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# A GENERALIZATION OF ČAPLYGIN'S INEQUALITY WITH APPLICATIONS TO SINGULAR BOUNDARY VALUE PROBLEMS

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**1. Introduction.** Let

$$Ly = y^{(n)} + p_1(t)y^{(n-1)} + \ldots + p_n(t)y = 0, \quad \alpha < t < \beta, \tag{1.1}$$

where $p_k \in C(\alpha, \beta)$ and $-\infty \leqq \alpha < \beta \leqq \infty$. A solution of (1.1) is a nontrivial function $y \in C^n(\alpha, \beta)$, a neighborhood of $\beta$ is an interval of the form $(\gamma, \beta)$, $\alpha \leqq \gamma < \beta$, and a neighborhood of $\alpha$ is an interval of the form $(\alpha, \gamma)$, $\alpha < \gamma \leqq \beta$. The endpoint $\beta$ $(\alpha)$ is said to be singular if it is $\infty$ $(-\infty)$ or if one of the functions $p_k$ in (1.1) is not integrable in a neighborhood of $\beta$ $(\alpha)$. An ordered set $(u_1, \ldots, u_n)$ of functions is called a *principal system at* $\beta$ $(\alpha)$ provided $u_k(t) > 0$ in some neighborhood of $\beta$ $(\alpha)$ and

$$\lim_{t \to \beta(\alpha)} \frac{u_k(t)}{u_{k+1}(t)} = 0 \qquad (k = 1, \ldots, n-1),$$

and is called a *fundamental principal system on* $[\alpha, \beta]$ provided $(u_1, \ldots, u_n)$ is a principal system at $\beta$ and $(u_n, \ldots, u_1)$ is a principal system at $\alpha$. Thus, $(e^{-t}, e^t)$ is a fundamental principal system of solutions of $y'' - y = 0$ on $[-\infty, \infty]$. In general the function $u_k$ in a fundamental principal system $(u_1, \ldots, u_n)$ of solutions of (1.1) is unique up to multiplication by an arbitrary constant $c_k$.

The Wronskian determinant of $k$ functions $u_1, \ldots, u_k$ is denoted by $W(u_1, \ldots, u_k)$, i.e.,

$$W(u_1, \ldots, u_k) = W(u_1, \ldots, u_k)(t)$$
$$= \det \begin{vmatrix} u_1(t) & \ldots u_k(t) \\ \cdot & \cdot \\ \cdot & \ldots \quad \cdot \\ \cdot & \cdot \\ u_1^{(k-1)}(t) & \ldots u_k^{(k-1)}(t) \end{vmatrix} \quad (W(u_1) \equiv u_1(t)).$$

A system $(u_1, \ldots, u_n)$ of functions for which

$$W(u_1, \ldots, u_k) \neq 0 \quad (t \in I;\ k = 1, \ldots, n),$$

is called a *Markov* system on the interval $I$. (Such a system was called a Pólya system in [**6**] and [**7**].)

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Let $(u_1, \ldots, u_n)$ be a fundamental principal system of solutions of (1.1) on $[\alpha, \beta]$ for which both $(u_1, \ldots, u_n)$ and $(u_n, \ldots, u_1)$ are Markov systems on $(\alpha, \beta)$. Define

$$(\mathscr{D}_\alpha{}^k f)(t) = \lim_{s\to t} \frac{W(u_1, \ldots, u_k, f)(s)}{W(u_1, \ldots, u_{k+1})(s)} \qquad (k = 0, \ldots, n-1),$$

$$(\mathscr{D}_\beta{}^k f)(t) = \lim_{s\to t} \frac{W(u_n, \ldots, u_{n-k+1}, f)(s)}{W(u_n, \ldots, u_{n-k})(s)} \qquad (k = 0, \ldots, n-1),$$

$(W(u_1, \ldots, u_k, f) \equiv f \equiv W(u_n, \ldots, u_{n-k+1}, f)$ when $k = 0)$. Define the $k$th *generalized derivative* of a function $f \in C^k(\alpha, \beta)$ by the formula

$$(\mathscr{D}^k f)(t) = \begin{cases} (\mathscr{D}_\alpha{}^k f)(\alpha) & \text{if } t = \alpha, \\ f^{(k)}(t) & \text{if } \alpha < t < \beta, \\ (\mathscr{D}_\beta{}^k f)(\beta) & \text{if } t = \beta. \end{cases}$$

In contexts where the dependence of the operators $\mathscr{D}_\alpha$, $\mathscr{D}_\beta$ on the fundamental principal system $(u_1, \ldots, u_n) \equiv S$ is of critical importance to the discussion we will write $\mathscr{D}_{\alpha S}$, $\mathscr{D}_{\beta S}$, $\mathscr{D}_S$, respectively. If $S$ and $T$ are two fundamental principal systems of solutions of (1.1) on $[\alpha, \beta]$, then it follows from the previous remark about uniqueness that there exist positive constants $b_k$, $c_k$ such that $\mathscr{D}_{\alpha S}{}^k f = c_k \mathscr{D}_{\alpha T}{}^k f$ and $\mathscr{D}_{\beta S}{}^k f = b_k \mathscr{D}_{\beta T}{}^k f$ for all admissible $f$. Thus, $(\mathscr{D}_S{}^k f)(t) = 0$ if and only if $(\mathscr{D}_T{}^k f)(t) = 0$. Define $f$ to have a $j$th *order zero* at $t \in [\alpha, \beta]$ if $(\mathscr{D}^k f)(t) = 0$ for $k = 0, \ldots, j-1$ and $(\mathscr{D}^j f)(t) \neq 0$. Here, $(\mathscr{D}^j f)(t) \neq 0$ means $(\mathscr{D}^j f)(t)$ does not exist, or exists and is not 0. Thus the idea of a zero of a solution of (1.1) at $\alpha$ or $\beta$ becomes meaningful even though $\alpha$ and $\beta$ may be singular points of the equation. It is important to keep in mind that zeros at singular points depend upon the operator $L$ as well as the function. Thus, the function $t$ does not have a zero at $\infty$ with respect to $L = D^2$ but has a double zero at $\infty$ with respect to $L = D^4$. The above definition of zero is equivalent (cf. section 2) for solutions of (1.1) to the definition used by Levin [**3**].

Define $Z_y I$ to be the number of zeros counting multiplicities, of the function $y$ in the set $I$, and $Z_y \gamma = Z_y[\gamma, \gamma]$. Equation (1.1) is called *disconjugate on an interval* $I \subset [\alpha, \beta]$ if for any solution $y$, $Z_y I \leqq n - 1$; and is called disconjugate at $\gamma$, $\alpha \leqq \gamma \leqq \beta$, if there exists a neighborhood $N$ of $\gamma$ such that (1.1) is disconjugate on $[-\infty, \infty]$, thus, the equation $y'' = 0$ is disconjugate on every proper subinterval of $[-\infty, \infty]$ but not on $[-\infty, \infty]$.

We assume throughout this paper that equation (1.1) is disconjugate on $[\alpha, \beta]$. Our main purpose is to prove and apply the following two theorems.

THEOREM 1.1. *Assume that* (1.1) *is disconjugate on* $[\alpha, \beta]$, $f \in C^{n-1}(\alpha, \beta)$ *and* $f^{(n)}(t)$ *exists for* $\alpha < t < \beta$. *If there exists a solution* $y$ *of* (1.1) (*may be the identical zero solution*) *such that* $Z_{f-y}[\alpha, \beta] \geqq n + 1$ and $Z_{f-y}(\alpha, \beta) \geqq 1$, *then there exists* $\xi$, $\alpha < \xi < \beta$, *such that* $(Lf)(\xi) = 0$.

THEOREM 1.2. *Assume that* (1.1) *is disconjugate on* $[\alpha, \beta]$. *If* $f \in C^n(\alpha, \beta)$, *there exists an integer* $r$, $0 \leqq r \leqq n$, *such that* $(-1)^{n-r}Lf \geqq 0$ *and*

$$(1.2) \qquad \begin{aligned} (\mathscr{D}^j f)(\alpha) &\geqq 0 \quad (j = 0, \ldots, r-1), \\ (\mathscr{D}^j f)(\beta) &\geqq 0 \quad (j = 0, \ldots, n-r-1), \end{aligned}$$

*then* $f(t) \geqq 0$ *for* $\alpha < t < \beta$. *Furthermore, if* $(-1)^{n-r}Lf > 0$ *or strict inequality holds for at least one value of* $j$ *in* (1.2), *then* $f(t) > 0$ *for* $\alpha < t < \beta$. (It is to be understood that all the conditions in (1.2) are to be taken at $\beta$ if $r = 0$ and at $\alpha$ if $r = n$).

Theorem 1.1, which is proven in section 2, is a generalization of a well-known theorem of Pólya [**4**]. Some consequences of this theorem are given in section 3. These include generalized finite Taylor expansion formulas with remainders in an integral form in one case and in a Lagrange form in another case.

Theorem 1.2 is a generalization of Čaplygin's inequality [**2**], which states that $Lz \geqq 0$, $z^{(k)}(\alpha) = 0$ $(k = 0, \ldots, n-1)$ implies $z \geqq 0$ on $[\alpha, \beta)$ provided $Ly = 0$ is disconjugate on $[\alpha, \beta)$. Levin [**3**] (cf., also [**6**; **7**]) has shown that the disconjugacy of (1.1) on $[\alpha, \beta]$ implies the existence of a fundamental principal system of solutions on $[\alpha, \beta]$. Thus, if (1.1) is disconjugate on $[\alpha, \beta]$ in the sense of Levin and its solutions can be found, then the appropriate behaviour of a function $f$ at $\alpha$ and/or $\beta$ with respect to these solutions and $(-1)^{n-r}Lf \geqq 0$ imply $f \geqq 0$. For example, $f(t) \geqq 0$, $-\infty < t < \infty$, provided either $f''(t) \geqq f(t)$, $-\infty < t < \infty$,

$$\lim_{t \to -\infty} e^t f(t) \geqq 0 \quad \text{and} \quad \lim_{t \to -\infty} e^{-t}[f'(t) + f(t)] \geqq 0,$$

or $f''(t) \leqq f(t)$, $-\infty < t < \infty$,

$$\lim_{t \to -\infty} e^t f(t) \geqq 0 \quad \text{and} \quad \lim_{t \to \infty} e^{-t} f(t) \geqq 0.$$

Note that (1.2) is invariant of the fundamental principal system used in evaluating the generalized derivatives, i.e., $\mathscr{D}_S{}^k f \geqq 0$ at a point with respect to one fundamental principal system $S$ implies $\mathscr{D}_T{}^k f \geqq 0$ at that point with respect to any other fundamental principal system $T$. In section 4, Theorem 1.2 is proven, and a refinement eliminating the requirement that the solutions of (1.2) be known in order to evaluate the generalized derivatives at $\alpha$ and $\beta$, is obtained for a large class of second order equations.

Theorem 1.2 is applied in section 5 to generalized boundary value problems of the form

$$(1.3) \qquad Ly = f(t, y, y', \ldots, y^{(n-1)}) \quad (\alpha < t < \beta),$$

$$(1.4) \qquad \begin{aligned} (\mathscr{D}^k y)(\alpha) &= a_{k+1} \quad (k = 0, \ldots, r-1), \\ (\mathscr{D}^k y)(\beta) &= b_{k+1} \quad (k = 0, \ldots, n-r-1), \end{aligned}$$

where both $\alpha$ and $\beta$ may be singular points. Corollary 5.1 is a result for the conventional boundary value problem

$$(1.5)\qquad \begin{aligned} y^{(n)} &= f(t, y) && (0 \leqq t < \infty), \\ y^{(k)}(0) &= a_{k+1} && (k = 0, \ldots, r-1), \\ \lim_{t\to\infty}(-1)^k t^{k-r} y^{(k)}(t) &= b_{k+1} && (k = 0, \ldots, n-r-1). \end{aligned}$$

For the special case $n = 2$, it implies the corresponding result of Bebernes and Jackson [**1**, Problem II].

**2. Zeros and Theorem 1.1.** The following Wronskian identities (cf., eg., Pólya and Szego [**5**, p. 113]) will be needed:

$$(2.1)\qquad W(vu_1, \ldots, vu_k) = v^k W(u_1, \ldots, u_k),$$

$$(2.2)\qquad W(u_1, \ldots, u_k) = u_1^k W((u_2/u_1)', \ldots, (u_k/u_1)') \quad (u_1 \neq 0),$$

$$(2.3)\qquad \frac{d}{dt}\frac{W(u_1, \ldots, u_{k-1}, u_{k+1})}{W(u_1, \ldots, u_k)} = \frac{W(u_1, \ldots, u_{k-1})W(u_1, \ldots, u_{k+1})}{W^2(u_1, \ldots, u_k)}.$$

Also, any principal system $(u_1, \ldots, u_n)$ of solutions at $\beta$ have the properties that

$$(2.4)\qquad W(u_{i_1}, \ldots, u_{i_k}) > 0 \quad (1 \leqq i_1 < \ldots < i_k \leqq n;\ k = 1, \ldots, n)$$

is some neighborhood of $\beta$, and

$$(2.5)\qquad \lim_{t\to\beta} \frac{W(u_{i_1}, \ldots, u_{i_k})}{W(u_{j_1}, \ldots, u_{j_k})} = 0 \quad (i_m \leqq j_m \text{ with strict inequality for at least one value of } m).$$

Pólya [**4**] proved that (1.1) is disconjugate on an interval $I \subset (\alpha, \beta)$ provided (1.1) has a Markov system on $I$. The existence of a principal system at a point is not sufficient to imply disconjugacy at that point (cf., Levin [**3**, p. 49]). However, it is true that a principal system $(u_1, \ldots, u_n)$ at $\beta$, as well as the system $(u_n, \ldots, u_1)$, are Markov systems at $\beta$ if and only if (1.1) is disconjugate at $\beta$. Thus, the disconjugacy of (1.1) on $(\alpha, \beta)$ implies that the denominators in the definition of $\mathscr{D}_\alpha^k$ and $\mathscr{D}_\beta^k$ do not vanish. We assume throughout this section that (1.1) is disconjugate on $[\alpha, \beta]$.

LEMMA 2.1. *If $Z_f\alpha = j$ ($j \leqq n$), then for any principal system $(u_n, \ldots, u_1)$ of solutions at $\alpha$,*

$$\lim_{t\to\alpha} (f(t)/u_k(t)) = 0 \qquad (k = 1, \ldots, j),$$

*and the corresponding limit with $k = j + 1$ is not zero when $j < n$.*

*Proof.* Clearly the lemma is true for all first order equations. Suppose that it is true for all disconjugate equations of order less than $n$ ($n > 1$). Let

$(u_1, \ldots, u_n)$ be a fundamental principal system of solutions of (1.1) on $[\alpha, \beta]$ and suppose that

$$(2.6) \qquad (\mathscr{D}_\alpha{}^k f)(\alpha) = \lim_{t\to\alpha} \frac{W(u_1, \ldots, u_k, f)}{W(u_1, \ldots, u_{k+1})} = 0 \qquad (k = 0, \ldots, j-1)$$

and the corresponding limit with $k = j$ is not 0 (provided $j \leqq n$). Now, (2.1) and (2.2) imply

$$(2.7) \qquad \frac{W(u_1, \ldots, u_{m-1}, f)}{W(u_1, \ldots, u_m)} = \frac{W(v_1, \ldots, v_{m-2}, F)}{W(v_1, \ldots, v_{m-1})} \qquad (m = 2, \ldots, n),$$

where

$$F = W(u_1, f), \quad v_k = W(u_1, u_{k+1}) \quad (k = 1, \ldots, n-1).$$

Since

$$W(v_1, \ldots, v_{k-1}) = u_1{}^{k-2} W(u_1, \ldots, u_k) \neq 0 \quad (k = 2, \ldots, n)$$

and

$$\lim_{t\to\alpha} (v_{k+1}(t)/v_k(t)) = 0 = \lim_{t\to\beta} (v_k(t)/v_{k+1}(t)) \qquad (k = 1, \ldots, n-2)$$

by (2.5), we conclude that $(v_1, \ldots, v_{n-1})$ is a fundamental principal system on $[\alpha, \beta]$ for the disconjugate $(n-1)$st order differential equation $My = W(v_1, \ldots, v_{n-1}, y) = 0$. But (2.6) and (2.7) imply that $F$ has a $(j-1)$st order zero at $\alpha$ with respect to $M$. Thus, the induction hypothesis implies

$$\lim_{t\to\alpha} (F(t)/v_k(t)) = 0 \qquad (k = 1, \ldots, j-1)$$

and the corresponding limit with $k = j$ is not zero (provided $j \leqq n$). But (2.2) and L'Hôpital's Rule in conjunction with another induction argument imply

$$\lim_{t\to\alpha} \frac{f(t)}{u_{k+1}(t)} = \lim_{t\to\alpha} \frac{(f(t)/u_1(t))'}{(u_{k+1}(t)/u_1(t))'} = \lim_{t\to\alpha} \frac{F(t)}{v_k(t)} = 0 \qquad (k = 1, \ldots, j-1),$$

and the corresponding limit with $k = j$ is not zero (provided $j \leqq n$). Thus, the conclusion holds for the particular principal system $(u_n, \ldots, u_1)$ at $\alpha$. It is a trivial manner to obtain the conclusion for all principal systems once it has been achieved for one principal system.

The converse of Lemma 2.1 is easy to prove for solutions $f$ of (1.1). Thus, our definition of zero and the definition of Levin [**3**] coincide for solutions of (1.1).

LEMMA 2.2. *For any $\tau$, $\alpha < \tau < \beta$, $Z_f\tau = j$ $(j \leqq n)$, if and only if*

$$(\mathscr{D}_\alpha{}^k f)(\tau) = 0 \quad (k = 0, \ldots, j-1),$$
$$(\mathscr{D}_\alpha{}^j f)(\tau) \neq 0 \quad (\text{if } j < n).$$

*Proof.* Since $(u_1, \ldots, u_n)$ is a Markov system on $(\alpha, \beta)$ and $\alpha < \tau < \beta$,

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there exist positive constants $c_k$ such that

$$\lim_{t\to\tau} W(u_1, \ldots, u_k)(t) = c_k \qquad (k = 1, \ldots, n).$$

Suppose that $f^{(m)}(\tau) = 0$ for $m = 0, \ldots, k$ $(k \leqq j)$. Then

$$(\mathscr{D}_\alpha{}^{k+1} f)(\tau) = c_k f^{(k)}(\tau)/c_{k+1}$$

which implies $f^{(k)}(\tau) = 0$ if $k < j$ and $f^{(k)}(\tau) \neq 0$ if $k = j$ and $j < n$, i.e., $Z_{f\tau} = j$. The converse is trivial.

LEMMA 2.3. *If* $(u_1, \ldots, u_n)$ *is a fundamental principal system of solutions of* (1.1) *on* $[\alpha, \beta]$, *then*

(2.8) $\quad W(u_1, \ldots, u_r, u_{n-s+1}, \ldots, u_n) > 0$ *on* $(\alpha, \beta)$ $(1 \leqq r, s \leqq n - 1;$
$r + s \leqq n).$

*Proof.* In [**6**] and [**7**] we showed that there exist functions $\xi_k \in C^{n-k}(\alpha, \beta)$ $(k = 2, \ldots, n)$, which are positive on $(\alpha, \beta)$, locally integrable in $[\alpha, \beta]$ but not integrable on $[\alpha, \beta]$ such that

(2.9) $$u_k(t) = u_1(t) I(t, \alpha; \xi_2, \ldots, \xi_k) \quad (k = 2, \ldots, n),$$

where

$$I(t, s; \xi_2, \ldots, \xi_k) = \int_s^t \xi_2(t_2) \int_s^{t_2} \xi_3(t_3) \ldots \int_s^{t_{k-1}} \xi_k(t_k) dt_k \ldots dt_2.$$

Thus, $r$ applications of (2.2) implies

(2.10) $$W(u_1, \ldots, u_r, u_{n-s+1}, \ldots, u_n) = u_1{}^{r+s} \xi_2{}^{r+s-1} \ldots \xi_r{}^{s+1} W(z_{n-r-s+1}, \ldots, z_{n-r}),$$

where

(2.11) $$z_1(t) = \xi_{r+1}(t),\ z_k(t) = \xi_{r+1}(t) I(t, \alpha; \xi_{r+2}, \ldots, \xi_{r+k}) (k = 2, \ldots, n - r).$$

But (2.11) clearly implies that $(z_1, \ldots, z_{n-r})$ is a Markov system on $(\alpha, \beta)$ and a fundamental principal system of solutions on $[\alpha, \beta]$ of the $(n - r)$th order equation $My = W(z_1, \ldots, z_{n-r}, y) = 0$, which must then be disconjugate on $[\alpha, \beta]$. Hence, $(z_{n-r}, \ldots, z_1)$ is a Markov system on $(\alpha, \beta)$, which implies that (2.10) cannot vanish in $(\alpha, \beta)$, and hence must be positive by (2.4).

*Proof of Theorem* 1.1. Since $Lf = Lf - Ly = L(f - y)$, the theorem is true provided the special case when $y \equiv 0$ is true. The proof of this case will be divided into two main parts.

First, suppose that $Z_f[\alpha, \beta) \geqq n + 1$ and $Z_f(\alpha, \beta) \geqq 1$. Let

$$g_k = \frac{W(u_1, \ldots, u_{k-1}, f)}{W(u_1, \ldots, u_k)} \qquad (k = 1, \ldots, n + 1; u_{n+1} \equiv 1),$$

and define $f$ to have at least $j(j \leqq n - k + 2)$ $g_k$-zeros at a point $\tau \in [\alpha, \beta)$ if

$$\lim_{t \to \tau} g_i(t) = 0 \qquad (i = k, k+1, \ldots, k+j-1). \tag{2.12}$$

The assumptions and Lemma 2.2 imply that $f$ has at least $n + 1$ $g_1$-zeros in $[\alpha, \beta)$ with at least one $g_1$-zero in $(\alpha, \beta)$. Suppose there exist at least $n + 3 - k$ $g_{k-1}$-zeros in $[\alpha, \beta)$ $(1 < k < n + 1)$ with at least one $g_{k-1}$-zero in $(\alpha, \beta)$. Let these $g_{k-1}$-zeros be located at the points $\zeta_1, \ldots, \zeta_m$, where $\alpha \leqq \zeta_1 < \ldots < \zeta_m < \beta$, with multiplicity at least $\nu_1, \ldots, \nu_m$ so that $\nu_1 + \ldots + \nu_m = n + 3 - k$. Since $\zeta_s$ is a $g_{k-1}$-zero of multiplicity at least $\nu_s$, $\zeta_s$ is clearly a $g_k$-zero of multiplicity at least $\nu_s - 1$. Furthermore, since

$$\lim_{t \to \zeta_s} g_{k-1}(t) = 0 = \lim_{t \to \zeta_{s+1}} g_{k-1}(t) \qquad (s = 1, \ldots, m-1),$$

Rolle's theorem implies there exists $\eta_s (\zeta_s < \eta_s < \zeta_{s+1})$ such that $(Dg_{k-1})(\eta_s) = 0$ $(s = 1, \ldots, m - 1)$. But then (2.3) implies

$$g_k(\eta_s) = \frac{W^2(u_1, \ldots, u_{k-1})}{W(u_1, \ldots, u_{k-2}) W(u_1, \ldots, u_k)} (Dg_{k-1})(\eta_s) = 0 \qquad (s = 1, \ldots, m-1).$$

So, strictly in between every two $g_{k-1}$-zeros $\zeta_s$, $\zeta_{s+1}$, there exists at least one $g_k$-zero $\eta_s$, which implies that the total number of $g_k$-zeros in $[\alpha, \beta)$ is at least

$$(\nu_1 - 1) + \ldots + (\nu_m - 1) + (m - 1) = n + 2 - k,$$

and clearly, $\zeta_m$ or $\eta_{m-1}$ is in $(\alpha, \beta)$. Thus the finite induction principal implies that there exist at least one $g_{n+1}$-zero $\xi$ in $(\alpha, \beta)$, i.e.,

$$0 = g_{n+1}(\xi) = (Lf)(\xi).$$

Now suppose that there are $r + 1$ zeros of $f$ in $[\alpha, \beta)$, $0 \leqq r < n$, and at least $n - r$ zeros at $\beta$. Consider the equation $My = W(u_1, \ldots, u_r, y) = 0$. Clearly the $r + 1$ zeros of $f$ in $[\alpha, \beta)$ with respect to $Ly = 0$ are also zeros of $f$ with respect to $My = 0$. Thus, by the first case of this proof, the part already established, there exists $\xi_1 \in (\alpha, \beta)$ such that

$$0 = (Mf)(\xi_1) = W(u_1, \ldots, u_r, f)(\xi_1). \tag{2.13}$$

The remainder of the proof in this case is by induction. Assume that there exists $\xi_j \in (\alpha, \beta)$, $1 \leqq j < n - r + 1$, such that

$$W(u_1, \ldots, u_r, f, u_{n-j+2}, \ldots, u_n)(\xi_j) = 0. \tag{2.14}$$

Here, if $j = 1$, then (2.14) is to be interpreted as (2.13), which has been established. Similar adjustments, which will not be made explicit, in the notation have to be made for the case $j = 1$ in what follows. Let

$$q(t) = \frac{W(u_1, \ldots, u_r, f, u_{n-j+2}, \ldots, u_n)(t)}{W(u_1, \ldots, u_r, u_{n-j+1}, \ldots, u_n)(t)}.$$

Lemma 2.3 and (2.3) imply $q \in C^1(\alpha, \beta)$, and (2.14) implies $q(\xi_j) = 0$.

If there exists $\eta \in (\alpha, \beta)$, $\eta \neq \xi_j$, such that $q(\eta) = 0$, or if $q(t) \to 0$ as $t \to \beta$, then Rolle's Theorem implies there exists $\xi_{j+1} \in (\alpha, \beta)$ such that $q'(\xi_{j+1}) = 0$, which implies by (2.3) and Lemma 2.3 that

$$W(u_1, \ldots, u_r, f, u_{n-j+1}, \ldots, u_n)(\xi_{j+1}) = 0. \tag{2.15}$$

We will show in what follows that there must always exist $\xi_{j+1} \in (\alpha, \beta)$ such that (2.15) holds.

Suppose that $|q(t)| > \epsilon > 0$ for $\tau \leqq t < \beta$, where $\xi_j < \tau < \beta$. For all $|c| > 1/\epsilon$, the differential equations

$$K_c y = \frac{W(u_1, \ldots, u_r, \overset{*}{u}_{n-j+1}, u_{n-j+2}, \ldots, u_n, y)}{W(u_1, \ldots, u_r, u^*_{n-j+1}, u_{n-j+2}, \ldots, u_n)} = 0,$$

where $u^*_{n-j+1}(t) = u_{n-j+1}(t) - cf(t)$, are well defined on $[\tau, \beta)$. Suppose there exists $c(|c| > 1/\epsilon)$ such that $K_c y = 0$ is not disconjugate on $(\tau, \beta)$. Then, there exists a solution

$$\phi(t) = c_1 u_1(t) + \ldots + c_{n-j+1} u^*_{n-j+1}(t) + \ldots + c_n u_n(t)$$

of $K_c y = 0$ with $r + j$ zeros in $(\tau, \beta)$. Also, $c_{n-j+1} \neq 0$, for otherwise $\phi$ would be a nontrivial solution of the equation

$$Ny = W(u_1, \ldots, u_r, y, u_{n-j+2}, \ldots, u_n) = 0,$$

which would violate the disconjugacy of $Ny = 0$ on $(\tau, \beta)$. Thus, $u^*_{n-j+1}$ agrees with the solution

$$\psi(t) = [-\phi(t) + c_{n-j+1} u^*_{n-j+1}(t)]/c_{n-j+1}$$

of the $(r + j - 1)$st order disconjugate equation $Ny = 0$ at $r + j$ points (counting multiplicities) of $(\tau, \beta)$. So by the first case of this proof, which has already been established, there exists $\zeta \in (\tau, \beta)$ such that $(Nu^*_{n-j+1})(\zeta) = 0$, i.e.

$$(u_1, \ldots, u_r, u_{n-j+1}, \ldots, u_n) - cW(u_1, \ldots, u_r, f, u_{n-j+2}, \ldots, u_n)](\zeta) = 0.$$

Thus,

$$|q(\zeta)| = 1/|c| < \epsilon < |q(\zeta)|,$$

which is a contradiction; hence, each of the differential equations $K_c y = 0$ $(|c| > 1/\epsilon)$ must be disconjugate on $(\tau, \beta)$. Since $f$ has at least $n - r > j - 1$ zeros at $\beta$, Lemma 2.1 implies

$$\lim_{t \to \beta} (f(t)/u_{n-j+1}(t)) = 0,$$

which implies

$$u^*_{n-j+1}(t) = u_{n-j+1}(t)[1 - cf(t)/u_{n-j+1}(t)] = u_{n-j+1}(t)[1 + o(1)], \text{ as } t \to \beta.$$

Thus, $(u_1, \ldots, u_r, u^*_{n-j+1}, u_{n-j+2}, \ldots, u_n)$ is a principal system of solutions of $K_c y = 0$ at $\beta$. Hence, (2.4) implies $W(u_1, \ldots, u_r, u_{n-j+1}, u^*_{n-j+2}, \ldots, u_n) > 0$

in some neighborhood $N(c)$ of $\beta$, i.e.,

$$|q(t)| < 1/|c| \quad \text{for} \quad t \in N(c) \cap N(-c).$$

Hence, once again $q(t) \to 0$, as $t \to \beta$, and Rolle's Theorem implies the existence of $\xi_{j+1}$ in $(\alpha, \beta)$ such that (2.15) holds.

Thus in every case, (2.14) implies (2.15); hence, by the principal of finite induction, there exists $\xi_{n-j+1} \in (\alpha, \beta)$ such that

$$W(u_1, \ldots, u_r, f, u_{r+1}, \ldots, u_n)(\xi_{n-r+1}) = 0,$$

i.e., $(Lf)(\xi_{n-r+1}) = 0$.

**3. Generalized mean value theorems.** Throughout this section we assume that (1.1) is disconjugate on $[\alpha, \beta]$ and $(u_1, \ldots, u_n)$ is an arbitrary but fixed fundamental principal system of solutions on $[\alpha, \beta]$. If

$$\text{(3.1)} \quad v_k(t) = W(u_1, \ldots, u_{k-1}, u_{k+1}, \ldots, u_n)(t)/W(u_1, \ldots, u_n)(t) \qquad (k = 1, \ldots, n),$$

$$\text{(3.2)} \quad h_j(t, s) = \begin{cases} \sum_{k=1}^{j-1} (-1)^{n-k} u_k(t)(v_k(s)/v_j(s))', & \alpha < s < t, \\ \sum_{k=j+1}^{n} (-1)^{n-k+1} u_k(t)(v_k(s)/v_j(s))', & t \leqq s < \beta, \end{cases}$$

$$\text{(3.3)} \quad \begin{cases} w_0(t) = \int_\alpha^\beta h_1(t, s)\left(\int_\beta^s v_1(\tau) f(\tau) d\tau\right) ds, \\ w_j(t) = \int_\alpha^\beta h_j(t, s)\left(\int_\alpha^s v_j(\tau) f(\tau) d\tau\right) ds \qquad (j = 1, \ldots, n), \end{cases}$$

(where $\sum_{k=1}^{j-1} \equiv 0$ if $j = 1$ and $\sum_{k=j+1}^{n} \equiv 0$ if $j = n$ in (3.2)) then (cf. [**6**; **7**]) $(v_n, \ldots, v_1)$ is a fundamental principal system on $[\alpha, \beta]$,

$$\text{(3.4)} \quad (-1)^{n-j-1} h_j(t, s) > 0 \qquad (s \neq t),$$

$$\text{(3.5)} \quad u_j(t) = (-1)^{n-j-1} \int_\alpha^\beta h_j(t, s) ds,$$

$$\text{(3.6)} \quad (Lw_j)(t) = -f(t) \qquad (\alpha < t < \beta),$$

$$\text{(3.7)} \quad w_j^{(k)}(t) = \sum_{r=1}^{n} c_r(t) u_r^{(k)}(t) \qquad (k = 0, \ldots, n-1),$$

where

$$c_r(t) = \begin{cases} (-1)^{n-r+1} \int_\alpha^t v_r(s) f(s) ds & (r = 1, \ldots, j; 1 \leqq j \leqq n), \\ (-1)^{n-r} \int_t^\beta v_r(s) f(s) ds & (r = j+1, \ldots, n; 0 \leqq j \leqq n-1). \end{cases}$$

The formula (3.7) follows from (3.3) by an appropriate integration by parts

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and the fact that $u_n(t)v_n(s) - \ldots + (-n)^{n-1}u_1(t)v_1(s)$ is the Cauchy function for (1.1).

LEMMA 3.1. *If $v_j f$ (when $1 \leqq j \leqq n$) is integrable on $[\alpha, t]$ and $v_{j+1}f$ (when $0 \leqq j \leqq n-1$) is integrable on $[t, \beta]$ for $\alpha < t < \beta$, then $w_j^{(n-1)}$ is locally absolutely continuous on $(\alpha, \beta)$ and*

$$(\mathscr{D}^k w_j)(\alpha) = 0 \quad (k = 0, \ldots, j-1; j = 1, \ldots, n),$$
$$(\mathscr{D}^k w_j)(\beta) = 0 \quad (k = 0, \ldots, n-j-1; j = 0, \ldots, n-1).$$

*Proof.* Consider just the situation at the point $\alpha$; the situation at $\beta$ can be handled similarly. The elementary properties of determinants and (3.7) imply

$$\frac{W(u_1, \ldots, u_k, w_j)}{W(u_1, \ldots, u_{k+1})} = \int_t^\beta f(s) \sum_{r=j+1}^n (-1)^{n-r} v_r(s) \frac{W(u_1, \ldots, u_k, u_r)(t)}{W(u_1, \ldots, u_{k+1})(t)} ds + o(1)$$

$(k = 0, \ldots, j-1)$. Let the functions $u_k$ be represented by (2.9); then

$$v_k(t) = v_n(t) I(t, \alpha; \xi_n, \ldots, \xi_{k+1}) \quad (k = 1, \ldots, n-1)$$

(cf. [**6**, Theorem 1.2, p. 294]), and if

$$\zeta_r(t, s) = I(t, \alpha; \xi_{k+2}, \ldots, \xi_r) I(s, \alpha; \xi_n, \ldots, \xi_{r+1}),$$

then (cf. [**6**, Lemma 2.2, pp. 298-299])

$$\sum_{r=k+1}^n (-1)^{n-r} \zeta_r(t, s) = (-1)^{n-k+1} I(s, t; \xi_n, \ldots, \xi_{k+2}). \tag{3.8}$$

Using (2.2) and (3.8), we conclude that

$$\left| \sum_{r=j+1}^n (-1)^{n-r} v_r(s) \frac{W(u_1, \ldots, u_k, u_r)(t)}{W(u_1, \ldots, u_{k+1})(t)} \right| = \left| \sum_{r=j+1}^n (-1)^{n-r} v_n(s) \zeta_r(t, s) \right|$$

$$= v_n(s) \left| I(s, t; \xi_n, \ldots, \xi_{k+2}) - I(s, \alpha; \xi_n, \ldots, \xi_{k+2}) + \sum_{r=k+2}^j (-1)^{k-r} \zeta_r(t, s) \right|$$

$$\leqq v_n(s) I(s, \alpha; \xi_n, \ldots, \xi_{j+1}) \left[ 2I(s, \alpha; \xi_j, \ldots, \xi_{k+2}) + \sum_{r=k+2}^j \zeta_r(t, s) \right]$$

$$\leqq v_j(s)[2I(s, \alpha; \xi_j, \ldots, \xi_{k+2}) + o(1), \text{ as } t \to \alpha$$

($\sum_{r=k+2}^j \equiv 0$ and $I(s, \alpha; \xi_j, \ldots, \xi_{k+2}) \equiv 1$ if $k = j-1$). Let $\epsilon > 0$ be given and choose $T = T_\epsilon > \alpha$ such that

$$2 \int_\alpha^T I(s, \alpha; \xi_j, \ldots, \xi_{k+2}) v_j(s) |f(s)| ds < \epsilon.$$

Then

$$|\mathscr{D}^k w_j)(\alpha)| \leqq \lim_{t \to \alpha} \left| \int_t^T \right| + \lim_{t \to \alpha} \left| \int_T^\beta \right| \leqq \epsilon,$$

since the second limit is zero by (2.5). Since $\epsilon$ is arbitrary, we conclude that $(\mathscr{D}^k w_j)(\alpha)$ exists and is zero $(k = 0, \ldots, j-1)$.

COROLLARY 3.1. *For any integer $r$, $0 \leqq r \leqq n$, and constants $a_0, \ldots, a_{r-1}$ $(r \geqq 1)$, $b_0, \ldots, b_{n-r-1}$ $(r \leqq n-1)$, Equation* (1.1) *has the unique solution*

$$y(t) = \sum_{k=1}^{r} a_{k-1}u_k(t) + \sum_{k=r+1}^{n} b_{n-k}u_k(t)$$

*satisfying*

$$(\mathscr{D}^k y)(\alpha) = a_k \quad (k = 0, \ldots, r-1),$$
$$(\mathscr{D}^k y)(\beta) = b_k \quad (k = 0, \ldots, n-r-1).$$

Lemma 3.1 and Corollary 3.1 imply a generalized finite Taylor expansion for functions; namely

$$f(t) = \sum_{k=1}^{r} (\mathscr{D}^{k-1}f)(\alpha)u_k(t) + \sum_{k=r+1}^{n} (\mathscr{D}^{n-k}f)(\beta)u_k(t) - R(t) \tag{3.9}$$

$(0 \leqq r \leqq n;\ \sum_{k=1}^{0} \equiv 0 \equiv \sum_{k=n+1}^{n})$, provided $f \in C^n(\alpha, \beta)$, the generalized derivatives of $f$ appearing in (3.9) exist, and $v_{r+1}(t)(Lf)(t)$ (if $0 \leqq r \leqq n-1$) is integrable at $\beta$ and $v_r(t)(Lf)(t)$ (if $1 \leqq r \leqq n$) is integrable at $\alpha$. Here the remainder term $R(t)$ has the exact integral representation

$$R(t) = \begin{cases} \displaystyle\int_\alpha^\beta h_{r+1}(t, s)\left(\int_\beta^s v_{r+1}(\tau)(Lf)(\tau)d\tau\right)ds & (0 \leqq r \leqq n-1), \\ \displaystyle\int_\alpha^\beta h_r(t, s)\left(\int_\alpha^s v_r(\tau)(Lf)(\tau)d\tau\right)ds & (1 \leqq r \leqq n). \end{cases}$$

Lemmas 2.1 and 3.1 imply

$$R(t) = \begin{cases} o(u_r(t)), & \text{as } t \to \alpha \quad (1 \leqq r \leqq n), \\ o(u_{r+1}(t)), & \text{as } t \to \beta \quad (0 \leqq r \leqq n-1). \end{cases}$$

Theorem 3.1, which follows, generalizes (3.9) to the multi-point situation at the expense of preciseness in the remainder term, i.e., there is a Lagrange type remainder term.

THEOREM 3.1 (Generalized Mean Value Theorem). *Let $m$, $0 \leqq m \leqq n-1$, and $r_k (k = 0, \ldots, m)$ be integers such that $\sum_{k=0}^{m} r_k = n$. If $\alpha \leqq t_0 < t_1 < \ldots < t_m \leqq \beta$, $f \in C^n(\alpha, \beta)$, $(\mathscr{D}^k f)(t_j)$ $(k = 0, \ldots, r_j - 1;\ j = 0, \ldots, m)$ exist, $g \in C(\alpha, \beta)$ and $g > 0$, then there exists $\xi$ $(= \xi(t))$ such that* $\min(t_0, t) < \xi < \max(t_m, t)$ *and*

$$f(t) = u(t) + w(t)(Lf)(\xi)[g(\xi)]^{-1} \quad (\alpha < t < \beta), \tag{3.10}$$

*where $Lu = 0 = Lw - g$ and*

$$(\mathscr{D}^k u)(t_j) - (\mathscr{D}^k f)(t_j) = 0 = (\mathscr{D}^k w)(t_j)$$
$$(k = 0, \ldots, r_j - 1;\ j = 0, \ldots, m).$$

*Proof.* Let $\alpha < t < \beta$, $t \neq t_r$ $(r = 0, \ldots, m)$. If $w(t) = 0$, then $w$ has $(n+1)$ zeros in $[\alpha, \beta]$ with at least one in $(\alpha, \beta)$; hence, Theorem 1.1 implies there exist $\zeta \in (\alpha, \beta)$ such that $(Lw)(\zeta) = 0$, which contradicts $(Lw)(\zeta) =$

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$g(\zeta) > 0$. So $f(t) = u(t) + cw(t)$ has a solution for $c$, namely,

$$c = [f(t) - u(t)]/w(t). \tag{3.11}$$

Thus, the function $z(s) = f(s) - u(s) - cw(s)$ has $(n + 1)$-zeros in $[\alpha, \beta]$, one of which is at $t \in (\alpha, \beta)$. Theorem 1.1 implies there exist $\xi$, min $(t_0, t) < \xi <$ max $(t_m, t)$, such that

$$0 = (Lz)(\xi) = (Lf)(\xi) - c(Lw)(\xi),$$

which implies

$$c = (Lf)(\xi)/g(\xi),$$

and proves the theorem upon substituting into (3.11).

**4. Proof of Theorem 1.2.** Let $g \in C(\alpha, \beta)$, $g > 0$, $v_r g$ be integrable at $\alpha$ and $v_{r+1}g$ be integrable at $\beta$. Then Lemma 3.1 and Corollary 3.1 imply that

$$u(t) = \sum_{k=1}^{r} (\mathscr{D}^{k-1}f)(\alpha)u_k(t) + \sum_{k=r+1}^{n} (\mathscr{D}^{n-k}f)(\beta)u_k(t),$$

$$w(t) = -\int_\alpha^\beta h_r(t, s) \left(\int_\alpha^s v_r(\tau)g(\tau)d\tau\right)ds, \; 1 \leqq r \leqq n,$$

satisfy the assumptions of Theorem 3.1 with $m = 1$, $t_0 = \alpha$, $t_1 = \beta$, $r_0 = r$ and $r_1 = n - r$. Hence, there exists $\xi$, $\alpha < \xi < \beta$, such that

$$f(t) = u(t) + w(t)(Lf)(\xi)[g(\xi)]^{-1} \quad (\alpha < t < \beta).$$

The assumptions imply $u(t) \geqq 0$ and $(-1)^{n-r}(Lf)(\xi) \geqq 0$, and (3.4) implies $(-1)^{n-r}w(t) > 0$; hence, $f(t) \geqq 0$. The case $r = 0$ is similar.

An unfortunate aspect of Theorem 1.2 in applications is the need to find a fundamental principal system of solutions of (1.1) in order to evaluate the generalized derivatives in (1.2). In some cases, one of which we shall now give, this problem can be partially circumvented.

Suppose that

$$L_2 z \equiv (r(t)z')' + p(t)z \geqq 0 \quad (\alpha < t < \beta),$$
$$z(\alpha) \geqq 0, \quad z'(\alpha) \geqq u_1'(\alpha)z(\alpha),$$

where $[\alpha, \beta]$ is bounded, $p \in C[\alpha, \beta]$, $r \in C'[\alpha, \beta]$, $r > 0$, and $u_1(t)$ is the minimal solution of $L_2 y = 0$ at $\beta$ normalized at $\alpha$, i.e., $u_1(\alpha) = 1$. Then, Theorem 1.2 implies $z(t) \geqq 0$ for $\alpha \leqq t \leqq \beta$ provided $L_2 y = 0$ is disconjugate on $[\alpha, \beta]$. Assume that

$$\begin{aligned} &R(t) \equiv \int_t^\beta r^{-1}(s)ds < \infty \qquad (\alpha \leqq t \leqq \beta), \\ &\mu \equiv R(\alpha) - \int_\alpha^\beta [R(\alpha) - R(t)]R(t)p_+(t)dt > 0 \quad (p_+(t) = \max(0, p(t))). \end{aligned} \tag{4.1}$$

Then Corollary 3 of Willett [8, p. 541] implies that $L_2 y = 0$ is disconjugate on $[\alpha, \beta]$, and the general procedure [**8**, pp. 542-544] can be easily adapted to $L_2 y = 0$ using $\beta$ as the fixed endpoint to imply that

$$(4.2) \qquad u_1'(\alpha) \leqq \nu \equiv \left[ \int_\alpha^\beta R(t) p_+(t) dt - 1 \right] \Big/ \mu r(\alpha).$$

This proves the following corollary.

COROLLARY 4.1. *If* (4.1) *holds*, $L_2 z \geqq 0$, $z(\alpha) \geqq 0$ *and* $z'(\alpha) \geqq \nu z(\alpha)$, *then* $z(t) \geqq 0$ *for* $\alpha \leqq t \leqq \beta$.

**5. Generalized boundary value problems.** Once again assume that (1.1) is disconjugate on $[\alpha, \beta]$, and that the notation of the previous two section holds; in particular, $(u_1, \ldots, u_n)$ is a fixed fundamental principal system of solutions of (1.1) on $[\alpha, \beta]$ and $(v_n, \ldots, v_1)$ is the companion fundamental principal system (3.1) on $[\alpha, \beta]$. Let $\mu_1^r(t) = u_r(t)$ and

$$\mu_k^r(t) - \sum_{m=1, m \neq r}^{n} |u_m^{(k-1)}(t)| v_m(t) v_r^{-1}(t) = \begin{cases} 0, & k = 2, \ldots, n-1, \\ v_r^{-1}(t), & k = n. \end{cases}$$

The generalized boundary value problem (1.3)-(1.4), which includes the generalized initial value problem ($r = 0$ or $r = n$), will be considered in this section.

THEOREM 5.1. *If* $f(t, y_1, \ldots, y_n) \in C((\alpha, \beta) \times R^n)$ *and if there exists a constant* $\delta$, $0 < \delta < \infty$, *such that*

$$(5.1) \qquad \int_\alpha^\beta v_r(t) \sup | f(t, u(t) + z_1(t), \ldots, u^{(n-1)}(t) + z_n(t)) | dt \leqq \delta,$$

*where the supremum is over all* $z_k \in C(\alpha, \beta)$ *such that* $|z_k(t)| \leqq \delta \mu_k^r(t)$ $(k = 1, \ldots, n)$, *and*

$$(5.2) \qquad u(t) = \sum_{k=1}^{r} a_k u_k(t) + \sum_{k=r+1}^{n} b_{n-k+1} \; {}_k(t),$$

*then* (1.3)-(1.4) *has a solution* $y \in C^n(\alpha, \beta)$ *such that*

$$(5.3) \quad y(t) = u(t) - \int_\alpha^\beta h_r(t, s) \left( \int_\alpha^s v_r(\tau) f(\tau, y(\tau), \ldots, y^{(n-1)}(\tau)) d\tau \right) ds.$$

*Proof.* The operator $T : X \to X$ defined by

$$(Tx)_k(t) = -D^{k-1} \int_\alpha^\beta h_r(t, s) \left( \int_\alpha^s v_r(\tau) f(\tau, u(\tau) + x_1(\tau), \ldots, u^{(n-1)}(\tau) + x_n(\tau)) d\tau \right) ds$$

$(k = 1, \ldots, n)$ is completely continuous on the space $X = \{(x_1, \ldots, x_n) : x_k \in C(\alpha, \beta)$ and $|x_k(t)| \leqq \delta \mu_k^r(t) (k = 1, \ldots, n)\}$, which implies that $T$ has a fixed point $(y, y', \ldots, y^{(n-1)})$ by the Schauder-Tychonoff Theorem. The details are essentially the same as in the proof of Theorem 4.2 of [**6**, pp. 311-312].

Some remarks concerning Theorem 5.1 can be made. First, (5.4) holds if $|f(t, y_1, \ldots, y_n)| \leqq \xi(t)$ for $\alpha < t < \beta$, $(y_1, \ldots, y_n) \in R^n$, and $\xi(t)v_r(t)$ is integrable on $[\alpha, \beta]$. Second, the generalized boundary value problem has a solution

$$(5.4)\quad y(t) = u(t) + \int_\alpha^\beta h_{r+1}(t, s)\left(\int_s^\beta v_{r+1}(\tau)f(\tau, y(\tau), \ldots, y^{(n-1)}(\tau))d\tau\right)ds$$

provided (5.1) with $v_r$ replaced by $v_{r+1}$ holds. Both (5.3) and (5.4) imply that

$$(5.5)\quad y^{(k)}(t) - u^{(k)}(t) = \begin{cases} o(\mu_{k+1}{}^{r+1}(t)), \text{ as } t \to \beta, \\ o(\mu_{k+1}{}^{r}(t)), \text{ as } t \to \alpha, \end{cases} (k = 0, \ldots, n-1).$$

THEOREM 5.2. *If $f(t, y) \in C((\alpha, \beta) \times R)$ and there exist functions $\xi, \zeta \in C^n(\alpha, \beta)$ such that $(\alpha < t < \beta)$*

(i) $\xi(t) \geqq \zeta(t)$,

(ii) $(-1)^{n-r}[L\xi - f(t, \xi)] \geqq 0 \geqq (-1)^{n-r}[L\zeta - f(t, \zeta)]$,

(iii) $(\mathscr{D}^k\xi)(\alpha) \geqq (\mathscr{D}^k\zeta)(\alpha)$ $(k = 0, \ldots, r-1)$ *and* $(\mathscr{D}^k\xi)(\beta) \geqq (\mathscr{D}^k\zeta)(\beta)$ $(k = 0, \ldots, n-r-1)$,

(iv) $(-1)^{n-r}f(t, y)$ *is nondecreasing in* $y$ *for* $\zeta(t) \leqq y \leqq \xi(t)$,

(v) $-\infty < (-1)^{n-r}\int_\alpha^\beta v_r(t)f(t, \zeta(t))dt \leqq (-1)^{n-r}\int_\alpha^\beta v_r(t)f(t, \xi(t))dt < \infty$,

*then for any* $a_1, \ldots, a_r, b_1, \ldots, b_{n-r}$ *such that*

$$(\mathscr{D}^k\xi)(\alpha) \geqq a_{k+1} \geqq (\mathscr{D}^k\zeta)(\alpha) \quad (k = 0, \ldots, r-1),$$
$$(\mathscr{D}^k\xi)(\beta) \geqq b_{k+1} \geqq (\mathscr{D}^k\zeta)(\beta) \quad (k = 0, \ldots, n-r-1),$$

*there exists a solution* $y \in C^n(\alpha, \beta)$ *of* $Ly = f(t, y)$ *satisfying*

$$(5.6)\quad \begin{aligned} (\mathscr{D}^k y)(\beta) &= b_{k+1} (k = 0, \ldots, n-r-1), \\ (\mathscr{D}^k y)(\alpha) &= a_{k+1} (k = 0, \ldots, r-1), \end{aligned}$$

*and*

$$\zeta(t) \leqq y(t) \leqq \xi(t) (\alpha < t < \beta).$$

*Proof.* Let

$$F(t, y) = \begin{cases} f(t, \xi(t)), & y > \xi(t), \\ f(t, y), & \xi(t) \geqq y \geqq \zeta(t), \\ f(t, \zeta(t)), & \zeta(t) > y, \end{cases}$$

so that the problem $Ly = F(t, y)$ and (5.6) has a solution $w(t)$ by Theorem 3.1. Let $z = \xi - w$ so that

$$(-1)^{n-r}Lz \geqq (-1)^{n-r}[f(t, \xi(t)) - F(t, w(t))] \geqq 0,$$

$(\mathscr{D}^k z)(\beta) \geqq 0$ $(k = 0, \ldots, n-r-1)$, $(\mathscr{D}^k z)(\alpha) \geqq 0$ $(k = 0, \ldots, r-1)$. Thus, Theorem 1.2 implies $z(t) \geqq 0$, $\alpha < t < \beta$. Similarly, $w(t) - \zeta(t) \geqq 0$, $\alpha < t < \beta$; hence, $w$ is actually a solution of $Ly = f(t, y)$ such that $\xi \geqq w \geqq \zeta$.

COROLLARY 5.1. *Assume that* $f(t, y) \in C((\alpha, \beta) \times R)$, $0 \leqq t < \infty$, $y \in R$, *and that* $0 \leqq r \leqq n - 1$ ($r = n$ *is trivial*). *If there exist functions* $\xi, \zeta \in C^n[0, \infty)$ *such that* $(0 < t < \infty)$

(i) $\xi(t) \geqq \zeta(t)$,

(ii) $(-1)^{n-r}[\xi^{(n)}(t) - f(t, \xi(t))] \geqq 0 \geqq (-1)^{n-r}[\zeta^{(n)}(t) - f(t, \zeta(t))]$,

(iii) $\xi^{(k)}(0) \geqq \zeta^{(k)}(0)$ $(k = 0, \ldots, r - 1)$ *and*

$$-\infty < \lim_{t\to\infty} ((-1)^k t^{k-r} \zeta^{(k)}(t)) \leqq \lim_{t\to\infty} ((-1)^k t^{k-r} \xi^{(k)}(t) < \infty$$
$$(k = 0, \ldots, n - r - 1),$$

(iv) $(-1)^{n-r} f(t, y)$ *is nondecreasing in* $y$ *for* $\zeta(t) \leqq y \leqq \xi(t)$,

(v) $-\infty < (-1)^{n-r} \int_0^\infty t^{n-r} f(t, \zeta(t))dt \leqq (-1)^{n-r} \int_0^\infty t^{n-r} f(t, \xi(t))dt < \infty$,

*then for any* $a_1, \ldots, a_r, b_1, \ldots, b_{n-r}$ *such that*

$$\xi^{(k)}(0) \geqq a_{k+1} \geqq \zeta^{(k)}(0) (k = 0, \ldots, r - 1), \textit{ and}$$

$$\lim_{t\to\infty} (-1)^k t^{k-r} \xi^{(k)}(t) \geqq b_{k+1} \geqq \lim_{t\to\infty} (-1)^k t^{k-r} \zeta^{(k)}(t) \qquad (k = 0, \ldots, n - r - 1),$$

*the boundary value problem*

$$y^{(n)} = f(t, y),$$
$$y^{(k)}(0) = a_{k+1} \qquad (k = 0, \ldots, r - 1),$$
$$\lim_{t\to\infty} (-1)^k t^{k-r} y^{(k)}(t) = b_{k+1} \qquad (k = 0, \ldots, n - r - 1),$$

*has a solution* $y \in C^n[0, \infty)$ *such that*

$$\zeta(t) \leqq y(t) \leqq \xi(t) \quad (0 < t < \infty).$$

*Proof.* Let

(5.7) $$u_k(t) = t^{k-1}/(k - 1)! = v_{n-k+1}(t) \quad (k = 1, \ldots, n),$$

so that $(u_1, \ldots, u_n)$ and $(v_n, \ldots, v_1)$ are corresponding fundamental principal systems on $[0, \infty)$ for $y^{(n)} = 0$. The corollary is a consequence of Theorem 5.2 and the following lemma.

LEMMA 5.1. *If the generalized derivatives are defined in terms of* (5.7) *and*

$$\infty > \lim_{t\to\infty} (-1)^k t^{k-r} z^{(k)}(t) \geqq 0 \qquad (k = 0, \ldots, n - r - 1),$$

*then*

$$(\mathscr{D}^{n-r-1} z)(\infty) \geqq 0 \textit{ and } (\mathscr{D}^k z)(\infty) = 0 \; (k = 0, \ldots, n - r - 2).$$

*Proof.* The lemma follows from the identity

$$\frac{W(u_n, \ldots, u_{n-k+1}, z)}{W(u_n, \ldots, u_{n-k})} = \sum_{j=0}^{k} (-1)^j z^{(j)} \frac{W(u_{n-k}, \ldots, u_{n-j-1}, u_{n-j+1}, \ldots, u_n)}{W(u_{n-k}, \ldots, u_n)}$$

$$= \sum_{j=0}^{k} (-1)^j t^{j+k+1-n} z^{(j)}(t)\gamma_j$$

$$(j = 0, \ldots, k; k = 0, \ldots, n - r - 1),$$

where $\gamma_j$ are nonnegative constants ($\gamma_j \geqq 0$ follows from (2.4)).

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