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Integer matrices obeying generalized incidence equations

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We consider integer matrices obeying certain generalizations of the incidence equations for (v, k, λ)-configurations and show that given certain other constraints, a constant multiple of the incidence matrix of a (v, k, λ)-configuration may be identified as the solution of the equation.

We define (v, k, λ)-configurations as usual (see [3]). If B is the $(0, 1)$ incidence matrix of a (v, k, λ)-configuration and if $A = bB$ where b is a positive integer, then

$$(1) \qquad \begin{cases} AA^T = b^2(k-\lambda)I + b^2\lambda J \\ AJ = bkJ \\ \lambda(v-1) = k(k-1) \, , \end{cases}$$

with J as usual the matrix with every element $+1$, and I the identity matrix. Ryser [2] proved a partial converse:

LEMMA 1. *If A is a $v \times v$ integer matrix satisfying equations* (1) *with $b = 1$, then A is the incidence matrix of a (v, k, λ)-configuration (and consequently has every entry 0 or 1).*

One might conjecture, in view of the powerful theorems of Ryser [2] and Bridges and Ryser [1], that an integer matrix satisfying (1) would necessarily be b times the incidence matrix of a (v, k, λ)-configuration. But the matrix

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$$A = \begin{bmatrix} 0 & 3 & 0 & 0 & 1 & 1 & 1 \\ -1 & 0 & 2 & 1 & 2 & 1 & 1 \\ 2 & 0 & 0 & 0 & 2 & 2 & 0 \\ 2 & 1 & 2 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 3 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & -1 & 2 & 0 \end{bmatrix}$$

satisfies (1) with $b = 2$, $v = 7$, $k = 3$ and $\lambda = 1$. So we need other conditions on the matrix A before we can ensure that every element is 0 or b . We shall prove:

THEOREM 2. *If A is a $v \times v$ matrix of non-negative integers which satisfies* (1), *and if every entry of A is less than or equal to b , then A is b times the incidence matrix of a (v, k, λ)-configuration.*

The corresponding result for non-positive A and negative b also holds.

By similar methods we shall obtain a result about more general equations:

THEOREM 3. *Let B be an integer matrix of order v which satisfies*

$$BB^T = (p-q)I + qJ$$
$$BJ = dJ$$

where p, q and d are constants and $d > 0$. Write w and z for the greatest and least elements of B respectively, and $\omega = |w|$.

If

$$z \le \frac{d}{v} = \delta \quad and \quad z \le \frac{\omega d+p}{d+\omega v} ,$$

then δ is an integer, $p = d\delta = v\delta^2$, and $B = \delta J$.

1. Proof of Theorem 2

LEMMA 4. *Let $B = (b_{ij})$ of order v be a matrix of non-negative integers such that $\sum_{j=1}^{v} b_{ij}^2 = p$, p a constant, for every i , and let*

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$BJ = dJ$, d a non-zero constant. If $b_{ij} \le \frac{p}{d}$ for every b_{ij} , or if $b_{ij} \ge \frac{p}{d}$ for every non-zero b_{ij} , then every entry of B is 0 or $\frac{p}{d}$.

Proof. $\sum_{j=1}^{v} b_{ij}^2 = p$ and $\sum_{j=1}^{v} b_{ij} = d$, so

$$d \sum_j b_{ij}^2 - p \sum_j b_{ij} = dp - dp = 0 ;$$

that is

$$\sum_j b_{ij}(db_{ij}-p) = 0 .$$

From the data every term in this summation has the same sign, so every term is zero. So $b_{ij} = 0$ or $\frac{p}{d}$.

COROLLARY 5. *If there is a matrix B satisfying the conditions of Lemma 4, then $d|p$.*

Corresponding results may be obtained for matrices of non-positive integers.

Proof of Theorem 2. The matrix A satisfies the conditions of Lemma 4 with $p = b^2k$ and $d = bk$. So every entry is 0 or b $\left(b = \frac{p}{d}\right)$. Consider $B = b^{-1}A$. B is an integer matrix satisfying Lemma 1, so it is the incidence matrix of a (v, k, λ)-configuration, and we have the result.

2. Proof of Theorem 3

Proof of Theorem 3. Clearly $p = \sum_i b_{ij}^2$ implies $p \ge 0$; and $d > 0$ implies $p > 0$. Consider the class of matrices

$$C_\alpha = B + \alpha J$$

where α is an integer and $\alpha \ge \omega$. Every element of every member of this class is non-negative and

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$$C_\alpha C_\alpha^T = (p-q)I + \left(\alpha^2 v+2\alpha d+q\right)J$$
$$C_\alpha J = (d+\alpha v)J .$$

Then using Lemma 4, if every non-zero element of C_α is less than or equal to β ,

$$\beta = \alpha + \frac{\alpha d+p}{d+\alpha v} ,$$

then every element is 0 or β .

We show that the conditions on z imply that every element is $\leq \beta$. For

$$z \leq \frac{\omega d+p}{d+\omega v}$$

implies

$$z(d+\omega v) \leq \omega d + p ;$$

since $z \leq \frac{d}{v}$ we have

$$zd + z\omega v + \gamma z v \leq \omega d + p + \gamma z v \leq \omega d + p + \gamma d$$

for any integer $\gamma \geq 0$, so

$$z \leq \frac{(\omega+\gamma)d+p}{d+(\omega+\gamma)v} .$$

This means (putting $\alpha = \omega + \gamma$) that for any admissable α ,

$$z + \alpha \leq \alpha + \frac{\alpha d+p}{d+\alpha v} ;$$

but $z + \alpha$ is the greatest element of C_α . Therefore, any element of C_α is 0 or $\alpha + \frac{\alpha d+p}{d+\alpha v}$, so any element of B is $-\alpha$ or $\frac{\alpha d+p}{d+\alpha v}$.

Corollary 5 tells us that

$$A(\gamma) = \frac{(\omega+\gamma)d+p}{d+(\omega+\gamma)v} = \frac{d+p(\omega+\gamma)^{-1}}{d(\omega+\gamma)^{-1}+v}$$

is integral for all integers $\gamma \geq 0$. Therefore $\lim_{\gamma\to\infty} A(\gamma)$ must be an integer, so $v|d$. Write $d = v\delta$:

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$$A(\gamma) = \frac{(\omega+\gamma)v\delta+p}{v\delta+(\omega+\gamma)v}$$

so $v|p$. Write $p = \varepsilon v$:

$$A(\gamma) = \frac{(\omega+\gamma)\delta+\varepsilon}{\delta+(\omega+\gamma)} .$$

Choose n any integer greater than $\delta + \omega$. Then

$$A(n-\delta-\omega) = \frac{(n-\delta)\delta+\varepsilon}{n}$$

so $n|(\varepsilon-\delta^2)$. But this is true for every large enough n ; hence $\varepsilon = \delta^2$. That is

$$d = v\delta$$

$$p = v\delta^2$$

so

$$p = d\delta = v\delta^2 .$$

Then we have

$$\frac{\alpha d+p}{d+\alpha v} = \frac{v\delta(\alpha+\delta)}{v(\delta+\alpha)} = \delta$$

for any γ , so every element of B is $-\alpha$ or δ . Now the row sum of B is $d = v\delta$ and the sum of the squares of the elements is $p = v\delta^2$; together these imply

$$B = \delta J$$

where $\delta = \frac{d}{v}$.

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[1] W.G. Bridges and H.J. Ryser, "Combinatorial designs and related systems", *J. Algebra* 13 (1969), 432-446.

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