \in R$. This implies (22), since for all $a, b \in R$ we have by (5) and (17)

$$\begin{aligned} [z, a] \circ b &= (z \circ a + (\bar{z} * \bar{a})(z \circ a)) \circ b \\ &= (z \circ a) \circ b + ((\bar{z} * \bar{a}) \circ b)(z \circ a) + (\bar{z} * \bar{a})((z \circ a) \circ b) \\ &= (z \circ a) \circ b + (\bar{z} * \bar{a})((z \circ a) \circ b). \end{aligned}$$

Now we are ready to prove the desired equality $Y_2 = Z_2$. Suppose first $z \in Z_2$. Then obviously the hypothesis of (22) is satisfied, and for all $a \in R$ we consequently have $[z, a] \in Z(R)$, whence $z \in Y_2$. Conversely, suppose $z \in Y_2$. Then, applying (18), we see that again the hypothesis of (22) holds. Hence

$(z \circ a) \circ b + (\bar{z} * \bar{a})((z \circ a) \circ b) = 0$ for all $a, b \in R$. But (16) now yields $(z \circ a) \circ b = 0$ for all $a, b \in R$, i.e. $z \in Z_2$. Our proof is complete.

References

1. N. Jacobson, *Lie Algebras*, Wiley, New York–London 1962.

2. S. A. Jennings, *The group ring of a class of infinite nilpotent groups*, Canad. J. Math. **7** (1955), 169–187.

3. S. A. Jennings, *Radical rings with nilpotent associated groups*, Trans. Roy. Soc. Can. **49** ser. III (1955), 31–38.

4. L. Kaloujnine, *Über gewisse Beziehungen zwischen einer Gruppe und ihren Automorphismen*, Ber. math. Tagung Berlin 1953.

Mathematisches Seminar der Universität
Olshausenstr. 40
D 2300 Kiel 1
Bundesrepublik Deutschland