Can. J. Math. Vol. **43** (5), 1991 pp. 960–974

STRONG BOUNDEDNESS AND STRONG CONVERGENCE IN SEQUENCE SPACES

MARTIN BUNTINAS AND NAZA TANOVIĆ-MILLER

ABSTRACT. Strong convergence has been investigated in summability theory and Fourier analysis. This paper extends strong convergence to a topological property of sequence spaces E. The more general property of strong boundedness is also defined and examined. One of the main results shows that for an FK-space E which contains all finite sequences, strong convergence is equivalent to the invariance property $E = \ell v_0 \cdot E$ with respect to coordinatewise multiplication by sequences in the space ℓv_0 defined in the paper. Similarly, strong boundedness is equivalent to another invariance $E = \ell v \cdot E$. The results of the paper are applied to summability fields and spaces of Fourier series.

1. **Introduction.** This paper defines strong boundedness and strong convergence in sequence spaces E and relates these properties to invariances of the form $E = D \cdot E$ with respect to coordinatewise multiplication by sequences in some space D. Such invariance statements have been investigated in relation to other types of convergence such as:

- sectional boundedness AB and sectional convergence AK [7], induced by the ordinary convergence I of numerical series;
- Cesàro sectional boundedness σB and Cesàro sectional convergence σK [2], induced by Cesàro convergence C_1;
- unrestricted sectional boundedness UAB and unrestricted sectional convergence UAK [11], [12], and absolute boundedness $|AB|$ and absolute convergence $|AK|$ [5], both induced by absolute convergence $|I|$;
- and other types of convergence [3], [4].

Strong boundedness $[AB]$ and strong convergence $[AK]$ in sequence spaces, which are considered in this paper, are induced by strong convergence $[I]$. This type of convergence is related to the other ones mentioned above by the implications

$$|I| \Rightarrow [I] \Rightarrow I \Rightarrow C_1.$$

The induced concepts in sequence spaces satisfy

$$|AB| \Rightarrow UAB \Rightarrow [AB] \Rightarrow AB \Rightarrow \sigma B$$

and

$$|AK| \Rightarrow UAK \Rightarrow [AK] \Rightarrow AK \Rightarrow \sigma K.$$

Research partially supported by U.S.–Yugoslav Joint Fund (NSF JF 803).
Received by the editors April 12, 1990.
AMS subject classification: Primary: 46A45; secondary: 42A16, 42A24.

The basic definitions are given in Section 2. The results in this paper are given for *FK*-spaces and *BK*-spaces containing the spaces of finite sequences ϕ although generalizations to other topological spaces are possible.

Section 3 gives some basic results related to $[AK]$ and $[AB]$. In Section 4, four specific spaces ℓv, ℓv_0, $[cs]$, and $[bs]$ are considered. Section 5 contains the main invariance statements:

- An *FK*-space E has the property of strong boundedness if and only if $E = \ell v \cdot E$.
- An *FK*-space E has the property of strong convergence if and only if $E = \ell v_0 \cdot E$.

The concept of strong convergence $[I]$ was investigated both in summability theory [8], [1], [9], [10] and Fourier analysis [13], [14], [15], [16], [17]. This generated new classes of interesting sequence spaces and spaces of Fourier series. In the last section we give examples and applications to convergence fields and to some important spaces of Fourier series.

Strong convergence of orders $0 < p < \infty$ $[I]_p$ were also investigated in the above mentioned papers. Invariance statements for these methods do not follow from our results however since statements corresponding to Theorem 4.2 do not hold for general orders.

2. **Definitions.** Let ω be the space of all real or complex sequences $x = (x_k)$. An *FK*-space is a subspace of ω with a complete metrizable locally convex topology with continuous coordinate functionals $f_k: x \longrightarrow x_k$ for all k. An *FK*-space whose topology is defined by a norm is a Banach space and is called a *BK*-space. Let e^k be the sequence with 1 in the k^{th} coordinate and 0 elsewhere and let ϕ be the linear span of $\{e^1, e^2, e^3, \ldots\}$. In this paper we consider only *FK*—and *BK*—spaces containing ϕ.

We write $\sum_{2^n} = \sum_{k=2^n}^{2^{n+1}-1}$ and $\max_{2^n} = \max_{2^n \le k < 2^{n+1}}$. We use the notation $x \cdot y := (x_k y_k)$ for the coordinatewise product of sequences x and y and, for subsets A and B of ω, we use $A \cdot B := \{x \cdot y \mid x \in A,\ y \in B\}$. If $A \subset \omega$ and F is an *FK*-space, we define the F-dual of A as the multiplier space $A^F = (A \longrightarrow F) := \{y \in \omega \mid x \cdot y \in F \text{ for all } x \in A\}$. Let $s^n := \sum_{k=1}^{n} e^k = (1, 1, \ldots, 1, 0, \ldots)$, $\sigma^n := \frac{1}{n} \sum_{k=1}^{n} s^k$, $d^n := \sum_{2^n} e^k$, and let $e := (1, 1, 1, \ldots)$ be the sequence of all *ones*. The n^{th} section of a sequence x is $s^n x := s^n \cdot x = (x_1, x_2, \ldots, x_n, 0, \ldots)$, the n^{th} Cesàro section is $\sigma^n x := \sigma^n \cdot x$, and the n^{th} dyadic section is $d^n x := d^n \cdot x = \sum_{2^n} x_k e^k$.

A sequence x in ω has the property AB of *sectional boundedness* in an *FK*-space E if the sections $s^n x$ of x form a bounded subset of E and it has the property σB if the Cesàro sections $\sigma^n x$ are bounded in E.

Let $\mathcal{H} := \{h \in \omega \mid h_k = 1 \text{ or } h_k = 0 \text{ for all } k\}$ and $\mathcal{H}_\phi := \mathcal{H} \cap \phi$. The *unconditional* (or *unrestricted*) *sections* of a sequence x are the sequences in the set $\mathcal{H}_\phi \cdot x$. The *absolute set* of x is $\mathcal{H} \cdot x$. Since $e \in \mathcal{H}$, we have $x \in \mathcal{H} \cdot x$. Let E be an *FK*-space and let $x \in \omega$. We say that x has the property UAB of *unconditional sectional boundedness* in E if $\mathcal{H}_\phi \cdot x$ is a bounded subset of E, we say that x has the property $|AB|$ of *absolute boundedness* if $\mathcal{H} \cdot x$ is a bounded subset of E, and x has the property $[AB]$ of *strong boundedness* if x has the property AB and $\{\mathcal{H} \cdot d^j x\}_{j=1}^{\infty}$ is a bounded subset of E.

For each *FK*-space E, we define the space E_{AB} consisting of all elements x of ω with the property AB in E. Similarly, for the properties σB, UAB, $|AB|$ and $[AB]$, we obtain spaces $E_{\sigma B}$, E_{UAB}, $E_{|AB|}$ and $E_{[AB]}$. These spaces are FK-spaces under appropriate topologies discussed in Section 3. They are not necessarily subspaces of E as is shown by the example $(c_0)_{UAB} = (c_0)_{AB} = \ell^\infty$; except $E_{|AB|}$, which is always a subspace of E since $e \in \mathcal{H}$. We say that an FK-space E has the property AB, UAB, $|AB|$ or $[AB]$ if E is a subset of E_{AB}, E_{UAB}, $E_{|AB|}$ or $E_{[AB]}$, respectively. Clearly we have $E_{|AB|} \subset E_{UAB} \subset E_{[AB]} \subset E_{AB} \subset E_{\sigma B}$.

A sequence x in an FK-space E has the property σK of *Cesàro sectional convergence* if the sections $\sigma^n x$ converge to x in the topology of E. If the sections $s^n x$ converge to x, we say that x has the property AK of *sectional convergence* and if, in addition, x has the property $[AB]$, we say that x has the property $[AK]$ of *strong convergence.*

The set $\mathcal{H}_\phi$ is a directed set under the relation $h'' \geq h'$ defined by $h''_k \geq h'_k$ for all k. A sequence x in an FK-space E containing ϕ has the property UAK in E if $\mathcal{H}_\phi \cdot x \subset E$ and the net $h \cdot x$, where h ranges over $\mathcal{H}_\phi$, converges to x under the topology of E. We say that x has the property $|AK|$ of *absolute sectional convergence* if $\mathcal{H} \cdot x \subset E$ and the net $h \cdot h' \cdot x$, where h ranges over $\mathcal{H}_\phi$, converges to $h' \cdot x$ uniformly in $h' \in \mathcal{H}$ under the topology of E.

We define E_{AK} to be the space of all elements x of E with the property AK in E. The same can be done for the properties σK, UAK, $|AK|$ and $[AK]$. The space E_{AD} is the closure of ϕ in E. Since $\phi \subset E$, we have the inclusions $\phi \subset E_{|AK|} \subset E_{UAK} \subset E_{[AK]} \subset E_{AK} \subset E_{\sigma K} \subset E_{AD} \subset E$. If $E_{AD} = E$, we say that E has the property of sectional density AD. If $y \in E$ whenever $|y_k| \leq |x_k|$ for some $x \in E$, we say that E is solid; this is equivalent to ℓ^∞-invariance: $E = \ell^\infty \cdot E$.

We finish this section with a list of some BK-spaces and their norms. The BK-spaces ℓ^∞, c and c_0 are the space of all bounded, convergent and null sequences x, respectively, under the sup norm $\|x\|_\infty := \sup_k |x_k|$;
bv is the BK-space of all sequences x of bounded variation under the norm

$$\|x\|_{bv} := \sum_{k=1}^{\infty} |x_k - x_{k+1}| + \|x\|_\infty;$$

$bv_0 = bv \cap c_0$ under the same norm; cs is the BK-space of sequences x with convergent series under the norm

$$\|x\|_{bs} := \sup_n \Big|\sum_{k=1}^{n} x_k\Big|;$$

ℓ^p, for $1 \leq p < \infty$, are the BK-spaces of sequences x with absolutely p-summable series under the norm

$$\|x\|_p := \Big(\sum_{k=1}^{\infty} |x_k|^p\Big)^{\frac{1}{p}};$$

the mixed $\ell^{p,q}$ spaces $(1 \leq p \leq \infty,\ 1 \leq q < \infty)$ consist of all x with

$$\|x\|_{p,q} := \Big(\sum_{j=0}^{\infty} (\|d^j x\|_p)^p\Big)^{\frac{1}{q}} < \infty,$$

https://doi.org/10.4153/CJM-1991-053-8 Published online by Cambridge University Press

and for $q = \infty$,

$$\|x\|_{p,\infty} := \sup_j \|d^j x\|_p.$$

Clearly $\ell^{p,p} = \ell^p$. Finally,

$$\ell^{p,o} := \{x \mid \lim_j \|d^j x\|_p = 0\}.$$

Clearly $(\ell^{p,\infty})_{AD} = \ell^{p,o}$.

3. **Basics.** Let E be an FK-space whose topology is defined by a collection P of seminorms. The spaces E_{AB} and $E_{\sigma B}$, both induced by matrix summability methods, are FK-spaces with topologies defined by the collections of seminorms $p_{AB}(x) := \sup_n p(s^n x)$ and $p_{\sigma B}(x) := \sup_n p(\sigma^n x)$, respectively, for $p \in P$, [3]. The space $E_{|AB|}$, induced by absolute summation, is an FK-space with the topology defined by $p_{|E|}(x) = \sup_{h \in \mathcal{H}} p(h \cdot x)$, $p \in P$, [5]. In between $E_{|AB|}$ and E_{AB} we have the space $E_{[AB]}$, induced by strong convergence, which is also an FK-space:

THEOREM 3.1. *Let E be an FK-space containing ϕ with topology defined by a collection of seminorms P. Then $E_{[AB]}$ is an FK-space with topology defined by the seminorms*

$$p_{[AB]}(x) = \sup_j p_{|E|}(d^j x) + p_{AB}(x)$$

for $p \in P$.

PROOF. Since E_{AB} is an FK-space defined by the collection of seminorms p_{AB} for $p \in P$ and since $p_{AB} \leq p_{[AB]}$, we have $E_{[AB]} = \{x \in E_{AB} \mid p_{[AB]}(x) < \infty \text{ for } p \in P\}$. The functions $p_{[AB]}$ are clearly lower semicontinuous extended seminorms on E_{AB}. By Garling's Theorem [7], p. 998, $E_{[AB]}$ is an FK-space. ∎

COROLLARY 3.2. *Let E be a BK-space containing ϕ with norm $\|\cdot\|_E$. Then $E_{[AB]}$ is a BK-space with norm*

$$\|x\|_{[AB]} = \sup_j \|d^j x\|_{|E|} + \sup_n \|s^n x\|_E.$$

The seminorms of the form $p_{|E|}$ are important because they simulate absolute convergence of series. We state some lemmas.

LEMMA 3.3. *Let E be an FK-space containing ϕ. For each seminorm p on E, $y \in \phi$, and $0 \leq a_k \leq b_k$, $k = 1, 2, 3, \ldots$, we have $p_{|E|}(a \cdot y) \leq p_{|E|}(b \cdot y)$.*

PROOF. For each sequence $h \in \mathcal{H}$, let $x_k = h_k a_k / b_k$, if $b_k \neq 0$, and $x_k = 0$, otherwise. Since $y \in \phi$ we have $h \cdot a \cdot y = x \cdot b \cdot y = \sum_{k=1}^n x_k b_k y_k e^k$ for some n. Rearrange the terms such that $h \cdot a \cdot y = \sum_{i=1}^n x_{k_i} b_{k_i} y_{k_i} e^{k_i}$ with $1 \geq x_{k_1} \geq x_{k_2} \geq \cdots \geq x_{k_n} \geq 0$. By partial summation, $h \cdot a \cdot y = \sum_{i=1}^n \left(x_{k_i} - x_{k_{i+1}}\right) \sum_{j=1}^i b_{k_j} y_{k_j} e^{k_j}$ with $x_{k_{n+1}} = 0$. Since each partial sum $\sum_{j=1}^i b_{k_j} y_{k_j} e^{k_j}$ is of the form $h^i \cdot b \cdot y$ for some $h^i \in \mathcal{H}$, we have $p(h \cdot a \cdot y) \leq$

$\sum_{i=1}^{n} |x_{k_i} - x_{k_{i+1}}| p(h^i \cdot b \cdot y) \leq \sum_{i=1}^{n} |x_{k_i} - x_{k_{i+1}}| p_{|E|}(b \cdot y) = x_{k_1} p_{|E|}(b \cdot y) \leq p_{|E|}(b \cdot y)$. Thus $p_{|E|}(a \cdot y) \leq p_{|E|}(b \cdot y)$. ∎

Separating a bounded sequence x into real and imaginary parts and then into the positive and negative parts $x = x^1 - x^2 + ix^3 - ix^4$, $0 \leq x_k^i \leq \|x\|_\infty$ $(i = 1,2,3,4,\ k = 1,2,3,\ldots)$, we obtain the following from Lemma 3.3.

LEMMA 3.4. *Let E be an FK-space containing ϕ. For each seminorm p on E and $x, y \in \phi$ we have $p_{|E|}(x \cdot y) \leq 4\|x\|_\infty p_{|E|}(y)$.*

LEMMA 3.5. *Let E be an FK-space containing ϕ. Let p be any seminorm on E and let $x \in \omega$. Then*

$$\sup_j p_{|E|}(d^j x) < \infty \tag{3.6}$$

if and only if

$$\sup_n p_{|E|}\left(\frac{1}{n}\sum_{k=1}^{n} k x_k e^k\right) < \infty. \tag{3.7}$$

PROOF. By Lemma (3.3), $p_{|E|}(d^j x) = p_{|E|}\left(\sum_{2^j} x_k e^k\right) \leq p_{|E|}\left(\sum_{2^j} \frac{k}{2^j} x_k e^k\right) \leq 2p_{|E|}\left(\frac{1}{2^{j+1}}\sum_{k=1}^{2^{j+1}} k x_k e^k\right)$. Conversely, suppose $2^m \leq n < 2^{m+1}$. Then again by Lemma 3.3, $p_{|E|}\left(\frac{1}{n}\sum_{k=1}^{n} k x_k e^k\right) \leq \frac{1}{n}\sum_{j=0}^{m} 2^{j+1} p_{|E|}(d^j x) \leq 4 \sup_j p_{|E|}(d^j x)$. ∎

THEOREM 3.8. *Let E be an FK-space containing ϕ. The following statements are equivalent for a sequence $x \in \omega$:*

[a] *x has the property $[AB]$ in E;*

[b] *x has the property AB and satisfies (3.6) for every continuous seminorm p on E;*

[c] *x has the property AB and satisfies (3.7) for every continuous seminorm p on E;*

[d] *x has the property σB and satisfies (3.6) for every continuous seminorm p on E;*

[e] *x has the property σB and satisfies (3.7) for every continuous seminorm p on E.*

PROOF. [b] is a restatement of [a], the definition of $[AB]$. [b] ⇔ [c] and [d] ⇔ [e] follow from Lemma 3.5. [c]⇒ [e] is clear since $AB \Rightarrow \sigma B$. [e] ⇒ [c]: Since $s^n x - \sigma^n x = \frac{1}{n}\sum_{k=1}^{n}(k-1)x_k e^k$, condition (3.7) implies the sequences $s^n x - \sigma^n x$ are bounded. If x also has the property σK, then the sections $s^n x$ are bounded. ∎

THEOREM 3.9. *Let E be an FK-space containing ϕ. Then E has the property $[AK]$ if and only if it has the properties AD and $[AB]$.*

PROOF. The property $[AK]$ means AK and $[AB]$. Since $AK \Rightarrow AD$, one implication is immediate. Conversely, it is well known that an FK-space has the property AK if and only if it has the properties AD and AB [18]. Thus AD and $[AB]$ imply AK; hence also $[AK]$. ∎

THEOREM 3.10. *Suppose E is an FK-space with the property* $[AB]$. *Then* $E_{[AK]} = E_{AK} = E_{\sigma K} = E_{AD}$.

PROOF. Clearly $E_{[AK]} \subset E_{AK} \subset E_{\sigma K} \subset E_{AD}$. If E has the property $[AB]$, then it has the property AB. Then $E_{AK} = E_{AD}$ [7]. This means that E_{AK} is a closed subspace of E and hence an FK-space with the properties AD and $[AB]$. By Theorem 3.9, E_{AK} has the property $[AK]$. ∎

However, for an FK-space with the property $[AB]$, the space $E_{|AK|}$ may be smaller than $E_{[AK]}$. The space ℓv, defined in Section 4, is an example.

COROLLARY 3.11. *If E is a solid FK-space containing ϕ, then*

$$E_{|AK|} = E_{[AK]} = E_{AK} = E_{\sigma K} = E_{AD}.$$

PROOF. By Theorem 2, Corollary 2, of [5], an FK-space is solid if and only if it has the property $|AB|$. This clearly implies $[AB]$ which by Theorem (3.10) yields all but the first equality. But by Theorem 6, Corollary 2, of [5], $E_{|AK|} = E_{AD}$. ∎

4. **The spaces ℓv, ℓv_0, $[cs]$, and $[bs]$.** The convergence field of the strong convergence method $[I]$ is $[cs] = \left\{x \in \omega \;\middle|\; \sum_{2^j} |x_k| = o(1)\ (j \longrightarrow \infty) \text{ and } \sum_k x_k \text{ exists} \right\}$ and the boundedness domain is $[bs] = \left\{x \in \omega \;\middle|\; \sum_{2^j} |x_k| = O(1)\ (j \longrightarrow \infty) \text{ and } \left|\sum_{k=1}^n x_k\right| = O(1)\ (n \longrightarrow \infty)\right\}$. The conditions $\sum_{2^j} |x_k| = o(1)$ $(j \longrightarrow \infty)$ and $\sum_{2^j} |x_k| = O(1)$ $(j \longrightarrow \infty)$ can be replaced by $\frac{1}{n}\sum_{k=1}^n |x_k| = o(1)$ $(n \longrightarrow \infty)$ and $\frac{1}{n}\sum_{k=1}^n |x_k| = O(1)$ $(n \longrightarrow \infty)$, respectively. These spaces are BK-spaces under the norm $\|x\|_{[bs]} = \sup_j \sum_{2^j} |x_k| + \sup_j \left|\sum_{k=1}^n x_k\right|$. The space $[cs]$, as well as more general spaces $[cs]_p$, $0 < p < \infty$, were defined by Hyslop [8] and Borwein [1] and then further investigated by Kuttner and Maddox [9]. Convergence factors were investigated by Kuttner and Thorpe [10]. It is shown in [9] that $([cs] \longrightarrow [cs]) = \ell v := \left\{x \in \omega \;\middle|\; \sum_j \max_{2^j} |x_k - \alpha_j| + \sum_j |\alpha_j - \alpha_{j+1}| < \infty\right\}$ where $\alpha_j = \alpha_j(x) := \frac{1}{2^j}\sum_{2^j} x_k$. The space ℓv is a BK-space under the norm

$$\|x\|_{\ell v} = \sum_j \max_{2^j} |x_k - \alpha_j| + \sum_j |\alpha_j - \alpha_{j+1}| + \sup_j |\alpha_j|.$$

An alternate criterion for $x \in \ell v$ is

$$\sum_j |x_{k_j} - x_{k_{j+1}}| < \infty \text{ for all lacunary sequences } (k_j). \tag{4.1}$$

That is, $x \in \ell v$ if and only if for every lacunary sequence (k_j), the subsequence x_{k_j} belongs to bv. This criterion clearly shows that $\ell v \subset c$. Furthermore, if $x \in bv$, then $\sum_j |x_{k_j} - x_{k_{j+1}}| \le \sum_j \{ |x_{k_j} - x_{k_j+1}| + |x_{k_j+1} - x_{k_j+2}| + \cdots + |x_{k_{j+1}-1} - x_{k_{j+1}}| \} \le \|x\|_{bv}$. Thus we obtain the following theorem.

THEOREM 4.2. $bv \subset \ell v \subset c$.

THEOREM 4.3. *The space ℓv has the property* $[AB]$.

PROOF. For each $h \in \mathcal{H}$, $x \in \ell v$, and $m = 1, 2, 3, \ldots$, we have

$$\begin{aligned} \|h \cdot d^m x\|\, \ell v &= \sum_j \max_{2^j} |h_k d_k^m x_k - \alpha_j(h \cdot d^m x)| \\ &\quad + \sum_j |\alpha_j(h \cdot d^m x) - \alpha_{j+1}(h \cdot d^m x)| + \sup_j |\alpha_j(h \cdot d^m x)| \\ &= \max_{2^m} |h_k x_k - \alpha_m(h \cdot x)| + 3|\alpha_m(h \cdot x)| \\ &\le \max_{2^m} |x_k| + 4|\alpha_m(h \cdot x)| \le 5\|x\|_\infty. \end{aligned}$$

That is, $\|d^m x\|_{|\ell v|} \le 5\|x\|_\infty$. Since $\ell v \subset \ell^\infty$, (3.6) is satisfied. To show that ℓv has the property AB, let $x \in \ell v$ and $2^m \le n < 2^{m+1}$. Then $\|s^n x\|_{\ell v} = \|s^{2^m-1}x + s^n d^m x\|_{\ell v} \le \|s^{2^m-1}x\|_{\ell v} + \|s^n d^m x\|_{\ell v} \le \|x\|_{\ell v} + \|d^m x\|_{|\ell v|} \le \|x\|_{\ell v} + 5\|x\|_\infty$ since $\|s^{2^m-1}x\|_{\ell v} = \sum_{j=0}^{m-1} \max_{2^j} |x_k - \alpha_j(x)| + \sum_{j=0}^{m-2} |\alpha_j(x) - \alpha_{j+1}(x)| + |\alpha_{m-1}(x)| + \sup_{j<m} |\alpha_j(x)| \le \|x\|_{\ell v}$, and $s^n \in \mathcal{H}$. ∎

Define $\ell v_0 := \ell v \cap c_0$.

THEOREM 4.4. $\ell v_{[AK]} = \ell v_{AK} = \ell v_{AD} = \ell v_0$.

PROOF. $\ell v_{AD} \subset (\ell^\infty)_{AD} = c_0$. Thus $\ell v_{AD} \subset \ell v_0$. Conversely, for $x \in \ell v_0$, we have $\|x - s^{2^m-1}x\|_{\ell v} = \sum_{j=m}^{\infty} \max_{2^j} |x_k - \alpha_j(x)| + |\alpha_m(x)| + \sum_{j=m}^{\infty} |\alpha_j(x) - \alpha_{j+1}(x)| + \sup_{i \ge m} |\alpha_j(x)| = o(1)\ (m \to \infty)$. Thus $\ell v_0 \subset \ell v_{AD}$. Since ℓv has the property $[AB]$, the other equalities follow from Theorem 3.10. ∎

THEOREM 4.5. *The space* $[cs]$ *has the property* $[AK]$.

PROOF. For each $x \in [cs]$ and $y \in \ell v$, let $T_x(y) = x \cdot y$. Since $([cs] \to [cs]) = \ell v$, T_x maps ℓv into $[cs]$. By the Closed Graph Theorem, T_x is continuous. Since ℓv has the property $[AB]$, $T_x(s^n y) = s^n x \cdot y$, $(n = 1, 2, 3, \ldots)$ and $T_x(h \cdot d^j y) = h \cdot d^j x \cdot y$, $(h \in \mathcal{H}$, $j = 0, 1, 2, \ldots)$ form bounded subsets of $[cs]$. Since this is true for all $x \in [cs]$ and $y \in \ell v$ and since $[cs] = \ell v \cdot [cs]$, the space $[cs]$ has the property $[AB]$. It remains to show that $[cs]$ has the property AD. But $\|x - s^{2^m-1}x\|_{[bs]} = \sup_{j \ge m} \sum_{2^j} |x_k| + \sup_n \left|\sum_{k=2^m}^{n} x_k\right| = o(1) + \|x - s^{2^m-1}x\|_{bs} = o(1)\ (m \to \infty)$. ∎

THEOREM 4.6. $[cs]_{[AB]} = [bs]$.

PROOF. If $x \in [bs]$, then $\|h \cdot d^j x\|_{[bs]} = \sum_{2^j} |x_k| + \|d^j h \cdot x\|_{bs} \le 2\sum_{2^j} |x_k| \le 2\|x\|_{[bs]}$. Also $\|s^n x\|_{[bs]} \le \|x\|_{bs}$. Thus $[bs] \subset [cs]_{[AB]}$. Conversely, if $x \in [cs]_{[AB]}$, then $\sup_{h \in \mathcal{H}} \|h \cdot d^j x\|_{[bs]} \ge \sum_{2^j} |x_k| = O(1)\ (j \to \infty)$. Also $[cs]_{[AB]} \subset [cs]_{AB} \subset cs_{AB} \subset bs$. Thus $[cs]_{[AB]} \subset [bs]$. ∎

THEOREM 4.7. $\ell v \cdot \ell v = \ell v$.

PROOF. Let $x, y \in \ell v$. By criterion (4.1) we have $\sum_k |x_{n_k} y_{n_k} - x_{n_{k+1}} y_{n_{k+1}}| \le \sum_k \{ |x_{n_k}|\, |y_{n_k} - y_{n_{k+1}}| + |y_{n_k}|\, |x_{n_k} - x_{n_{k+1}}| \} \le \|x\|_\infty \sum_k |x_{n_k} - y_{n_{k+1}}| + \|y\|_\infty \sum_k |x_{n_k} - x_{n_{k+1}}| < \infty$ for every lacunary subsequence. Thus $\ell v \cdot \ell v \subset \ell v$. Since $e \in \ell v$, $\ell v \cdot \ell v \supset \ell v \cdot e = \ell v$. ∎

COROLLARY 4.8. $\ell v \cdot \ell v_0 = \ell v_0$.

THEOREM 4.9. $\ell v_0 \cdot \ell v_0 = \ell v_0$.

PROOF. $\ell v_0 \cdot \ell v_0 \subset \ell v_0$ is clear from above. Since ℓv_0 has the property AK, we have $\ell v_0 = bv_0 \cdot \ell v_0$ ([7], Theorem 4). Since $bv_0 \subset \ell v_0$, we have $\ell v_0 \subset \ell v_0 \cdot \ell v_0$. ∎

Recall that $\ell^{\infty,1}$ is the space of all sequences satisfying $\sum_j \max_{2^j} |x_k| < \infty$. Clearly $\sum_j \max_{2^j} |x_k - \alpha_j(x)| < \infty$ and $\sum_j |\alpha_j(x)| < \infty$ if and only if $\sum_j \max_{2^j} |x_k| < \infty$. Thus $\ell^{\infty,1} = \{x \in \ell v \big| \sum_j |\alpha_j(x)| < \infty\}$.

THEOREM 4.10. $\ell v_{|AB|} = \ell^{\infty,1}$.

PROOF. Let $x \in \ell^{\infty,1}$ and $h \in \mathcal{H}$. Then $\|h \cdot x\|_{\ell v} = \sum_j \max_{2^j} |h_k x_k - \alpha_j(h \cdot x)| + \sum_j |\alpha_j(h\cdot x) - \alpha_{j+1}(h\cdot x)| + \sup_j |\alpha_j(h\cdot x)| \le \sum_j \max_{2^j} |x_k| + 4\sum_j |\alpha_j(x)| \le 5\sum_j \max_{2^j} |x_k|$. Thus $x \in \ell v_{|AB|}$. Conversely, if $\mathcal{H} \cdot x \in \ell v$, then $\sum_j |\alpha_j(h \cdot x) - \alpha_{j+1}(h \cdot x)| < \infty$ for all $h \in \mathcal{H}$. Letting h alternate between 0 and 1 on dyadic blocks we get $\sum_j |\alpha_j(h\cdot x) - \alpha_{j+1}(h\cdot x)| = |\alpha_2(x)| + |\alpha_4(x)| + \cdots$ or $|\alpha_1(x)| + |\alpha_3(x)| + \cdots$. Adding we obtain $\sum_j |\alpha_j(x)| < \infty$. ∎

5. **Multiplier results.**

THEOREM 5.1. *Let E be an FK-space containing ϕ. The following statements are equivalent:*

[a] *E has the property* $[AB]$;

[b] $E = \ell v \cdot E$;

[c] $E_{[AK]} = \ell v_0 \cdot E$;

[d] $\ell v_0 \cdot E \subset E$.

PROOF. $[a] \Rightarrow [c]$: Suppose E has the property $[AB]$. By Theorem 3.10 $E_{[AK]} = E_{AK} = E_{AD}$ and by Theorem 4 of [7], $E_{AK} = bv_0 \cdot E_{AK}$. Since $bv_0 \subset \ell v_0$, we have $E_{[AK]} \subset \ell v_0 \cdot E$. Let $y \in \ell v_0$ and $x \in E$. It is sufficient to show $y \cdot x \in E_{AD}$ by showing that $s^{2^n-1} y \cdot x$ is a Cauchy sequence in E. For $n > m$ we have $s^{2^n-1} y \cdot x - s^{2^m-1} y \cdot x = \sum_{j=m}^{n-1} \sum_{2^j} y_k x_k e^k = \sum_{j=m}^{n-1} \sum_{2^j} \big(y_k - \alpha_j(y)\big) x_k e^k + \sum_{j=m}^{n-1} \alpha_j(y) \sum_{2^j} x_k e^k = \sum_{j=m}^{n-1} \sum_{2^j} \big(y_k - \alpha_j(y)\big) x_k e^k + \sum_{j=m-1}^{n-1} \big(\alpha_j(y) - \alpha_{j+1}(y)\big) s^{2^{j+1}-1} x + \alpha_n(y) s^{2^{n-1}-1} x - \alpha_{m-1}(y) s^{2^{m-1}-1} x$. Thus for each continuous seminorm p on E, we have by Lemma 3.4

$$
\begin{aligned}
p(s^{2^n-1}&y \cdot x - s^{2^m-1} y \cdot x) \\
&\le \sum_{j=m}^{n-1} 4 \max_{2^j} |y_k - \alpha_j(y)| p_{|E|}(d^j x) \\
&\quad + \sup_j p(s^j x) \Big\{ \sum_{j=m-1}^{n-1} |\alpha_j(y) - \alpha_{j+1}(y)| + |\alpha_n(y)| + |\alpha_{m-1}(y)| \Big\} \\
&\le 4 \sup_j p_{|E|}(d^j x) \sum_{j=m}^{n-1} \max_{2^j} |y_k - \alpha_j(y)| \\
&\quad + \sup_j p(s^j x) \Big\{ \sum_{j=m-1}^{n-1} |\alpha_j(y) - \alpha_{j+1}(y)| + |\alpha_n(y)| + |\alpha_{m-1}(y)| \Big\}.
\end{aligned}
$$

https://doi.org/10.4153/CJM-1991-053-8 Published online by Cambridge University Press

Since $y \in \ell v_0$, this tends to 0 as $m, n \longrightarrow \infty$. $[c] \Rightarrow [d]$ is obvious. $[d] \Rightarrow [b]$: Since $\ell v \subset c$, every $y \in \ell v$ is of the form $y = z + w$, where $z \in \ell v_0$ and $w = (\lim_k y_k)e$. If $z \cdot E \subset E$, then $y \cdot E \subset E$ since $w \cdot E \subset E$. Thus $\ell v \cdot E \subset E$. Since $e \in \ell v$, $E \subset \ell v \cdot E$. $[b] \Rightarrow [a]$: Suppose $E = \ell v \cdot E$. For each $y \in \ell v$ and $x \in E$, let $T_x(y) = x \cdot y$. T_x is continuous and maps ℓv into E. Following the proof of Theorem 4.5, E has the property $[AB]$. ∎

THEOREM 5.2. *Let E be an FK-space containing ϕ. Then E has the property $[AK]$ if and only if $E = \ell v_0 \cdot E$.*

THEOREM 5.3. *Let E be an FK-space containing ϕ. Then*

$$E_{[AB]} = (\ell v_0 \longrightarrow E_{[AK]}) = (\ell v_0 \longrightarrow E).$$

PROOF. By Theorem 5.1 [c], $\ell v_0 \cdot E_{[AB]} = E_{[AK]}$. Thus $E_{[AB]} \subset (\ell v_0 \longrightarrow E_{[AK]}) \subset (\ell v_0 \longrightarrow E)$. Conversely, suppose $x \cdot \ell v_0 \subset E$. Then $T_x(y) := x \cdot y$ is a continuous map from ℓv_0 into E. Let p be a continuous seminorm on E. Then there exists $M > 0$ such that for all $h \in \mathcal{H}$, $p(h \cdot d^j x) \leq M \|h \cdot d^j\|_{\ell v}$. As in the proof of Theorem 4.3, $\|h \cdot d^j\|_{\ell v} \leq 5\|e\|_\infty = 5$. Thus $p_{|E|}(d^j x) \leq 5M$. Similarly, $p(s^n x) \leq M\|s^n\|_{\ell v} \leq M(\|e\|_{\ell v} + 5\|e\|_\infty) = 6M$. Thus $x \in E_{[AB]}$. ∎

6. **Examples and applications in summability theory and Fourier analysis.** For standard sequence spaces E such as c_0, ℓ^∞, cs, bs, bv, ℓ^p, etc., especially for those that are solid, the spaces $E_{[AB]}$ and $E_{[AK]}$ are easily determined. The following theorem collects some of these statements, the proofs of which are almost immediate.

THEOREM 6.1. [a] *If E is a BK-space and $c_0 \subset E \subset \ell^\infty$, then $E_{[AB]} = \ell^\infty$ and $E_{[AK]} = c_0$;*

[b] $bs_{[AB]} = [bs]$ *and* $cs_{[AK]} = bs_{[AK]} = [cs]$;

[c] $bv_{[AB]} = bv \cap \ell^{1,\infty}$ *and* $bv_{[AK]} = bv_0 \cap \ell^{1,o}$;

[d] $\ell^p_{[AB]} = \ell^p_{[AK]} = \ell^p \quad (1 \leq p < \infty)$;

[e] $\ell^{p,q}_{[AB]} = \ell^{p,q}_{[AK]} = \ell^{p,q} \quad (1 \leq p < \infty,\ 1 \leq q \leq \infty)$.

For an infinite matrix of real or complex numbers $T = (t_{nk})$, let c_T denote the convergence field of T, that is, $c_T = \{x \in \omega : Tx \in c\}$. We say that a matrix T is series-sequence conservative if $c_T \supset cs$. The following result extends the equality $cs_{[AK]} = [cs]$ to the convergence fields of all such matrices T.

THEOREM 6.2. *If a matrix T is series-sequence conservative, then $(c_T)_{[AK]} = [cs]$.*

PROOF. By assumption $cs \subset c_T$ and hence by Theorem 6.1, $[cs] = cs_{[AK]} \subset (c_T)_{[AK]}$. Conversely if $x \in (c_T)_{[AK]}$, then $x \in (c_T)_{AK}$ and the sequence $(d^j x)$ is bounded in $(c_T)_{|AK|}$. By Proposition 4 in [3] $(c_T)_{AK} = cs$ and by Theorem 10 in [5], $(c_T)_{|AB|} = \ell^1$. Hence $x \in cs$ and $(d^j x)$ is a bounded sequence in ℓ^1. Therefore $x \in [cs]$. ∎

We shall now apply the concepts of strong boundedness $[AB]$ and strong convergence $[AK]$ to the spaces of Fourier coefficients of various classes of 2π-periodic functions. Let L^p $(p \geq 1)$ be the Banach space of all real or complex valued 2π-periodic functions such that $|f|^p$ is integrable, under the standard norm $\|f\|_{L^p} = \left(\frac{1}{2\pi}\int |f|^p\right)^{\frac{1}{p}}$ where the interval of integration is of the length 2π. Let C be the Banach space of all continuous real or complex valued 2π-periodic functions with the norm $\|f\|_C = \sup_x |f(x)|$.

For $f \in L^1$ let $\hat{f}(k)$, $k \in \mathbb{Z}$, denote the k^{th} complex Fourier coefficient of f, $\hat{f} = (\hat{f}(k))_{k\in\mathbb{Z}}$ and let s_nf and σ_nf, $n = 0, 1, \ldots$, denote respectively the n^{th} partial sum and the n^{th} Cesàro partial sum of the Fourier series of f. If E is a subspace of L^1, let $\hat{E}$ denote the class of all sequences of Fourier coefficients of functions in E, i.e., $\hat{E} = \{\hat{f} : f \in E\}$. Although the results in the preceding sections of this paper are for spaces of one-way sequences, they can be easily extended to the spaces $\hat{E}$ of two-way sequences. If E is a linear space, then $\hat{E}$ is a linear sequence space, and if E is a Banach space, then $\hat{E}$ is a Banach space under the induced norm $\|\hat{f}\|_{\hat{E}} := \|f\|_E$ and conversely. Given a Banach space $E \subset L^1$ we shall try to determine the corresponding subspaces of strongly bounded and strongly convergent Fourier series, in the topology of E, by determining the spaces $\hat{E}_{[AB]}$ and $\hat{E}_{[AK]}$.

Two classical spaces of functions in Fourier analysis determined by two methods of pointwise convergence, ordinary I and absolute $|I|$, are the spaces of uniformly and absolutely convergent Fourier series

$$\mathcal{U} = \{f \in C : s_nf \longrightarrow f \quad I \text{ uniformly}\} \text{ and } \mathcal{A} = \{f \in C : s_nf \longrightarrow f \quad |I| \text{ a.e.}\}.$$

They are Banach spaces, under the norms

$$\|f\|_{\mathcal{U}} := \sup_n \|s_nf\|_C \text{ and } \|f\|_{\mathcal{A}} := \sum_{k\in\mathbb{Z}} |\hat{f}(k)| = \|\hat{f}\|_{\ell^1}.$$

It is well known that $\mathcal{A} \subset \mathcal{U} \subset C \subset L^\infty$ properly, where L^∞ is the space of essentially bounded measurable 2π-periodic functions.

The space L^1 is also determined by pointwise convergence types, namely by C_1 and $[C_1]$. That is, by Féjer's Theorem, and Marcinkiewicz–Zygmund's Theorem, we have

$$L^1 = \{f \in L^1 : s_nf \longrightarrow f \quad C_1 \text{ a.e.}\} = \{f \in L^1 : s_nf \longrightarrow f \quad [C_1] \text{ a.e.}\}.$$

We shall also consider the space M of 2π-periodic Radon measures under the norm $\|f\|_M := \sup_n \|\sigma_nf\|_{L^1}$.

In view of the concepts of ordinary, Cesàro, strong Cesàro, absolute boundedness and absolute convergence in sequence spaces and the above duality between the function spaces E and the spaces of Fourier coefficients $\hat{E}$, each of these classical function spaces bears other descriptions. For example, by classical results (see [6], [19]),

$$\widehat{\mathcal{U}} = \widehat{C}_{AK}, \quad \widehat{\mathcal{A}} = \widehat{C}_{|AB|} = \widehat{C}_{|AK|}$$

$$\widehat{L^p} = \widehat{L^p}_{AK} \text{ for } p > 1, \quad \widehat{L^1} = \widehat{L^1}_{\sigma K} \text{ and } M = L^1_{\sigma B}.$$

Applying the concept of strong sectional boundedness and convergence, the following theorem shows that none of these classical spaces, except $E = L^2$ and $E = \mathcal{A}$, coincides with the space $E_{[AB]}$ or $E_{[AK]}$. The standard sequence spaces appearing in these statements are to be interpreted as the spaces of two-way sequences.

THEOREM 6.3. [a] *If* $1 < p \leq 2$, *then* $\widehat{L^p}_{[AB]} = \widehat{L^p} \cap \ell^{2,\infty}$ *and* $\widehat{L^p}_{[AK]} = \widehat{L^p} \cap \ell^{2,o}$;
[b] $\widehat{L^1}_{[AB]} = \hat{M} \cap \ell^{2,\infty}$ *and* $\widehat{L^1}_{[AK]} = \widehat{L^1} \cap \ell^{2,o}$;
[c] *If* $p > 2$, *then* $\widehat{L^p} \cap \ell^{q,\infty} \subset \widehat{L^p}_{[AB]}$ *and* $\widehat{L^p} \cap \ell^{q,o} \subset \widehat{L^p}_{[AK]}$, *where* $1/p + 1/q = 1$;
[d] *If* E *is a Banach space and* $\mathcal{A} \subset E \subset L^\infty$, *then* $\hat{E}_{[AB]} = \hat{E}_{\sigma B} \cap \ell^{1,\infty}$ *and* $\hat{E}_{[AK]} = \hat{E}_{\sigma K} \cap \ell^{1,\infty}$;
[e] $\hat{M}_{[AB]} = \hat{M} \cap \ell^{2,\infty}$ *and* $\hat{M}_{[AK]} = \hat{M} \cap \ell^{2,o}$.

COROLLARY 6.4. $\widehat{L^2}_{[AB]} = \widehat{L^2}_{[AK]} = \ell^2 = \widehat{L^2}$, $\hat{\mathcal{A}}_{[AB]} = \hat{\mathcal{A}}_{[AK]} = \ell^1 = \hat{\mathcal{A}}$.

COROLLARY 6.5. *Let* E *be a Banach space and* $\mathcal{A} \subset E \subset L^\infty$. *If* E *has* σB, *then* $\hat{E}_{[AB]} = \hat{E} \cap \ell^{1,\infty}$. *If* E *has* σK, *then* $\hat{E}_{[AK]} = \hat{E} \cap \ell^{1,o}$.

PROOF. [a]: Suppose $1 < p \leq 2$. If $\hat{f} \in \widehat{L^p}_{[AB]}$, then we have $\hat{f} \in \widehat{L^p}$ and $\|d^n \hat{f}\|_{|\widehat{L^p}|} = O(1)\,(n \longrightarrow \infty)$ since $\widehat{L^p}_{AB} = \widehat{L^p}$ for $p > 1$. Therefore the sequence $(d^n \hat{f})$ is bounded in the topology of $\widehat{L^p}_{|AB|}$. But by Theorem 11 in [5], $\widehat{L^p}_{|AB|} = \ell^2$ for $1 \leq p \leq 2$. Consequently, $\|d^n \hat{f}\|_{\ell^2} = O(1)\,(n \longrightarrow \infty)$. Thus $\widehat{L^p}_{[AB]} \subset \widehat{L^p} \cap \ell^{2,\infty}$. Conversely, suppose $\hat{f} \in \widehat{L^p} \cap \ell^{2,\infty}$. Then $\hat{f} \in \widehat{L^p}_{AB}$. Furthermore, since $\widehat{L^2} = \ell^2$, for each $h \in \mathcal{H}$ and $1 < p \leq 2$, we have

$$\|d^n \hat{f} h\|_{\widehat{L^p}} \leq \|d^n \hat{f} h\|_{\widehat{L^2}} = \|d^n \hat{f} h\|_{\ell^2} \leq \|d^n \hat{f}\|_{\ell^2},$$

so that by the assumption that $\hat{f} \in \ell^{2,\infty}$, we have $\|d^n \hat{f}\|_{|\widehat{L^p}|} = O(1)\ (n \longrightarrow \infty)$. Consequently, $\widehat{L^p} \cap \ell^{2,\infty} \subset \widehat{L^p}_{[AB]}$. The equality $\widehat{L^p}_{[AK]} = \widehat{L^p} \cap \ell^{2,o}$ follows by the same argument, replacing AB and $|AB|$ by AK and $|AK|$, and $O(1)$ by $o(1)$.
[b]: The corresponding statements for $\widehat{L^1}_{[AB]}$ and $\widehat{L^1}_{[AK]}$ follow similarly, using σB and σK, applying Theorem 3.8 and recalling that $\widehat{L^1}_{\sigma B} = \widehat{M}$ and $\widehat{L^1}_{\sigma K} = \widehat{L^1}$.
[c]: Suppose $p > 2$ and let $1/p + 1/q = 1$. If $\hat{f} \in \widehat{L^p} \cap \ell^{q,\infty}$, then $\hat{f} \in \widehat{L^p}_{AB}$ and moreover $\|d^n \hat{f}\|_q = O(1)\ (n \longrightarrow \infty)$. By the Hausdorff-Young Theorem there exists a constant K_p such that for each $h \in \mathcal{H}$ and for each n,

$$\|d^n \hat{f} h\|_{\widehat{L^p}} \leq K_p \|d^n \hat{f} h\|_{\ell^q} \leq K_p \|d^n \hat{f}\|_{\ell^q}.$$

Hence $\|d^n \hat{f}\|_{|\widehat{L^p}|} = O(1)\ (n \longrightarrow \infty)$ and then $\hat{f} \in \widehat{L^p}_{[AB]}$. The corresponding inclusion $\widehat{L^p} \cap \ell^{q,o} \subset \widehat{L^p}_{[AK]}$ can be proved similarly.
[d]: Suppose E is a Banach space and $\mathcal{A} \subset E \subset L^\infty$. If $\hat{f} \in \hat{E}_{[AB]}$, then clearly $\hat{f} \in \hat{E}_{\sigma B}$ and $\|d^n \hat{f}\|_{|\hat{E}|} = O(1)\ (n \longrightarrow \infty)$. Hence the sequence $(d^n \hat{f})$ is bounded in $\hat{E}_{|AB|}$. But by Theorem 11 in [5], $\hat{E}_{|AB|} = \hat{E}_{|AK|} = \ell^1$ and consequently $\|d^n \hat{f}\|_{\ell^1} = O(1)\ (n \longrightarrow \infty)$. Thus $\hat{f} \in \hat{E}_{\sigma B} \cap \ell^{1,\infty}$. Conversely if $\hat{f} \in \hat{E}\sigma B \cap \ell^{1,\infty}$, then $\hat{f}$ has σB in E and $\|d^n \hat{f}\|_{|\hat{E}|} \leq \|d^n \hat{f}\|_{|\hat{\mathcal{A}}|} = \|d^n \hat{f}\|_{\ell^1} = O(1)\,(n \longrightarrow \infty)$, again referring to Theorem 11 in [5]. By Theorem 3.8 it follows that $\hat{f} \in \hat{E}_{[AB]}$. The second equality is proved the same way.
[e]: Since $\phi \subset \widehat{L^1}$ and $\widehat{L^1}$ is a closed subspace of $\hat{M}$, we have $\hat{M}_{[AK]} = \widehat{L^1}_{[AK]}$ and $\hat{M}_{[AB]} = \widehat{L^1}_{[AB]}$. ∎

We shall now apply the concepts of strong boundedness and convergence to some classes of functions recently introduced in Fourier analysis. They are determined by other types of pointwise convergence, namely the strong convergence of index $p \geq 1$, $[I]_p$, and the absolute convergence of index $p \geq 1$, $|I|_p$. The first extends the concept of strong convergence $[I]$ and the second extends the concept of absolute convergence $|I|$, to higher indices $p > 1$. Namely, for $p \geq 1$, they can be defined as follows (see [13] through [17]):

$$s_n \longrightarrow t \quad [I]_p \text{ if and only if } s_n \longrightarrow t\, I \text{ and } \frac{1}{n+1}\sum_{k=0}^{n} k^p |s_k - s_{k-1}|^p = o(1) \quad (n \longrightarrow \infty)$$

and

$$s_n \longrightarrow t \quad |I|_p \text{ if and only if } s_n \longrightarrow t \quad I \text{ and } \sum_k k^{p-1}|s_k - s_{k-1}|^p < \infty.$$

They are related by the following implication (see [15])

$$|I|_p \Rightarrow [I]_p \Rightarrow I \text{ and } [I]_{p'} \text{ for } p' > p \geq 1.$$

These notions were applied to trigonometric and Fourier series in several recent papers [13] through [17], which led to the study of the related spaces of functions [13], [14], [17]:

$$S^p = \{f \in L^1 : s_n f \longrightarrow f \quad [I]_p \text{ a.e.}\}, \quad \mathbb{S}^p = \{f \in C : s_n f \longrightarrow f \quad [I]_p \text{ uniformly}\}$$
$$A^p = \{f \in L^1 : s_n f \longrightarrow f \quad |I|_p \text{ a.e.}\}, \quad \mathcal{A}^p = \{f \in C : s_n f \longrightarrow f \quad |I|_p \text{ uniformly}\}.$$

Clearly $A^2 = \mathcal{A}^1 = A$, but $\mathbb{S}^1 \subset S^1$ properly. We denote $\mathbb{S}^1$ and S^1 by $\mathbb{S}$ and S, respectively. For each $p > 1$ $\mathbb{S}^p \subset S^p \subset \bigcap_{1 \leq r < \infty} L^r$ properly, but $S^p \not\subset L^\infty$. The classes $\mathbb{S}^p$ and S^p decrease as p increases, while the classes $\mathcal{A}^p$ are mutually incomparable and the same is true for A^p. Furthermore, $A^p \subset \mathbb{S}^p \subset \mathcal{U}$ and $A^p \subset S^p \subset L^p$ properly. For these and other properties of these spaces see [13], [14], and [17]. By the results obtained there they also can be described as follows. For $p \geq 1$ let

$$s^p := \left\{x : \frac{1}{2n+1}\sum_{|k| \leq n} |k|^p |x_k|^p = o(1) \quad (n \longrightarrow \infty)\right\}$$
$$a^p := \left\{x : \sum_{k \in \mathbf{Z}} |k|^{p-1}|x_k|^p < \infty\right\}.$$

Then

$$\hat{S} = \widehat{L^1} \cap s^1 \text{ and } \hat{S} \subset s^1 \text{ properly,}$$
$$\widehat{S^p} = s^p = \left\{x : n^{p-1}\sum_{|k| \leqq n} |x_k|^p = o(1) \quad (n \longrightarrow \infty)\right\} \text{ for } p > 1,$$
$$\widehat{\mathbb{S}^p} = \hat{C} \cap s^p \text{ for } p \geq 1,$$
$$\widehat{A^p} = a^p \text{ and } \widehat{\mathcal{A}^p} = \hat{C} \cap a^p \text{ for } p \geq 1.$$

They are Banach spaces under the corresponding norms:

$$\|f\|_S = \|f\|_{L^1} + \|f\|_{[1]}; \quad \|f\|_{S^p} = \|f\|_{[p]} \text{ for } p > 1,$$
$$\|f\|_{\mathbb{S}^p} = \|f\|_{\mathcal{U}} + \|f\|_{[p]} \text{ for } p \geq 1,$$
$$\|f\|_{A^p} = \|f\|_{|p|} \text{ and } \|f\|_{\mathcal{A}^p} = \|f\|_{\mathcal{U}} + \|f\|_{|p|} \text{ for } p \geq 1,$$

where

$$\|f\|_{[p]} = \sup_n \Big(\frac{1}{2n+1} \sum_{|k|\le n} (|k|+1)^p |\hat f(k)|^p \Big)^{\frac1p} \text{ and}$$

$$\|f\|_{|p|} = \Big(|\hat f(0)|^p + \sum_{k\in\mathbb{Z}, k\neq 0} |k|^{p-1} |\hat f(k)|^p \Big)^{\frac1p}.$$

Let us also define for $p \ge 1$ the sequence space

$$s^{p,\infty} := \Big\{ x : \frac{1}{2n+1} \sum_{|k|\le n} |k|^p |x_k|^p = O(1) \quad (n\to\infty) \Big\}.$$

The following theorems show that the Banach spaces S^p and A^p can be characterized as the spaces of integrable functions whose Fourier series are strongly $[I]$ convergent in the topology of S^p, respectively A^p, and that the spaces $\mathbb{S}^p$ and $\mathcal{A}^p$ are precisely the spaces of continuous functions whose Fourier series are strongly $[I]$ convergent in the corresponding topology of $\mathbb{S}^p$, respectively $\mathcal{A}^p$.

THEOREM 6.6. [a] $\widehat{S^p}_{[AK]} = \widehat{S^p} = s^p$ *and* $\widehat{S^p}_{[AB]} = s^{p,\infty}$ *for* $p > 1$;
[b] $\hat S_{[AK]} = \hat S = \widehat{L^1} \cap s^1$ *and* $\hat S_{[AB]} = \widehat{L^1} \cap s^{1,\infty}$;
[c] $\widehat{\mathbb{S}^p}_{[AK]} = \widehat{\mathbb{S}^p} = \hat C \cap s^p$ *and* $\widehat{\mathbb{S}^p}_{[AB]} = \hat C \cap s^{p,\infty}$ *for* $p \ge 1$.

COROLLARY 6.7. $\widehat{S^p} = \ell v_0 \cdot \widehat{S^p}$ *for* $p \ge 1$ *and* $s^p = \ell v_0 \cdot s^p$ *for* $p > 1$.

PROOF. We first remark that by Theorems 2 and 3 in [14] $\|s^n f - f\|_{S^p} = o(1)$ $(n \to \infty)$ for each $p \ge 1$, so that $\widehat{S^p} = \widehat{S^p}_{AK}$ for each $p \ge 1$.
[a]: Suppose $p > 1$. Then $\widehat{S^p} = s^p$ is clearly solid and by the above remark it has AD. Thus by Corollary 3.11 $\widehat{S^p}_{[AK]} = s^p$. Now $\hat f \in \widehat{S^p}_{AB}$ if and only if $\|f\|_{[p]} < \infty$, so that for $p > 1$ $\widehat{S^p}_{AB} = s^{p,\infty}$. Hence $\widehat{S^p}_{[AB]} \subset s^{p,\infty}$. Conversely if $\hat f \in s^{p,\infty}$, then clearly $\|d^n \hat f\|_{|\widehat{S^p}|} \le \|f\|_{[p]}$ for each n, where $d^n \hat f$ is to be interpreted accordingly for the two-way sequence $\hat f$. Moreover clearly $\|s^n \hat f\|_{\widehat{S^p}} \le \|f\|_{[p]}$ since $p > 1$. Thus $s^{p,\infty} \subset \widehat{S^p}_{[AB]}$.
[b]: By the remark above we have $\hat S = \hat S_{AK}$. Moreover for each $\hat f \in \hat S$ we have

$$\|d^n \hat f\|_{|\hat S|} \le \|d^n \hat f_{|\widehat{L^1}|} + \|d^n \hat f\|_{[1]} \le 2\|d^n \hat f\|_{\ell^1} = o(1) \quad (n \to \infty).$$

Thus $\hat S = \hat S_{[AK]}$. The corresponding statement for $\hat S_{[AB]}$ is proved similarly.
[c]: Clearly $\widehat{\mathbb{S}^p}_{[AK]} \subset \hat C$ and $\widehat{\mathbb{S}^p}_{[AK]} \subset \widehat{\mathbb{S}^p}_{AK} \subset s^p$ for $p \ge 1$. Hence $\widehat{\mathbb{S}^p}_{[AK]} \subset \hat C \cap s^p$. Conversely if $\hat f \in \hat C \cap s^p$, then $\hat f \in \widehat{\mathbb{S}^p}$ and, by Theorem 2 in [13], $\|s^n f - f\|_{\mathbb{S}^p} = o(1)$ $(n \to \infty)$, so that $\hat f \in \widehat{\mathbb{S}^p_{AK}}$. Moreover as in the proof of [b] and by Hölder's inequality

$$\|d^n \hat f\|_{|\widehat{\mathbb{S}^p}|} = O(2^{n(1-1/p)}) \|d^n \hat f\|_{\ell^p} = o(1) \quad (n \to \infty).$$

Therefore $\hat C \cap s^p \subset \widehat{\mathbb{S}^p}_{[AK]}$ for each $p \ge 1$. The corresponding proof that $\widehat{\mathbb{S}^p}_{[AB]} = \hat C \cap s^{p,\infty}$ is similar. ∎

The following result for the Banach spaces A^p and $\mathcal{A}^p$ can be proved almost immediately from the discussed properties of these spaces.

THEOREM 6.8. *Suppose $p \geq 1$. Then*
[a] $\widehat{A^p}_{[AK]} = \widehat{A^p}_{[AB]} = a^p = \widehat{A^p}$;
[b] $\widehat{A^p}_{[AK]} = \widehat{\mathcal{A}^p}_{[AB]} = \hat{C} \cap a^p = \widehat{\mathcal{A}^p}$.

REMARK 6.9. For $p = 1$, Theorem 6.8 reduces to $\widehat{A^1}_{[AK]} = \widehat{A^1}_{[AB]} = \ell^1$.

COROLLARY 6.10. $\widehat{A^p} = \ell v_0 \cdot \widehat{A^p}$ *for each* $p \geq 1$.

Our next result shows that $\widehat{S^p}$ is a proper subspace of $\widehat{L^p}_{[AK]}$.

THEOREM 6.11. $\widehat{S^p} \subset \widehat{L^p}_{[AK]}$ *properly for each* $p \geq 1$.

PROOF. Since $\widehat{S^p} \subset \widehat{L^p}$ clearly $\widehat{S^p}_{[AK]} \subset \widehat{L^p}_{[AK]}$. By Theorem 6.6 $\widehat{S^p}_{[AK]} = \widehat{S^p}$ for each $p \geq 1$. Hence $\widehat{S^p} \subset \widehat{L^p}_{[AK]}$. To see that this inclusion is proper we consider the following examples of cosine series

$$\sum_{m=1}^{\infty} \frac{1}{m} \cos 2^m x \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k} \cos kx.$$

The first series is lacunary and by Theorem 8.20 of Chapter 5 in [19] it converges a.e. to a function $f \in \bigcap_{1 \leq r < \infty} L^r$. Moreover, clearly $\|d^n \hat{f}\|_{\ell^1} = \frac{1}{n} = o(1)$ $(n \longrightarrow \infty)$ so that $\hat{f} \in \widehat{L^p}_{[AK]}$ for each $p > 1$. However for $p > 1$ clearly $\hat{f} \notin s^p$. Hence $\hat{f} \notin \widehat{S^p}$ and consequently $\widehat{S^p} \subset \widehat{L^p}_{[AK]}$ properly for $p > 1$. The second series converges a.e. to a function $g \in L^2$. Since $\hat{g} \in \ell^2$ we have $\hat{g} \in \widehat{L^2}_{[AK]} \subset \widehat{L^1}_{[AK]}$. However, obviously $\hat{g} \notin \hat{S}$ since $\hat{g} \notin s^1$. Therefore $\hat{S} \subset \widehat{L^1}_{[AK]}$ also properly. ∎

REFERENCES

1. D. Borwein, *On strong and absolute summability*, Proc. Glasgow Math. Assoc. **4**(1960), 122–139.
2. M. Buntinas, *Convergent and bounded Cesàro sections in FK-spaces*, Math. Z. **121**(1971), 191–200.
3. ———, *On Toeplitz sections in sequence spaces*, Math. Proc. Cambridge Philos. Soc. **78**(1975), 451–460.
4. ———, *Strong summability in Fréchet spaces with applications to Fourier series*, J. Approximation Theory **67**(1991), to appear.
5. M. Buntinas and N. Tanović-Miller, *Absolute boundedness and absolute convergence in sequences spaces,* Proc. Amer. Math. Soc. **111**(1991), 967–979.
6. R. E. Edwards, *Fourier Series: A Modern Introduction,* vols. 1 and 2, Holt, Rinehart and Winston, 1967.
7. D. J. H. Garling, *On topological sequence spaces,* Proc. Cambridge Philos. Soc. **63**(1967), 997–1019.
8. J. M. Hyslop, *Note on the strong summability of series,* Proc. Glasgow Math. Assoc. **1**(1951/53), 16–20.
9. B. Kuttner and I. J. Maddox, *On strong convergence factors*, Quart. J. Math. Oxford (2)**16**(1965), 165–182.
10. B. Kuttner and B. Thorpe, *Strong convergence*, J. für die reine und angewandte Math. **311/312**(1979), 42–56.
11. J. J. Sember, *On unconditional section boundedness in sequence spaces*, Rocky Mountain J. Math. **7**(1977), 699–706.
12. J. Sember and M. Raphael, *The unrestricted section properties of sequences*, Can. J. Math. **31**(1979), 331–336.
13. I. Szalay and N. Tanović-Miller, *On Banach spaces of absolutely and strongly convergent Fourier series*, Acta Math. Hung. **55**(1990), 149–160.
14. ———, *On Banach spaces of absolutely and strongly convergent Fourier series*, II, Acta Math. Hung., to appear.
15. N. Tanović-Miller, *On strong convergence of trigonometric and Fourier series,* Acta Math. Hung. **42**(1983), 35–43.

16. ———, *Strongly convergent trigonometric series as Fourier series*, Acta Math. Hung. **47**(1986), 127–135.
17. ———, *On Banach spaces of strongly convergent trigonometric series*, J. Math. Anal. and Appl. **46**(1990), 110–127.
18. K. Zeller, *Allgemeine Eigenschaften von Limitierungsverfahren*, Math. Z. **53**(1951), 463–487.
19. A. Zygmund, *Trigonometric Series*. Cambridge University Press, 1968.

Department of Mathematical Sciences
Loyola University of Chicago
Chicago, Illinois 60626 USA

Department of Mathematics
University of Sarajevo
71000 Sarajevo, Yugoslavia