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WEIGHTED ESTIMATES FOR SOLUTIONS OF THE GENERAL STURM–LIOUVILLE EQUATION AND THE EVERITT–GIERTZ PROBLEM. I

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Abstract Consider the equation

$$-(r(x)y'(x))' + q(x)y(x) = f(x), \quad x \in \mathbb{R}, \tag{$*$}$$

where $f \in L_p(\mathbb{R})$, $p \in (1, \infty)$, and

$$r > 0, \quad q \geqslant 0, \quad 1/r \in L_1^{\text{loc}}(\mathbb{R}), \quad q \in L_1^{\text{loc}}(\mathbb{R}),$$
$$\lim_{|d|\to\infty} \int_{x-d}^{x} \frac{\mathrm{d}t}{r(t)} \int_{x-d}^{x} q(t)\,\mathrm{d}t = \infty \quad \forall x \in \mathbb{R}.$$

By a solution of $(*)$, we mean any function y absolutely continuous together with (ry') and satisfying $(*)$ almost everywhere on $\mathbb{R}$. In addition, we assume that $(*)$ is correctly solvable in the space $L_p(\mathbb{R})$, i.e.

(1) for any function $f \in L_p^{\text{loc}}(\mathbb{R})$, there exists a unique solution $y \in L_p(\mathbb{R})$ of $(*)$;

(2) there exists an absolute constant $c_1(p) > 0$ such that the solution $y \in L_p(\mathbb{R})$ of $(*)$ satisfies the inequality

$$\|y\|_{L_p(\mathbb{R})} \leqslant c_1(p)\|f\|_{L_p(\mathbb{R})} \quad \forall f \in L_p(\mathbb{R}). \tag{$**$}$$

We study the following problem on the strengthening estimate $(**)$. Let a non-negative function $\theta \in L_p^{\text{loc}}(\mathbb{R})$ be given. We have to find minimal additional restrictions for θ under which the following inequality holds:

$$\|\theta y\|_{L_p(\mathbb{R})} \leqslant c_2(p)\|f\|_{L_p(\mathbb{R})} \quad \forall f \in L_p(\mathbb{R}).$$

Here, y is a solution of $(*)$ from the class $L_p(\mathbb{R})$, and $c_2(p)$ is an absolute positive constant.

Keywords: linear differential equations; second-order equation; Everitt–Giertz problem

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1. Introduction

In this paper we continue the study developed in [**22**,**23**,**25**–**27**]. We consider the equation

$$-(r(x)y'(x))' + q(x)y(x) = f(x), \quad x \in \mathbb{R}, \tag{1.1}$$

where $f \in L_p$ ($L_p(\mathbb{R}) := L_p$), $p \in (1, \infty)$ and

$$r > 0, \quad q \geqslant 0, \quad 1/r \in L_1^{\mathrm{loc}}, \quad q \in L_1^{\mathrm{loc}} \quad (L_1^{\mathrm{loc}} := L_1^{\mathrm{loc}}(\mathbb{R})), \tag{1.2}$$

$$\lim_{|d|\to\infty} \int_{x-d}^{x} \frac{\mathrm{d}t}{r(t)} \int_{x-d}^{x} q(t)\,\mathrm{d}t = \infty \quad \forall x \in \mathbb{R}. \tag{1.3}$$

Throughout what follows, by a solution of (1.1) we mean any function y that is absolutely continuous together with (ry') and satisfies (1.1) almost everywhere in $\mathbb{R}$.

In addition to (1.2)–(1.3), we always assume that (1.1) is correctly solvable in L_p, $p \in (1, \infty)$. The latter condition means that requirements (I) and (II) hold (see [**31**, Chapter III, § 6.2]).

(I) For any $f \in L_p$, there exists a unique solution $y \in L_p$ of (1.1).

(II) There exists an absolute constant $c(p) > 0$ such that the solution $y \in L_p$ of (1.1) satisfies the inequality

$$\|y\|_p \leqslant c(p)\|f\|_p \quad \forall f \in L_p \ (\|f\|_p := \|f\|_{L_p}). \tag{1.4}$$

Exact restrictions to r and q that guarantee (I) and (II) are known (see [**26**] and § 2). Furthermore, in connection with (1.4), we adopt the following conventions: by the symbol y we denote only solutions of (1.1) belonging to the class L_p; the symbols c, $c(\cdot)$ stand for absolute positive constants that are not essential for exposition and may differ even within a single chain of computations.

We now return to (1.1). Our general goal is to study possibilities for strengthening estimate (1.4). To be more precise, let θ denote an arbitrary non-negative function from L_p^{loc} ($L_p^{\mathrm{loc}}(\mathbb{R}) := L_p^{\mathrm{loc}}$). Our specific goal is *to find minimal additional restrictions to θ under which the solutions y of (1.1) satisfy the inequality*

$$\|\theta y\|_p \leqslant c(p)\|f\|_p \quad \forall f \in L_p. \tag{1.5}$$

Note that in a particular case (1.5) for $\theta = q$, (1.1) may (or may not) have an interesting feature, depending on $p \in [1, \infty)$ and properties of the functions r and q. In order to describe this feature, we introduce the following definition.

Definition 1.1. Suppose that (1.1) is correctly solvable in the space L_p, $p \in [1, \infty)$. We say that this equation is separable in L_p if the following inequality holds:

$$\|(ry')'\|_p + \|qy\|_p \leqslant c(p)\|f\|_p \quad \forall f \in L_p, \tag{1.6}$$

or, equivalently,

$$\|qy\|_p \leqslant c(p)\|f\|_p \quad \forall f \in L_p. \tag{1.7}$$

(Definition 1.1 is sometimes referred to within this paper as the 'separability problem' or 'separability conditions'.)

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The problem on separability was posed by Everitt and Giertz in [**35**, **36**] (in terms of different operators) and is therefore called the Everitt–Giertz problem. In [**35**, **36**], the first examples of inseparable operators and the first study of sufficient conditions for separability (of the Sturm–Liouville operator in L_2) were given. These results were then strengthened and developed by the authors themselves [**37**–**44**], and in successive papers [**1**–**15**, **17**–**24**, **28**–**30**, **33**, **34**, **46**, **47**, **49**–**56**, **58**–**63**]. We want to emphasize the fact that until now unconditional criteria for the validity of (1.5) and (1.7) have been found only in particular cases (see [**3**, **21**, **22**, **24**, **28**, **30**, **45**, **49**, **52**, **53**]), and therefore the study of (1.5) and (1.7) continues to be of interest. All the works cited above represent the literature on this question as a whole, i.e. not necessarily in connection with (1.1) on the Sturm–Liouville operator.

Note that in spite of the abundance of outstanding results and the obvious interest in (1.5) and (1.7), no analytical survey paper has been dedicated to this subject. Therefore, below, in order to position our work among the above cited papers, we give only the most general additional necessary information. To stay within a limited framework, we only discuss (1.7). Thus, we first want to emphasize that the decisive role in studying the separability problem is played by (1.1). This equation (or the corresponding Sturm–Liouville operator) is the main testing ground for almost all the innovations in the separability research. In particular, this is the main explanation of the fact that a significant part of the work cited above is directly related to (1.1). The research dedicated to (1.1) can, in turn, be subdivided into two non-intersecting groups. The first group contains the majority of the papers on (1.1). This group can be characterized by the fact that separability conditions are expressed in the form of certain requirements on the coefficients r and q of (1.1). As for the second group, which consists of the papers [**3**, **19**–**21**, **23**, **29**, **30**, **51**, **56**, **61**], here separability conditions are expressed in terms of requirements on certain local integral averages of the functions r and q. (The first investigation in the second group is due to Otelbaev; see [**51**, **56**].) Each of these approaches has its advantages and disadvantages, which will be made clear after obtaining unconditional criteria for the separability of (1.1) in L_p for $p \in (1, \infty)$ (for $p = 1$, (1.7) holds automatically; see [**22**, **30**, **45**, **53**]).

We now begin the discussion of our present research. Since we are studying (1.1), according to our classification, this research belongs to the second group of papers. In particular, we study conditions for the separability of (1.1) in L_p, remaining in the framework of the approach to (1.7) that was developed in the papers [**3**, **19**, **20**, **23**, **30**, **45**, **53**, **61**].

Our general goal consists of extending the methods of [**23**], in which $r = 1$ and $1 \leqslant q \in L_1^{\text{loc}}$, to the case of (1.1) (with conditions (1.2) and (1.3)) correctly solvable in L_p, $p \in (1, \infty)$. Similar problems for the cases $p = 1$ and $p = \infty$ will be considered in forthcoming papers. Note that the main result of this paper contains necessary and (nearly) sufficient conditions for the validity of inequalities (1.5) and (1.7) (see Theorems 3.3 and 3.4). Unfortunately, although these assertions are general and precise, they are not sufficient for the study of (1.5) and (1.7) in concrete cases. The point is that the requirements of Theorems 3.3 and 3.4 are expressed in terms of certain auxiliary functions (averages of

Otelbaev type of the functions r and q; see § 3). Exact values of these auxiliary functions can be found only in exceptional cases. However, to apply Theorems 3.3 and 3.4 to concrete equations, it is enough to have two-sided, sharp-by-order estimates for these averages. The second part of this paper contains a solution of the problem on the proof of such inequalities.

We emphasize that, by combining all the results, we obtain an efficient tool for the study of inequalities (1.5) and (1.7). As an example of such an application, we consider (1.1) with the coefficients

$$r(x) = 1 + x^2, \quad q(x) = \mathrm{e}^{|x|} + \mathrm{e}^{|x|}\cos(\mathrm{e}^{\alpha|x|}), \quad x \in \mathbb{R},\ \alpha > 0. \tag{1.8}$$

In the second part of the paper, we obtain the following fact.

Proposition 1.2. *Equation (1.1) with coefficients (1.8) is correctly solvable in* L_p *for all* $p \in (1, \infty)$*, regardless of* $\alpha \in (0, \infty)$*. The inequality*

$$\|\mathrm{e}^{|x|}y\|_p \leqslant c(p)\|f\|_p \quad \forall f \in L_p \tag{1.9}$$

holds if and only if $\alpha \geqslant \frac{1}{2}$.

The paper has the following structure. Most of the facts used in the following are given in § 2, all our results are given in § 3, and the proofs are given in § 4.

2. Preliminaries

We adopt the following convention: the requirements (1.2) and (1.3) are assumed to be valid and do not appear in the statements.

Lemma 2.1 (Chernyavskaya and Shuster [22]). *The equation*

$$(r(x)z'(x))' = q(x)z(x), \quad x \in \mathbb{R}, \tag{2.1}$$

has a fundamental system of solution (FSS) with the properties

$$u(x) > 0, \quad v(x) > 0, \quad u'(x) \leqslant 0, \quad v'(x) \geqslant 0, \quad x \in \mathbb{R}, \tag{2.2}$$

$$r(x)[v'(x)u(x) - u'(x)v(x)] = 1, \quad x \in \mathbb{R}, \tag{2.3}$$

$$\lim_{x\to-\infty} \frac{v(x)}{u(x)} = \lim_{x\to\infty} \frac{u(x)}{v(x)} = 0. \tag{2.4}$$

Corollary 2.2 (Chernyavskaya and Shuster [22]). *Equation (2.1) has no solutions* $z \in L_p$ *apart from* $z \equiv 0$.

Throughout the following, by the symbols $\{u, v\}$ we denote only the FSS from Lemma 2.1.

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Theorem 2.3 (Chernyavskaya and Shuster [22], Davies and Harrell [32]). *For $\{u, v\}$, there exists the following Davies–Harrell representation:*

$$u(x) = \sqrt{\rho(x)} \exp\left(-\frac{1}{2}\int_{x_0}^{x} \frac{\mathrm{d}\xi}{r(\xi)\rho(\xi)}\right), \quad x \in \mathbb{R}, \tag{2.5}$$

$$v(x) = \sqrt{\rho(x)} \exp\left(\frac{1}{2}\int_{x_0}^{x} \frac{\mathrm{d}\xi}{r(\xi)\rho(\xi)}\right), \quad x \in \mathbb{R}, \tag{2.6}$$

where $\rho(x) = u(x)v(x)$, $x \in \mathbb{R}$, x_0 is the unique solution of the equation $u(x) = v(x)$ in $\mathbb{R}$. In addition, for the Green function $G(x,t)$ corresponding to (1.1),

$$G(x,t) = \begin{cases} u(x)v(t), & x \geqslant t, \\ u(t)v(x), & x \leqslant t, \end{cases} \tag{2.7}$$

and, for its 'diagonal value' $G(x,t)|_{t=x} = \rho(x)$, $x \in \mathbb{R}$, there are the following relations:

$$G(x,t) = \sqrt{\rho(x)\rho(t)} \exp\left(-\frac{1}{2}\left|\int_{x}^{t} \frac{\mathrm{d}\xi}{r(\xi)\rho(\xi)}\right|\right), \quad x, t \in \mathbb{R}, \tag{2.8}$$

$$\int_{-\infty}^{0} \frac{\mathrm{d}\xi}{r(\xi)\rho(\xi)} = \int_{0}^{\infty} \frac{\mathrm{d}\xi}{r(\xi)\rho(\xi)} = \infty. \tag{2.9}$$

Remark 2.4. Equations (2.5), (2.6) and (2.8) were given for $r \equiv 1$ in [**32**] and in [**22**] under the conditions (1.2) and (1.3).

Lemma 2.5 (Chernyavskaya and Shuster [22]). *For any given $x \in \mathbb{R}$, each of the equations in $d \geqslant 0$,*

$$\int_{x-d}^{x} \frac{\mathrm{d}t}{r(t)} \int_{x-d}^{x} q(t)\,\mathrm{d}t = 1 \quad \text{and} \quad \int_{x}^{x+d} \frac{\mathrm{d}t}{r(t)} \int_{x}^{x+d} q(t)\,\mathrm{d}t = 1, \tag{2.10}$$

has a unique finite positive solution.

Denote the solutions of (2.10) by $d^{(-)}(x)$ and $d^{(+)}(x)$, respectively. For $x \in \mathbb{R}$, we introduce the functions

$$\varphi(x) = \int_{x-d^{(-)}(x)}^{x} \frac{\mathrm{d}t}{r(t)}, \qquad \psi(x) = \int_{x}^{x+d^{(+)}(x)} \frac{\mathrm{d}t}{r(t)}, \tag{2.11}$$

$$h(x) = \frac{\varphi(x)\psi(x)}{\varphi(x) + \psi(x)} = \left(\int_{x-d^{(-)}(x)}^{x+d^{(+)}(x)} q(t)\,\mathrm{d}t\right)^{-1}. \tag{2.12}$$

Theorem 2.6 (Chernyavskaya and Shuster [22]). *We have Otelbaev's inequalities*

$$2^{-1}h(x) \leqslant \rho(x) \leqslant 2h(x), \quad x \in \mathbb{R}. \tag{2.13}$$

Remark 2.7. *A priori* sharp-by-order estimates of the function ρ were first obtained by Otelbaev in [**57**] (under some additional requirements on r and q). Therefore, all relations of the form (2.13) will be called Otelbaev inequalities. Note that the auxiliary function used in [**57**] is, probably, more complicated than the function h in (2.13).

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Lemma 2.8 (Chernyavskaya and Shuster [22]). *For every $x \in \mathbb{R}$, the equation in $d \geqslant 0$,*

$$\int_{x-d}^{x+d} \frac{\mathrm{d}t}{r(t)h(t)} = 1, \tag{2.14}$$

has a unique finite positive solution. Denote this solution by $d(x)$. The function $d(x)$ is continuous for $x \in \mathbb{R}$. In addition, $(|x| - d(x)) \to \infty$ as $|x| \to \infty$.

Lemma 2.9 (Chernyavskaya and Shuster [22]). *Let $x \in \mathbb{R}$, $t \in [x-d(x), x+d(x)]$. Then,*

$$\mathrm{e}^{-2}\rho(x) \leqslant \rho(t) \leqslant \mathrm{e}^2\rho(x), \quad (4\mathrm{e}^2)^{-1}h(x) \leqslant h(t) \leqslant (4\mathrm{e}^2)h(x), \quad \mathrm{e} = \exp(1). \tag{2.15}$$

Definition 2.10 (Chernyavskaya and Shuster [24]). Suppose that we are given $x \in \mathbb{R}$, a positive continuous function $\varkappa(t)$, $t \in \mathbb{R}$, and a sequence $\{x_n\}_{n\in\mathbb{N}'}$, $\mathbb{N}' = \{\pm 1, \pm 2, \dots\}$. Consider the segments $\Delta_n = [\Delta_n^-, \Delta_n^+]$, $\Delta_n^\pm = x_n \pm \varkappa(x_n)$, $n \in \mathbb{N}'$. We say that the segments $\{\Delta_n\}_{n=1}^\infty$ (respectively, $\{\Delta_n\}_{n=-\infty}^{-1}$) form an $\mathbb{R}(x,\varkappa)$-covering of $[x,\infty)$ (respectively, of $(-\infty, x]$) if the following requirements hold:

(1) $\Delta_n^+ = \Delta_{n+1}^-$ if $n \geqslant 1$ (respectively, $\Delta_{n-1}^+ = \Delta_n^-$ if $n \leqslant -1$),

(2) $\Delta_1^- = x$ (respectively, $\Delta_{-1}^+ = x$), $\bigcup_{n=1}^\infty \Delta_n = [x,\infty)$ (respectively, $\bigcup_{u=-\infty}^{-1} \Delta_n = (-\infty, x]$).

Lemma 2.11 (Chernyavskaya and Shuster [24]). *Suppose that a positive continuous function $\varkappa(t)$ for $t \in \mathbb{R}$ satisfies the relation*

$$\lim_{t\to\infty}(t - \varkappa(t)) = \infty \quad (\textit{respectively}, \lim_{t\to-\infty}(t + \varkappa(t)) = -\infty). \tag{2.16}$$

For every $x \in \mathbb{R}$ there then exists an $\mathbb{R}(x,\varkappa)$-covering of $[x,\infty)$ (respectively, an $\mathbb{R}(x,\varkappa)$-covering of $(-\infty, x]$).

Remark 2.12. Assertions similar to Lemma 2.11 were introduced by Otelbaev (see [**51**]).

Lemma 2.13 (Chernyavskaya and Shuster [22]). *For every $x \in \mathbb{R}$ there exist $R(x,d)$-coverings of $[-\infty, x)$ and $[x,\infty)$.*

We introduce the set $\mathcal{D}_p$ and the operator $\mathcal{L}_p$:

$$\mathcal{D}_p = \{y \in L_p \colon y, ry' \in AC^{\mathrm{loc}}(\mathbb{R}),\ -(ry')' + qy \in L_p\}, \tag{2.17}$$

$$\mathcal{L}_p y = -(ry') + qy, \quad y \in \mathcal{D}_p. \tag{2.18}$$

The linear operator $\mathcal{L}_p$ is called a maximal Sturm–Liouville operator, and (I) and (II) (see § 1) are, obviously, equivalent to the problem of existence and boundedness of the operator $\mathcal{L}_p^{-1} \colon L_p \to L_p$, i.e. the problem of continuous invertibility of the operator $\mathcal{L}_p$ (see [**16**]).

Theorem 2.14 (Chernyavskaya and Shuster [26]). *Let $p \in [1, \infty)$, and let $G\colon L_p \to L_p$ be the Green operator*

$$(Gf)(x) = \int_{-\infty}^{\infty} G(x,t)f(t)\,\mathrm{d}t \quad \forall f \in L_p,\ x \in R. \tag{2.19}$$

Then, (1.1) is correctly solvable in L_p if and only if the operator $G\colon L_p \to L_p$ is bounded. In the latter case, for any $f \in L_p$, the solution of $y \in L_p$ of (1.1) is of the form $y = Gf$. In particular, $\mathcal{L}_p^{-1} = G$.

Theorem 2.15 (Chernyavskaya and Shuster [26]). *Let $p \in (1, \infty)$. Then, (1.1) is correctly solvable in L_p if and only if*

$$B = \sup_{x\in\mathbb{R}}(h(x)d(x)) < \infty. \tag{2.20}$$

Theorem 2.16 (Oinarov [53]). *Suppose that condition (1.2) holds and that $\inf_{x\in\mathbb{R}} q(x) > 0$. The operator $G\colon L_p \to L_p$ is then bounded for all $p \in [1, \infty)$.*

Remark 2.17. In connection to Theorem 2.16, see [**22**, Theorem 2.3] and [**26**, Corollary 1.9].

Let μ, θ be almost everywhere finite, measurable, positive functions defined in an interval (a, b), $-\infty \leqslant a < b \leqslant \infty$. We introduce the integral operators

$$(Kf)(x) = \mu(x)\int_x^b \theta(t)f(t)\,\mathrm{d}t, \quad x \in (a,b), \tag{2.21}$$

$$(\tilde{\mathcal{K}}f)(x) = \mu(x)\int_a^x \theta(t)f(t)\,\mathrm{d}t, \quad x \in (a,b). \tag{2.22}$$

Theorem 2.18 (Kufner and Persson [48]). *For $p \in (1, \infty)$, the operator $K\colon L_p(a,b) \to L_p(a,b)$ is bounded if and only if*

$$H_p(a,b) = \sup_{x\in(a,b)} H_p(a,b,x) < \infty,$$

where

$$H_p(a,b,x) = \left[\int_a^x \mu(t)^p\,\mathrm{d}t\right]^{1/p}\left[\int_x^b \theta(t)^{p'}\,\mathrm{d}t\right]^{1/p'}, \quad p' = \frac{p}{p-1}. \tag{2.23}$$

In addition,

$$H_p(a,b) \leqslant \|K\|_{L_p(a,b)\to L_p(a,b)} \leqslant (p)^{1/p}(p')^{1/p'}H_p(a,b). \tag{2.24}$$

Theorem 2.19 (Kufner and Persson [48]). *For $p \in (1, \infty)$, the operator $\tilde{K}\colon L_p(a,b) \to L_p(a,b)$ is bounded if and only if*

$$\tilde{H}_p(a,b) = \sup_{x\in(a,b)} \tilde{H}_p(a,b,x) < \infty,$$

where

$$\tilde{H}_p(a,b,x) = \left[\int_a^x \theta(t)^{p'}\,\mathrm{d}t\right]^{1/p'}\left[\int_x^b \mu(t)^p\,\mathrm{d}t\right]^{1/p}, \quad p' = \frac{p}{p-1}. \tag{2.25}$$

In addition,

$$\tilde{H}_p(a,b) \leqslant \|K\|_{L_p(a,b)\to L_p(a,b)} \leqslant (p)^{1/p}(p')^{1/p'}\tilde{H}_p(a,b). \tag{2.26}$$

Note that, apart from the facts listed above, in § 4 we use several assertions (mainly of a technical nature) that are given there in the course of our exposition.

3. Main results

Note that if a pair of functions (in the following, just a 'pair') $\{r, q\}$ satisfies conditions (1.2) and (1.3), then, for every $\lambda \geqslant 0$, the pair $\{r, q_\lambda\}$, where $q_\lambda = q + \lambda$, satisfies the same conditions. We adopt the following convention: throughout what follows instead of the notation for auxiliary functions,

$$d^{(-1)},\ d^{(+)},\ \varphi,\ \psi,\ h,\ d \tag{3.1}$$

(see (2.10), (2.11), (2.12), (2.14)) constructed by a pair $\{r, q_\lambda\}$ for $\lambda \geqslant 0$, we use the notation

$$d_\lambda^{(-1)},\ d_\lambda^{(+)},\ \varphi_\lambda,\ \psi_\lambda,\ h_\lambda,\ d_\lambda, \tag{3.2}$$

respectively. We reserve the notation of (3.1) for the pair $\{r, q_\lambda\}$ with $\lambda = 0$ if this is specially mentioned.

Definition 3.1. Let a pair $\{r, q_\lambda\}$ be given. Assume that for some $b > 0$ there exist $a \geqslant 1$ and $\lambda \geqslant 0$ such that, for all $x \in \mathbb{R}$, the following relations hold:

$$a^{-1}h_\lambda(x)d_\lambda(x) \leqslant h_\lambda(t)d_\lambda(t) \leqslant ah_\lambda(x)d_\lambda(x) \quad \text{if } |t-x| \leqslant bd_\lambda(x), \tag{3.3}$$

$$\int_{x-bd_\lambda(x)}^{x+bd_\lambda(x)} \frac{\mathrm{d}t}{d_\lambda(t)} \geqslant \frac{b}{a} \quad \forall x \in \mathbb{R}, \tag{3.4}$$

$$\lim_{|x|\to\infty}(|x| - bd_\lambda(x)) = \infty. \tag{3.5}$$

The value

$$\gamma(b) = a\exp\left(-\frac{b}{500a^2}\right)$$

is then called the exponent of the pair $\{r, q\}$ corresponding to the number $b > 0$.

In (3.3)–(3.5), we formalize *a priori* properties of any pair $\{r, q\}$ satisfying conditions (1.2) and (1.3). We also note that if the number $\gamma(b)$ is small enough, then our main assertion (see Theorem 3.4) gives the complete answer to the question of (1.7). These facts are fixed in the following assertion.

Theorem 3.2. *Let a pair $\{r, q\}$ and a number $\tilde{\gamma} > 4\mathrm{e}^2$ be given. There then exist $b > 0$ and the exponent $\gamma(b)$ of this pair such that $\gamma(b) = \tilde{\gamma}$.*

Throughout the following, by the symbol $\gamma(b)$, $b > 0$, we denote the exponent of the pair $\{r, q\}$ formed by the coefficients of (1.1).

The next two theorems constitute the main result of this part of the paper.

Theorem 3.3. *Suppose that (1.1) is correctly solvable in the space L_p, $p \in (1, \infty)$, and inequality (1.5) holds. Then,*

$$m_p(r, q, \theta) = \sup_{x \in \mathbb{R}} h(x) d(x)^{1/p'} \left[\int_{x-d(x)}^{x+d(x)} \theta^p(t)\, \mathrm{d}t \right]^{1/p} < \infty. \tag{3.6}$$

In particular, (1.7) holds only if $m_p(r, q, q) < \infty$.

Theorem 3.4. *Suppose that (1.1) is correctly solvable in the space L_p, $p \in (1, \infty)$, and at least one of the exponents $\gamma(b)$, $b > 0$, of the pair $\{r, q\}$ satisfies the inequality $\gamma(b) \leqslant \gamma_0$, $\gamma_0 = \exp(-\frac{1}{500})$. The estimate (1.5) then holds if $m_p(r, q, \theta) < \infty$. In particular, if $m_p(r, q, q) < \infty$, then (1.1) is separable in the space $L_p(\mathbb{R})$.*

Remark 3.5. The first study of (1.5) and (1.7), with the help of inequalities of type (3.3) and a functional of type (3.6), was carried out by Otelbaev for $r \equiv 1$ (see [**51**, **56**]).

Remark 3.6. Recall that the main questions related to applying Theorems 3.3 and 3.4 to concrete equations will be considered in the future in part II of this paper.

4. Proofs

Proof of Theorem 3.2. We need some auxiliary assertions.

Lemma 4.1 (Chernyavskaya and Shuster [22, p. 1422]). *For $x \in \mathbb{R}$, we have the inequality*

$$|d(x + s) - d(x)| \leqslant |s| \quad \text{if } |s| \leqslant d(x). \tag{4.1}$$

Lemma 4.2. *Let $\varepsilon \in [0, 1)$ and $x \in \mathbb{R}$. Then,*

$$(1 - \varepsilon) d(x) \leqslant d(t) \leqslant (1 + \varepsilon) d(x) \leqslant (1 - \varepsilon)^{-1} d(x) \quad \text{if } |t - x| \leqslant \varepsilon d(x). \tag{4.2}$$

Proof. Set

$$x + s = t, \quad |s| \leqslant \varepsilon d(x), \quad \varepsilon \in [0, 1), \quad x \in \mathbb{R}.$$

From (4.1), it then follows that

$$\begin{aligned} |d(t) - d(x)| = |d(x + s) - d(x)| \leqslant |s| \leqslant \varepsilon d(x) \\ \Rightarrow \quad \left| \frac{d(t)}{d(x)} - 1 \right| \leqslant \varepsilon \quad \text{for } |t - x| \leqslant \varepsilon d(x) \\ \Rightarrow \quad (4.2). \end{aligned}$$

□

Remark 4.3. Lemmas of type 4.1 and 4.2 were first obtained by Otelbaev (see [**51**]).

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We now turn to the proof of Theorem 3.2. We find the exponent of the pair $\{r, q\}$ for $b = \varepsilon \in (0,1)$. Let $x \in \mathbb{R}$ and $\lambda = 0$. According to (4.2) and (2.15), we then have (here we use the notation of (3.2)) that

$$\frac{1-\varepsilon}{4\mathrm{e}^2} \leqslant \frac{h_\lambda(t)d_\lambda(t)}{h_\lambda(x)d_\lambda(x)} \leqslant \frac{4\mathrm{e}^2}{1-\varepsilon} \quad \text{if } |t-x| \leqslant \varepsilon d_\lambda(x). \tag{4.3}$$

Therefore, following (3.3), $a := 4\mathrm{e}^2(1-\varepsilon)^{-1}$. We show that, with such a choice of parameters a and b, we also have (3.4). Below we use (4.2):

$$\int_{x-bd_\lambda(x)}^{x+bd_\lambda(x)} \frac{\mathrm{d}t}{d_\lambda(t)} = \int_{x-\varepsilon d_\lambda(x)}^{x+\varepsilon d_\lambda(x)} \frac{d_\lambda(x)}{d_\lambda(t)} \frac{\mathrm{d}t}{d_\lambda(x)} \geqslant 2\varepsilon(1-\varepsilon) \geqslant \frac{\varepsilon(1-\varepsilon)}{4\mathrm{e}^2} = \frac{b}{a}.$$

Finally, for $b = \varepsilon \in [0,1]$, (3.5) holds by Lemma 2.8. Hence, by Definition 3.1, we have that

$$\gamma(b) = \gamma(\varepsilon) = \frac{4\mathrm{e}^2}{1-\varepsilon} \exp\left(-\frac{\varepsilon(1-\varepsilon)^2}{500(4\mathrm{e}^2)^2}\right).$$

It is easy to see that the function $\gamma(\varepsilon)$, $\varepsilon \in [0,1)$, is continuous, and as ε increases it monotonically increases from $\gamma(0) = 4\mathrm{e}^2$ to ∞, which proves the theorem. □

Proof of Theorem 3.3. Clearly, (1.5) is equivalent to boundedness of the operator $\theta\mathcal{L}_p^{-1}\colon L_p \to L_p$, $p \in (1,\infty)$, i.e. to the inequality

$$\|\theta\mathcal{L}_p^{-1}\|_{p\to p} < \infty \tag{4.4}$$

(see (I) and (II) in § 1, (2.17), (2.18) and Theorem 2.14). Below, we show that (4.4) implies the inequalities

$$\infty > \|\theta\mathcal{L}_p^{-1}\|_{p\to p} \geqslant c^{-1} m_p(r,q,\theta),$$

and thus proves Theorem 3.3.

Lemma 4.4. *Let $x \in \mathbb{R}$, $\Delta(x) = [x-d(x), x+d(x)]$,*

$$f_x(t) = \begin{cases} 1 & \text{if } t \in \Delta(x), \\ 0 & \text{if } t \notin \Delta(x), \end{cases}$$

and let $y_x(t)$ be the solution in the class L_p of (1.1) with $f \equiv f_x$. We then have the estimate

$$\|\theta y_x\|_p \geqslant c^{-1} h(x) d(x) \left[\int_{\Delta(x)} \theta(t)^p \,\mathrm{d}t\right]^{1/p}. \tag{4.5}$$

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Proof. Let $t \in \Delta(x)$. Below, we use Theorem 2.14, (2.8), (2.13), (2.14) and (2.15):

$$\begin{aligned} y_x(t) &= \int_{-\infty}^{\infty} G(t,\xi) f_x(\xi)\,\mathrm{d}\xi \\ &= \int_{\Delta(x)} G(t,\xi)\,\mathrm{d}\xi \\ &= \int_{\Delta(x)} \sqrt{\rho(t)\rho(\xi)} \exp\left(-\frac{1}{2}\left|\int_t^{\xi} \frac{\mathrm{d}s}{r(s)\rho(s)}\right|\right)\mathrm{d}\xi \\ &\geqslant c^{-1}\rho(x)\int_{\Delta(x)} \exp\left(-\int_{\Delta(x)} \frac{\mathrm{d}s}{r(s)h(s)}\right)\mathrm{d}\xi \\ &\geqslant c^{-1}h(x)d(x). \end{aligned}$$

This inequality implies (4.5):

$$\|\theta y_x\|_p^p \geqslant \int_{\Delta(x)} |\theta(r) y_x(\xi)|^p \,\mathrm{d}\xi \geqslant c^{-1}(h(x)d(x))^p \int_{\Delta(x)} \theta(\xi)^p\,\mathrm{d}\xi.$$

□

By (4.5), we now obtain that

$$\begin{aligned} \infty > \|\theta\mathcal{L}_p^{-1}\|_{p\to p}^p &= \sup_{f\in L_p} \frac{\|\theta\mathcal{L}_p^{-1}f\|_p^p}{\|f\|_p^p} \geqslant \sup_{x\in\mathbb{R}} \frac{\|\theta\mathcal{L}_p^{-1}f_x\|_p^p}{\|f_x\|_p^p} \\ &= \sup_{x\in\mathbb{R}} \frac{\|\theta y_x\|_p^p}{2d(x)} \geqslant c^{-1}\sup_{x\in\mathbb{R}} h(x)^p d(x)^{p-1}\int_{\Delta(x)} \theta^p(t)\,\mathrm{d}t \quad \Rightarrow \quad m_p(r,q,\theta) < \infty. \end{aligned}$$

□

Proof of Theorem 3.4. Below, we study the equation

$$-(r(x)y'(x))' + (q(x)+\lambda)y(x) = f(x), \quad x \in \mathbb{R}, \tag{4.6}$$

where $\lambda \geqslant 0$ and the functions r and q satisfy conditions (1.2) and (1.3). Note that, by the assumption of the theorem, (4.6) is correctly solvable in L_p, $p \in (1,\infty)$, for $\lambda = 0$, and for $\lambda > 0$ this equation is correctly solvable in L_p, $p \in [1,\infty)$, by Theorems 2.14 and 2.16. This implies that the semi-axis $[0,\infty)$ is the resolvent set for the operator $\mathcal{L}_p$ (see (2.17) and (2.18)). Define $R_\lambda = (\mathcal{L}_p + \lambda E)^{-1}$, where $\lambda \geqslant 0$, $E\colon L_p \to L_p$ is the identity operator. We now apply Hilbert's formula

$$R_\mu - R_\lambda = (\lambda - \mu) R_\lambda \cdot R_\mu$$

to our case for $\mu = 0$ and $\lambda > 0$ to obtain that

$$R_0 = R_\lambda + \lambda R_\lambda \cdot R_0 = R_\lambda(E + \lambda R_0), \quad \lambda > 0,$$

or

$$\mathcal{L}_p^{-1} = (\mathcal{L}_p + \lambda E)^{-1}[E + \lambda\mathcal{L}_p^{-1}], \quad \lambda > 0$$

(which is the same). Since $\|\mathcal{L}_p^{-1}\|_{p\to p} \leqslant cB < \infty$ (see Theorem 2.15), this implies that

$$\|\theta\mathcal{L}_p^{-1}\|_{p\to p} \leqslant \|\theta(\mathcal{L}_p+\lambda E)^{-1}\|_{p\to p}(1+c\lambda B). \tag{4.7}$$

Thus, we are reduced to proving (under the assumptions of the theorem) the inequality

$$\|\theta(\mathcal{L}_p+\lambda E)^{-1}\|_{p\to p} \leqslant c(\lambda)m_p(r,q,\theta). \tag{4.8}$$

(In (4.8), the parameter λ is chosen according to the assumption of the theorem such that the inequality $\gamma(b) \leqslant \gamma_0 = \exp(-\frac{1}{500})$ holds for at least one $b > 0$ for some $a \geqslant 1$, $\lambda \geqslant 0$; see Definition 3.1.)

In connection with (4.8), consider the operator $(\mathcal{L}_p+\lambda E)^{-1}$ in more detail. Our notation is as follows. Throughout the following an FSS of the equation

$$(r(x)z'(x))' = (q(x)+\lambda)z(x) := q_\lambda(x)z(x), \quad x\in\mathbb{R},\ \lambda \geqslant 0,$$

with the properties (2.2)–(2.4) is denoted by $\{u_\lambda, v_\lambda\}$, $\rho_\lambda := u_\lambda\cdot v_\lambda$, and, finally, the Green function $G(x,t,\lambda)$ (see (2.7)) of (4.6) is denoted by $G_\lambda(t,x)(t,x\in\mathbb{R})$. Let $f\in L_p$, $p\in(1,\infty)$. By Theorem 2.14 and (2.7), we then have that

$$\begin{aligned}
[(\mathcal{L}_p+\lambda E)^{-1}f](x) &= (G_\lambda f)(x)\\
&= \int_{-\infty}^{\infty} G_\lambda(x,t)f(t)\,\mathrm{d}t\\
&= u_\lambda(x)\int_{-\infty}^{x} v_\lambda(t)f(t)\,\mathrm{d}t + v_\lambda(x)\int_x^{\infty} u_\lambda(t)f(t)\,\mathrm{d}t\\
&:= (G_{1,\lambda}f)(x) + (G_{2,\lambda}f)(x), \quad x\in\mathbb{R}.
\end{aligned}$$

Here

$$(G_{1,\lambda}f)(x) = u_\lambda(x)\int_{-\infty}^{x} v_\lambda(t)f(t)\,\mathrm{d}t, \quad x\in\mathbb{R}, \tag{4.9}$$

$$(G_{2,\lambda}f)(x) = v_\lambda(x)\int_{x}^{\infty} u_\lambda(t)f(t)\,\mathrm{d}t, \quad x\in\mathbb{R}, \tag{4.10}$$

$$\begin{aligned}
\|\theta(\mathcal{L}+\lambda E)^{-1}\|_{p\to p} &= \|\theta G_\lambda\|_{p\to p}\\
&= \|\theta(G_{1,\lambda}+G_{2,\lambda})\|_{p\to p}\\
&\leqslant \|\theta G_{1,\lambda}\|_{p\to p} + \|\theta G_{2,\lambda}\|_{p\to p}.
\end{aligned} \tag{4.11}$$

To extend (4.11), we use Lemma 4.6, which is a straightforward consequence of (4.9) and (4.10), Lemma 2.1, and Theorems 2.18 and 2.19. We first introduce some more notation. Let $f_1(x)$ and $f_2(x)$ be positive continuous functions defined for $x\in(a,b)$, $-\infty \leqslant a < b \leqslant \infty$. If there exists a constant $c\in[1,\infty)$ such that

$$c^{-1}f_1(x) \leqslant f_2(x) \leqslant cf_1(x) \quad \forall x\in(a,b),$$

then we write $f_1(x) \asymp f_2(x)$, $x\in(a,b)$.

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Lemma 4.5. *We have that*

$$\|\theta G_{1,\lambda}\|_{p\to p} \asymp \sup_{x\in\mathbb{R}} \mu_\lambda^{[p]}(x), \quad \lambda \geqslant 0, \tag{4.12}$$

$$\|\theta G_{2,\lambda}\|_{p\to p} \asymp \sup_{x\in\mathbb{R}} \nu_\lambda^{[p]}(x), \quad \lambda \geqslant 0, \tag{4.13}$$

where

$$\mu_\lambda^{[p]}(x) = \left[\int_{-\infty}^{x} v_\lambda^{p'}(t)\,\mathrm{d}t\right]^{1/p'} \left[\int_x^\infty (\theta(t)u_\lambda(t))^p\,\mathrm{d}t\right]^{1/p}, \quad x \in \mathbb{R}, \tag{4.14}$$

and

$$\nu_\lambda^{[p]}(x) = \left[\int_{-\infty}^{x} (\theta(t)v_\lambda(t))^p\,\mathrm{d}t\right]^{1/p} \left[\int_x^\infty u_\lambda(t)^{p'}\,\mathrm{d}t\right]^{1/p'}, \quad x \in \mathbb{R}. \tag{4.15}$$

Lemma 4.6. *Define*

$$J(x,t) = \exp\left(-\left|\int_x^t \frac{\mathrm{d}\xi}{r(\xi)\rho_\lambda(\xi)}\right|\right), \quad x,t \in \mathbb{R}.$$

For $x \in \mathbb{R}$ and $\lambda \geqslant 0$ we then have the inequalities

$$\mu_\lambda^{[p]}(x) \leqslant \begin{cases} \left[\int_{-\infty}^{x} \rho_\lambda(t)J(x,t)\,\mathrm{d}t\right]^{1/p'} \left[\int_x^\infty \rho_\lambda(t)\theta^p(t)J(x,t)^{p-1}\,\mathrm{d}t\right]^{1/p} & \text{if } p \in (1,2], \\ \left[\int_{-\infty}^{x} \rho_\lambda(t)J(x,t)^{p'-1}\,\mathrm{d}t\right]^{1/p'} \left[\int_x^\infty \rho_\lambda(t)\theta(t)^p J(x,t)\,\mathrm{d}t\right]^{1/p} & \text{if } p \in (2,\infty), \end{cases} \tag{4.16}$$

$$\nu_\lambda^{[p]}(x) \leqslant \begin{cases} \left[\int_{-\infty}^{x} \rho_\lambda(t)\theta(t)^p J(x,t)^{p-1}\,\mathrm{d}t\right]^{1/p} \left[\int_x^\infty \rho_\lambda(t)J(x,t)\,\mathrm{d}t\right]^{1/p'} & \text{if } p \in (1,2], \\ \left[\int_{-\infty}^{x} \rho_\lambda(t)J(x,t)\theta(t)^p\,\mathrm{d}t\right]^{1/p} \left[\int_x^\infty \rho_\lambda(t)J(x,t)^{p'-1}\,\mathrm{d}t\right]^{1/p'} & \text{if } p \in (2,\infty). \end{cases} \tag{4.17}$$

Proof. Let $p \in (1,2]$ and $\varkappa = (p'-p)(p'+p)^{-1}$ (where $\varkappa$ is an element of $[0,1)$). We now use (2.2), (2.5) and (2.6) to give that

$$\begin{aligned}
\mu_\lambda^{[p]}(x) &\leqslant \left[\int_{-\infty}^{x} v_\lambda(t)^{(1-\varkappa)p'}\,\mathrm{d}t\right]^{1/p'} \left[\int_x^\infty \theta^p(t)v_\lambda^{\varkappa p}(t)u_\lambda^p(t)\,\mathrm{d}t\right]^{1/p} \\
&= \left[\int_{-\infty}^{x} \rho_\lambda(t)J(x,t)\,\mathrm{d}t\right]^{1/p'} \exp\left(\frac{1}{p'}\int_{x_0}^{x} \frac{\mathrm{d}\xi}{r(\xi)\rho_\lambda(\xi)}\right) \\
&\quad \times \left[\int_x^\infty \theta^p(t)\rho_\lambda(t)J(x,t)^{p-1}\,\mathrm{d}t\right]^{1/p} \exp\left(-\frac{1}{p'}\int_{x_0}^{x} \frac{\mathrm{d}\xi}{r(\xi)\rho_\lambda(\xi)}\right) \\
&= \left[\int_{-\infty}^{x} \rho_\lambda(t)J(x,t)\,\mathrm{d}t\right]^{1/p'} \left[\int_x^\infty \theta^p(t)\rho_\lambda(t)J(x,t)^{p-1}\,\mathrm{d}t\right]^{1/p} \quad \Rightarrow \quad (4.16).
\end{aligned}$$

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Similarly, for $p \in (2,\infty)$ and $\varkappa = (p-p')(p+p')^{-1}$ (where $\varkappa$ is an element of $(0,1)$), we have that

$$\begin{aligned}
\mu_\lambda^{[p]}(x) &\leqslant \bigg[\int_{-\infty}^x v_\lambda(t)^{p'} u_\lambda^{\varkappa p'}(t)\,\mathrm{d}t\bigg]^{1/p'} \bigg[\int_x^\infty \theta(t)^p \mu_\lambda^{(1-\varkappa)p}(t)\,\mathrm{d}t\bigg]^{1/p} \\
&= \bigg[\int_{-\infty}^x \rho_\lambda(t) J(x,t)^{p'-1}\,\mathrm{d}t\bigg]^{1/p'} \exp\bigg(\frac{1}{p}\int_{x_0}^x \frac{\mathrm{d}\xi}{r(\xi)\rho_\lambda(\xi)}\bigg) \\
&\quad\times \bigg[\int_x^\infty \theta(t)^p \rho_\lambda(t) J(x,t)\,\mathrm{d}t\bigg]^{1/p} \exp\bigg(-\frac{1}{p}\int_{x_0}^x \frac{\mathrm{d}\xi}{r(\xi)\rho_\lambda(\xi)}\bigg) \\
&= \bigg[\int_{-\infty}^x \rho_\lambda(t) J(x,t)^{p'-1}\,\mathrm{d}t\bigg]^{1/p'} \bigg[\int_x^\infty \theta(t)^p \rho_\lambda(t) J(x,t)\,\mathrm{d}t\bigg]^{1/p} \quad \Rightarrow \quad (4.16).
\end{aligned}$$

Inequality (4.17) is proved in a similar way. □

Below, we need the following lemma.

Lemma 4.7 (Chernyavskaya and Shuster [26]). *For given $x \in \mathbb{R}$ and $\lambda \geqslant 0$, we introduce the functions*

$$\begin{aligned}
F_1^{[\lambda]}(\eta) &= \int_{x-\eta}^x \frac{\mathrm{d}t}{r(t)} \int_{x-\eta}^x (q(t)+\lambda)\,\mathrm{d}t, \\
F_2^{[\lambda]}(\eta) &= \int_x^{x+\eta} \frac{\mathrm{d}t}{r(t)} \int_x^{x+\eta} (q(t)+\lambda)\,\mathrm{d}t, \\
F_3^{[\lambda]}(\eta) &= \int_{x-\eta}^{x+\eta} \frac{\mathrm{d}t}{r(t)h_\lambda(t)}.
\end{aligned}$$

The following assertions then hold.

(1) *The inequality $\eta \geqslant d_\lambda^{(-)}(x)$ $(0 \leqslant \eta \leqslant d_\lambda^{(-)}(x))$ holds if and only if $F_1^{[\lambda]}(\eta) \geqslant 1$ $(F_1^{[\lambda]}(\eta) \leqslant 1)$.*

(2) *The inequality $\eta \geqslant d_\lambda^{(+)}(x)$ $(0 \leqslant \eta \leqslant d_\lambda^{(+)}(x))$ holds if and only if $F_2^{[\lambda]}(\eta) \geqslant 1$ $(F_2^{[\lambda]}(\eta) \leqslant 1)$.*

(3) *The inequality $\eta \geqslant d_\lambda(x)$ $(0 \leqslant \eta \leqslant d_\lambda(x))$ holds if and only if $F_3^{[\lambda]}(\lambda) \geqslant 1$ $(F_3^{[\lambda]}(\eta) \leqslant 1)$.*

Lemma 4.8. *We have the inequality (see (3.6))*

$$m_p(r,q_\lambda,\theta) \leqslant m_p(r,q,\theta), \quad \lambda \geqslant 0. \tag{4.18}$$

Proof. The following relations are obvious (see Lemma 2.5):

$$\int_{x-d^{(-)}(x)}^x \frac{\mathrm{d}t}{r(t)} \int_{x-d^{(-)}(x)}^x (q(t)+\lambda)\,\mathrm{d}t \geqslant \int_{x-d^{(-)}(x)}^x \frac{\mathrm{d}t}{r(t)} \int_{x-d^{(-)}(x)}^x q(t)\,\mathrm{d}t = 1.$$

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Hence (see Lemma 4.7), $d_\lambda^{(-)}(x) \leqslant d^{(-)}(x)$, $x \in \mathbb{R}$, and, similarly, $d_\lambda^{(+)}(x) \leqslant d^{(+)}(x)$, $x \in \mathbb{R}$. Therefore, $\varphi_\lambda(x) \leqslant \varphi(x)$, $\psi_\lambda(x) \leqslant \psi(x)$ for $x \in \mathbb{R}$, since, say,

$$\psi_\lambda(x) = \int_x^{x+d_\lambda^{(+)}(x)} \frac{\mathrm{d}t}{r(t)} \leqslant \int_x^{x+d^{(+)}(x)} \frac{\mathrm{d}t}{r(t)} = \psi(x), \quad x \in \mathbb{R}.$$

Then, clearly, $h_\lambda(x) \leqslant h(x)$ for $x \in \mathbb{R}$, since

$$\frac{1}{h_\lambda(x)} = \frac{1}{\varphi_\lambda(x)} + \frac{1}{\psi_\lambda(x)} \geqslant \frac{1}{\varphi(x)} + \frac{1}{\psi(x)} = \frac{1}{h(x)}, \quad x \in \mathbb{R}.$$

This implies (see Lemma 4.7) that $d_\lambda(x) \leqslant d(x)$, $x \in \mathbb{R}$, since

$$1 = \int_{x-d(x)}^{x+d(x)} \frac{\mathrm{d}t}{r(t)h(t)} \leqslant \int_{x-d(x)}^{x+d(x)} \frac{\mathrm{d}t}{r(t)h_\lambda(t)}, \quad x \in \mathbb{R}.$$

The obtained estimates imply that

$$\begin{aligned} m_p(r, q_\lambda, \theta) &= h_\lambda(x) d_\lambda(x)^{1/p'} \left[\int_{x-d_\lambda(x)}^{x+d_\lambda(x)} \theta(t)^p \, \mathrm{d}t \right]^{1/p} \\ &\leqslant h(x) d(x)^{1/p} \left[\int_{x-d(x)}^{x+d(x)} \theta(t)^p \, \mathrm{d}t \right]^{1/p} \\ &= m_p(r, q, \theta). \end{aligned}$$

□

Lemma 4.9. *Let a pair $\{r, q\}$ be such that, for some $\lambda \geqslant 0$ and $\alpha \in (0,1)$, and for all $x, t \in \mathbb{R}$, the following inequalities hold:*

$$c^{-1} \exp\left(-\alpha \left| \int_x^t \frac{\mathrm{d}\xi}{\rho_\lambda(\xi) r(\xi)} \right| \right) \leqslant \frac{\rho_\lambda(t) d_\lambda(t)}{\rho_\lambda(x) d_\lambda(x)} \leqslant c \exp\left(\alpha \left| \int_x^t \frac{\mathrm{d}\xi}{r(\xi)\rho_\lambda(\xi)} \right| \right). \tag{4.19}$$

Then, for $p \in (1, \infty)$ we have the estimates

$$\|\theta G_{1,\lambda}\|_{p \to p} \leqslant c(\lambda, p) m_p(r, q, \theta), \tag{4.20}$$

$$\|\theta G_{2,\lambda}\|_{p \to p} \leqslant c(\lambda, p) m_p(r, q, \theta). \tag{4.21}$$

Proof. Below, we check (4.21) for $p \in (1, 2]$. Inequality (4.21) for $p \in (2, \infty)$ and the estimate (4.20) are established in a similar way. Let $x \in \mathbb{R}$, $\lambda \geqslant 0$.

Define (see (4.17))

$$H_\lambda(x) = \int_x^\infty \rho_\lambda(t) \exp\left(-\int_x^t \frac{\mathrm{d}\xi}{r(\xi)\rho_\lambda(\xi)}\right) \mathrm{d}t, \tag{4.22}$$

$$S_p^{[\lambda]}(x) = \int_{-\infty}^x \theta(t)^p \rho_\lambda(t) \exp\left(-(p-1)\int_t^x \frac{\mathrm{d}\xi}{r(\xi)\rho_\lambda(\xi)}\right) \mathrm{d}t. \tag{4.23}$$

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Our next goal is to get uniform estimates of $H_\lambda(x)$ and $S_p^{[\lambda]}(x)$ with respect to $x \in \mathbb{R}$. We start with $H_\lambda(x)$. The next chain of computations is based on properties of an $\mathbb{R}(x, d_\lambda)$-covering of $[x, \infty)$ (see Lemma 2.13) and inequality (2.15):

$$\begin{aligned} H_\lambda(x) &= \sum_{n=1}^{\infty} \int_{\Delta_n} \rho_\lambda(t) \exp\left(-\int_x^t \frac{\mathrm{d}\xi}{r(\xi)\rho_\lambda(\xi)}\right) \mathrm{d}t \\ &\leqslant c \sum_{n=1}^{\infty} \rho_\lambda(x_n) d_\lambda(x_n) \exp\left(-\int_{\Delta_1^-}^{\Delta_n^-} \frac{\mathrm{d}\xi}{r(\xi)\rho_\lambda(\xi)}\right). \end{aligned} \tag{4.24}$$

From the upper estimate in (4.19), Definition 2.10 and (2.13) and (2.14), we obtain, for every $n \geqslant 1$, that

$$\begin{aligned} \rho_\lambda(x_n) d_\lambda(x_n) &\leqslant c\rho_\lambda(x) d_\lambda(x) \exp\left(\alpha \int_x^{x_n} \frac{\mathrm{d}\xi}{r(\xi)\rho_\lambda(\xi)}\right) \\ &= c\rho_\lambda(x) d_\lambda(x) \exp\left(\alpha \int_{\Delta_1^-}^{\Delta_n^-} \frac{\mathrm{d}\xi}{r(\xi)\rho_\lambda(\xi)}\right) \exp\left(\alpha \int_{\Delta_n^-}^{x_n} \frac{\mathrm{d}\xi}{r(\xi)\rho_\lambda(\xi)}\right) \\ &\leqslant c\rho_\lambda(x) d_\lambda(x) \exp\left(\alpha \int_{\Delta_1^-}^{\Delta_n^-} \frac{\mathrm{d}\xi}{r(\xi)\rho_\lambda(\xi)}\right) \exp\left(2\alpha \int_{\Delta_n} \frac{\mathrm{d}\xi}{r(\xi)h_\lambda(\xi)}\right) \\ &= c\rho_\lambda(x) d_\lambda(x) \exp\left(\alpha \int_{\Delta_1^-}^{\Delta_n^-} \frac{\mathrm{d}\xi}{r(\xi)\rho_\lambda(\xi)}\right). \end{aligned} \tag{4.25}$$

Note that Definition 2.10 and (2.14) imply the inequalities

$$\int_{\Delta_1^-}^{\Delta_n^-} \frac{\mathrm{d}\xi}{r(\xi)h_\lambda(\xi)} = \sum_{k=1}^{n-1} \int_{\Delta_k} \frac{\mathrm{d}\xi}{r(\xi)h_\lambda(\xi)} = n-1 \quad \text{if } n \geqslant 2. \tag{4.26}$$

We now extend (4.24), taking into account (4.24), (4.25) and (2.13), as

$$\begin{aligned} H_\lambda(x) &\leqslant c\rho_\lambda(x) d_\lambda(x) \sum_{n=1}^{\infty} \exp\left(-(1-\alpha) \int_{\Delta_1^-}^{\Delta_n^-} \frac{\mathrm{d}\xi}{r(\xi)\rho_\lambda(\xi)}\right) \\ &\leqslant c\rho_\lambda(x) d_\lambda(x) \sum_{n=1}^{\infty} \exp\left(-\frac{1-\alpha}{2} \int_{\Delta_1^-}^{\Delta_n^-} \frac{\mathrm{d}\xi}{r(\xi)h_\lambda(\xi)}\right) \\ &= c\rho_\lambda(x) d_\lambda(x) \sum_{n-1}^{\infty} \exp\left(-\frac{1-\alpha}{2}(n-1)\right) \\ &= c\rho_\lambda(x) d_\lambda(x). \end{aligned} \tag{4.27}$$

We now turn to $S_p^{[\lambda]}(x)$. Below, we estimate this integral using the same tools as in the proof of (4.27). Obvious differences are technical. Say, instead of the upper estimate in (4.19) we use the lower one, and instead of an $\mathbb{R}(x, d_\lambda)$-covering of $[x, \infty)$, we use an

$\mathbb{R}(x, d_\lambda)$-covering of $(-\infty, x]$. Therefore, we do not comment on the following computations:

$$\begin{aligned}
S_p^{[\lambda]}(x) &= \sum_{n=-\infty}^{-1} \int_{\Delta_n} \theta^p(t)\rho_\lambda(t) \exp\left(-(p-1)\int_t^x \frac{\mathrm{d}\xi}{r(\xi)\rho_\lambda(\xi)}\right)\mathrm{d}t \\
&\leqslant c \sum_{n=-\infty}^{-1} \rho_\lambda(x_n)\left[\int_{\Delta_n} \theta^p(t)\,\mathrm{d}t\right] \exp\left(-(p-1)\int_{\Delta_n^+}^{\Delta_{-1}^+} \frac{\mathrm{d}\xi}{r(\xi)\rho_\lambda(\xi)}\right) \\
&= c \sum_{n=-\infty}^{-1} \left[h_\lambda^p(x_n) d_\lambda^{p-1}(x_n)\int_{\Delta_n} \theta^p(t)\,\mathrm{d}t\right]\left(\frac{\rho_\lambda(x_n)}{h_\lambda(x_n)}\right)^p \\
&\qquad\qquad \times \left(\frac{1}{(\rho_\lambda(x_n)d_\lambda(x_n))^{p-1}} \exp\left(-(p-1)\int_{\Delta_n^+}^{\Delta_{-1}^+} \frac{\mathrm{d}\xi}{r(\xi)\rho_\lambda(\xi)}\right)\right) \\
&\leqslant cm^P(r, q_\lambda, \theta) \sum_{n=-\infty}^{-1} \frac{1}{(\rho_\lambda(x_n)d_\lambda(x_n))^{p-1}} \exp\left(-(p-1)\int_{\Delta_n^+}^{\Delta_{-1}^+} \frac{\mathrm{d}\xi}{r(\xi)\rho_\lambda(\xi)}\right) \\
&\leqslant c\frac{m_p^p(r, q_\lambda\theta)}{(\rho_\lambda(x)d_\lambda(x))^{p-1}} \sum_{n=-\infty}^{-1} \exp\left(-(1-\alpha)(p-1)\int_{\Delta_n^+}^{\Delta_{-1}^+} \frac{\mathrm{d}\xi}{r(\xi)\rho_\lambda(\xi)}\right) \\
&\leqslant c\frac{m_p^p(r, q_\lambda, \theta)}{(\rho_\lambda(x)d_\lambda(x))^{p-1}} \sum_{n=-\infty}^{-1} \exp\left(-\frac{1-\alpha}{2}(p-1)(|n|-1)\right) \\
&= c\frac{m_p^p(r, q_\lambda, \theta)}{(\rho_\lambda(x)d_\lambda(x))^{p-1}}.
\end{aligned} \tag{4.28}$$

Inequality (4.21) follows from Lemmas 4.6, 4.7, 4.9 and (4.27) and (4.28). □

Remark 4.10. In all the following lemmas, except for Lemma 4.12, we always assume that the pair $\{r, q\}$ has the exponent $\gamma(b)$ with internal parameters $a \geqslant 1$ and $\lambda \geqslant 0$ (see Definition 3.1). Therefore, for brevity, we formulate our statements by writing $\gamma(b) = \gamma(b, a, \lambda)$.

Lemma 4.11. *Suppose that the pair $\{r, q\}$ has the exponent $\gamma(b) = \gamma(b, a, \lambda) \leqslant \gamma_0$, $\gamma_0 = \exp(-\frac{1}{500})$ (see Theorem 3.4). Then, $b \geqslant 1$.*

Proof. Assume the contrary, that $b \in (0, 1)$. Then,

$$\exp\left(\frac{1}{500}\right) = \frac{1}{\gamma_0} \leqslant \frac{1}{\gamma(b)} \leqslant \frac{1}{a}\exp\left(\frac{b}{500a^2}\right) \leqslant \exp\left(\frac{b}{500}\right) < \exp\left(\frac{1}{500}\right),$$

which is a contradiction. Hence, $b \geqslant 1$. □

Lemma 4.12. *Suppose that the pair $\{r, q\}$ has the exponent $\gamma(b) = \gamma(b, a, \lambda) \leqslant \gamma_0$. Then, for all $x \in R$ the following inequalities hold:*

$$\frac{1}{4\mathrm{e}^2 a} \leqslant \frac{d_\lambda(t)}{d_\lambda(x)} \leqslant 4\mathrm{e}^2 a \quad \text{if } |t-x| \leqslant d_\lambda(x). \tag{4.29}$$

Proof. Since $b \geqslant 1$ by Lemma 4.11, for $t \in [x-d_\lambda(x), x+d_\lambda(x)]$ from (3.3) and (2.15), we obtain the relations

$$\frac{1}{4\mathrm{e}^2 a} \leqslant \frac{1}{a}\frac{h_\lambda(x)}{h_\lambda(t)} \leqslant \frac{d_\lambda(t)}{d_\lambda(x)} \leqslant a\frac{h_\lambda(x)}{h_\lambda(t)} \leqslant 4\mathrm{e}^2 a.$$

□

Lemma 4.13. *Suppose that the pair $\{r, q\}$ has the exponent $\gamma(b) = \gamma(b, a, \lambda) \leqslant \gamma_0$. Then, for all $t, x \in R$ the following inequality holds:*

$$\left|\int_x^t \frac{\mathrm{d}\xi}{r(\xi)\rho_\lambda(\xi)}\right| \geqslant \frac{1}{16\mathrm{e}^2 a}\left|\int_x^t \frac{\mathrm{d}\xi}{d_\lambda(\xi)}\right| - \frac{1}{2}. \tag{4.30}$$

Proof. Let $t \geqslant x$ (the case $t \leqslant x$ is treated in a similar way), let $\{\Delta_n\}_{n=1}^\infty$ be an $\mathbb{R}(x, d_\lambda)$-covering of $[x, \infty)$, and let $t \in \Delta_n$. Together with (4.29), this leads to the relations

$$\int_{\Delta_n^-}^t \frac{\mathrm{d}\xi}{r(\xi)\rho_\lambda(\xi)} \geqslant 0,$$
$$\int_{\Delta_n^-}^t \frac{\mathrm{d}\xi}{d_\lambda(\xi)} \leqslant \int_{\Delta_n} \frac{\mathrm{d}\xi}{d_\lambda(\xi)} = \int_{\Delta_n} \frac{d_\lambda(x_n)}{d(\xi)}\frac{\mathrm{d}\xi}{d_\lambda(x_n)} \leqslant \int_{\Delta_n} 4\mathrm{e}^2 a\frac{\mathrm{d}\xi}{d_\lambda(x_n)} = 8\mathrm{e}^2 a.$$

These relations imply the following obvious estimates:

$$\int_{\Delta_n^-}^t \frac{\mathrm{d}\xi}{r(\xi)\rho_\lambda(\xi)} \geqslant \frac{1}{16\ell^2 a}\int_{\Delta_n^-}^t \frac{\mathrm{d}\xi}{d_\lambda(\xi)} - \frac{1}{2}, \quad t \in \Delta_n. \tag{4.31}$$

In particular, for $n = 1$ from the inequality $\Delta_1^- = x$, we see that (4.31) coincides with (4.30). Now Let $t \in \Delta_n$, $n \geqslant 2$.

Below we use Definition 2.10, Lemmas 2.13, 2.5, 2.8, 4.11 and (4.31) to obtain that

$$\begin{aligned}
\int_x^t \frac{\mathrm{d}\xi}{r(\xi)\rho_\lambda(\xi)} &= \sum_{k=1}^{n-1}\int_{\Delta_n} \frac{\mathrm{d}\xi}{r(\xi)\rho_\lambda(\xi)} + \int_{\Delta_n^-}^t \frac{\mathrm{d}\xi}{r(\xi)\rho_\lambda(\xi)} \\
&\geqslant \frac{1}{2}\sum_{k=1}^{n-1}\int_{\Delta_k} \frac{\mathrm{d}\xi}{r(\xi)h_\lambda(\xi)} + \int_{\Delta_n^-}^t \frac{\mathrm{d}\xi}{r(\xi)\rho_\lambda(\xi)} \\
&= \frac{1}{2}\sum_{k=1}^{n-1} 1 + \int_{\Delta_n^-}^t \frac{\mathrm{d}\xi}{r(\xi)\rho_\lambda(\xi)} \\
&= \frac{1}{2}\sum_{k=1}^{n-1}\frac{1}{2}\int_{\Delta_k} \frac{d_\lambda(\xi)}{d_\lambda(x_k)}\frac{\mathrm{d}\xi}{d_\lambda(\xi)} + \int_{\Delta_{n^-}}^t \frac{\mathrm{d}\xi}{r(\xi)\rho_\lambda(\xi)}
\end{aligned}$$

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$$\geqslant \frac{1}{4}\sum_{k=1}^{n-1}\frac{1}{4\mathrm{e}^2 a}\int_{\Delta_k}\frac{\mathrm{d}\xi}{d_\lambda(\xi)}+\frac{1}{16\mathrm{e}^2 a}\int_{\Delta_n^-}^{t}\frac{\mathrm{d}\xi}{d_\lambda(\xi)}-\frac{1}{2}$$
$$=\frac{1}{16\mathrm{e}^2 a}\bigg\{\sum_{k=1}^{n-1}\int_{\Delta_k}\frac{\mathrm{d}\xi}{d_\lambda(\xi)}+\int_{\Delta_n^-}^{t}\frac{\mathrm{d}\xi}{d_\lambda(\xi)}\bigg\}-\frac{1}{2}$$
$$=\frac{1}{16\mathrm{e}^2 a}\int_{x}^{t}\frac{\mathrm{d}\xi}{d_\lambda(\xi)}-\frac{1}{2}.$$

□

Lemma 4.14. *Suppose that the pair $\{r, q\}$ has the exponent $\gamma(b) = \gamma(b, a, \lambda) \geqslant \gamma_0$, and let $x \in \mathbb{R}$. There then exist $\mathbb{R}(x, bd_\lambda)$-coverings of $(-\infty, x]$ and $[x, \infty)$.*

Proof. This follows from Lemma 2.8, condition (3.5) and Lemma 2.11.

Lemma 4.15. *Suppose that the pair $\{r, q\}$ has the exponent $\gamma(b) = \gamma(b, a, \lambda) \leqslant \gamma_0$, and let $x \in \mathbb{R}$. Denote by $\{\Delta_n\}_{n=-\infty}^{-1}$ and $\{\Delta_n\}_{n=1}^{\infty}$ the $\mathbb{R}(x, bd_\lambda)$-coverings of $(-\infty, x]$ and $[x, \infty)$, respectively. Then, if $t \in \Delta_n$, $|n| \geqslant 1$, we have the inequalities*

$$\frac{1}{a^{2|n|}} \leqslant \frac{h_\lambda(t)d_\lambda(t)}{h_\lambda(x)d_\lambda(x)} \leqslant a^{2|n|}, \quad |n| \geqslant 1, \tag{4.32}$$

$$\bigg|\int_x^t \frac{\mathrm{d}\xi}{d_\lambda(\xi)}\bigg| \geqslant \frac{b}{a}(|n|-1), \quad |n| \geqslant 1. \tag{4.33}$$

Proof. Let $t \in \Delta_n$, $n \geqslant 1$ (the case $n \leqslant -1$ is treated in a similar way). Then, by (3.3) we obtain the inequalities

$$\left.\begin{aligned}\frac{1}{a} \leqslant \frac{h_\lambda(t)d_\lambda(t)}{h_\lambda(x_n)d_\lambda(x_n)} \leqslant a,\\ \frac{1}{a} \leqslant \frac{h_\lambda(x_n)d_\lambda(x_n)}{h_\lambda(\Delta_n^-)d_\lambda(\Delta_n^-)} \leqslant a\end{aligned}\right\} \quad \Rightarrow \quad \frac{1}{a^2} \leqslant \frac{h_\lambda(t)d_\lambda(t)}{h_\lambda(\Delta_n^-)d_\lambda(\Delta_n^-)} \leqslant a^2 \quad \text{for } t \in \Delta_n,\ n \geqslant 1. \tag{4.34}$$

□

In particular, for $n = 1$, from (4.34) and the equality $\Delta_1^- = x$, we obtain (4.32). Now let $t \in \Delta_n$, $n \geqslant 2$, and $k = \overline{1, n-1}$. Once again, we use (3.3) and obtain that

$$\left.\begin{aligned}\frac{1}{a} \leqslant \frac{h_\lambda(\Delta_k^+)d_\lambda(\Delta_k^+)}{h_\lambda(x_k)d_\lambda(x_k)} \leqslant a,\\ \frac{1}{a} \leqslant \frac{h_\lambda(x_k)d_\lambda(x_k)}{h_\lambda(\Delta_k^-)d_\lambda(\Delta_k^-)} \leqslant a\end{aligned}\right\} \quad \Rightarrow \quad \frac{1}{a^2} \leqslant \frac{h_\lambda(\Delta_k^+)d_\lambda(\Delta_k^+)}{h_\lambda(\Delta_k^-)d_\lambda(\Delta_k^-)} \leqslant a^2 \quad \text{for } k = \overline{1, n-1}. \tag{4.35}$$

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By (4.34), (4.35) and Definition 2.10, we now have that

$$\begin{aligned}\frac{h_\lambda(t)d_\lambda(t)}{h_\lambda(x)d_\lambda(x)} &= \frac{h_\lambda(t)d_\lambda(t)}{h_\lambda(\Delta_1^-)d_\lambda(\Delta_1^-)}\\ &= \left[\prod_{k=1}^{n-1}\frac{h_\lambda(\Delta_k^+)d_\lambda(\Delta_k^+)}{h_\lambda(\Delta_k^-)d_\lambda(\Delta_k^-)}\right]\frac{h_\lambda(t)d_\lambda(t)}{h_\lambda(\Delta_n^-)d_\lambda(\Delta_n^-)}\\ &\leqslant \left[\prod_{k=1}^{n-1}a^{\pm 2}\right]\cdot a^{\pm 2} = a^{\pm 2|n|} \quad\Rightarrow\quad (4.32).\end{aligned}$$

We now turn to (4.33). Let $t \geqslant x$ (the case $t \leqslant x$ is treated in a similar way). For $n = 1$, the estimate (4.33) is obvious. For $n \geqslant 2$, using Definition 2.10 and (3.4), we obtain that

$$\int_x^t \frac{\mathrm{d}\xi}{d_\lambda(\xi)} = \sum_{k=1}^{n-1}\int_{\Delta_k}\frac{\mathrm{d}\xi}{d_\lambda(\xi)} + \int_{\Delta_n^-}^t \frac{\mathrm{d}\xi}{d_\lambda(\xi)} \geqslant \sum_{k=1}^{n-1}\frac{b}{a} = \frac{b}{a}(n-1).$$

□

We finally turn to the proof of the theorem. To this end, we show that if the pair $\{r, q\}$ has the exponent $\gamma(b) = \gamma(b, a, \lambda) \leqslant \gamma_0 = \exp(-\frac{1}{500})$, then, for the given λ and $\alpha = \frac{1}{2}$, inequalities (4.19) hold. First note that, under our assumptions, we have the relations

$$\begin{aligned}&a\mathrm{e}^{-(1/64\mathrm{e}^2)(b/a^2)} < \mathrm{e}^{-(1/500)(b/a^2)} < \mathrm{e}^{-1/500} < 1\\ &\qquad\Rightarrow\quad a^{2(n-1)} \leqslant \mathrm{e}^{(1/32\mathrm{e}^2)(b/a^2)(n-1)} \quad \text{for } n \geqslant 1\\ &\qquad\Rightarrow\quad 4a^{2n} \leqslant c_0\mathrm{e}^{(1/32)(b/a^2)(n-1)-1/4}, \quad c_0 = 4a^2\mathrm{e}^{-1/4},\ n \geqslant 1.\end{aligned} \tag{4.36}$$

Let $t, x \in \mathbb{R}$ be given, and let $t \geqslant x$. There then exists a segment Δ_n, $n \geqslant 1$, from an $\mathbb{R}(x, bd_\lambda)$-covering of $[x, \infty)$ such that $t \in \Delta_n$. By (4.32) and (2.13), we have that

$$\frac{\rho_\lambda(t)d_\lambda(t)}{\rho_\lambda(x)d_\lambda(x)} \leqslant 4\frac{h_\lambda(t)d_\lambda(t)}{h_\lambda(x)d_\lambda(x)} \leqslant 4a^{2n}, \quad n \geqslant 1. \tag{4.37}$$

On the other hand, from (4.30) and (4.33), it follows that

$$\begin{aligned}\frac{1}{2}\int_x^t \frac{\mathrm{d}\xi}{r(\xi)\rho_\lambda(\xi)} &\geqslant \frac{1}{32\mathrm{e}^2}\frac{1}{a}\int_x^t\frac{\mathrm{d}\xi}{d_\lambda(\xi)} - \frac{1}{4}\\ &\geqslant \frac{1}{32\mathrm{e}^2}\frac{b}{a^2}(n-1) - \frac{1}{4}, \quad n \geqslant 1.\end{aligned} \tag{4.38}$$

For $t \geqslant x$, $x \in \mathbb{R}$, this implies, taking into account (4.36), (4.38) and (4.37), that

$$\begin{aligned}\frac{\rho_\lambda(t)d_\lambda(t)}{\rho_\lambda(x)d_\lambda(x)} &\leqslant 4a^{2n} \leqslant c_0\exp\left(\frac{1}{32}\frac{b}{a^2}(n-1) - \frac{1}{4}\right)\\ &\leqslant c_0\exp\left(\frac{1}{2}\int_x^t\frac{\mathrm{d}\xi}{r(\xi)\rho_\lambda(\xi)}\right) \quad\Rightarrow\quad (4.19).\end{aligned}$$

The lower estimate in (4.19) and the case $t \leqslant x$ are treated in a similar way, and inequalities (4.19) are proved. The theorem now follows from (4.20) and (4.21). □

References

1. T. T. Amanova, On the separability of a differential operator, *Izv. Akad. Nauk Kazakh. SSR Ser. Fiz.-Mat.* **3** (1981), 48–51.
2. F. V. Atkinson, On some results of Everitt and Giertz, *Proc. R. Soc. Edinb.* A **71** (1973), 151–158.
3. B. L. Baĭdel′dinov, On the problem of the solvability of the singular Sturm–Liouville equation in weighted L_p-class, *Izv. Akad. Nauk Kazakh. SSR Ser. Fiz.-Mat.* **5** (1991), 15–20.
4. A. Birgebaev, Separability of a differential operator in L_p, *Izv. Akad. Nauk Kazakh. SSR Ser. Fiz.-Mat.* **5** (1981), 1–5.
5. A. Birgebaev and M. Otelbaev, Separability of a third-order nonlinear differential operator, *Izv. Akad. Nauk Kazakh. SSR Ser. Fiz.-Mat.* **3** (1984), 11–13.
6. K. H. Boimatov, Separability theorems for the Sturm–Liouville operator, *Mat. Zametki* **14** (1973), 349–359.
7. K. H. Boimatov, Theorems on the separation property, *Dokl. Akad. Nauk SSSR* **213** (1973), 1009–1011.
8. K. H. Boimatov, The domain of definition of a Sturm–Liouville operator, *Diff. Uravn.* **12**(7) (1976), 1151–1160.
9. K. H. Boimatov, Separability theorems, weighted spaces and their application to boundary value problems, *Dokl. Akad. Nauk SSSR* **247**(3) (1979), 532–536.
10. K. K. Boimatov, Separation theorems, weighted spaces and their applications, in *Studies in the theory of differentiable functions of several variables and its applications, X*, Proceedings of the Steklov Institute of Mathematics, Volume 156, pp. 37–36 (American Mathematical Society, Providence, RI, 1984).
11. K. K. Boimatov, Coercive estimates and separability for second-order elliptic differential equations, *Dokl. Akad. Nauk SSSR* **301**(5) (1988), 1033–1036.
12. K. K. Boimatov, Coercive estimates and separability for second-order nonlinear elliptic differential operators, *Mat. Zametki* **46**(6) (1989), 110–112.
13. K. K. Boimatov and P. I. Lizorkin, Estimates for the growth of solutions of differential equations, *Diff. Uravn.* **25**(4) (1989), 578–588.
14. K. K. Boimatov and A. Sharifov, Coercive estimates and separability for differential operators of arbitrary order, *Russ. Math. Surv.* **44**(3) (1989), 147–148.
15. R. C. Brown, Separation and disconjugacy, *J. Ineq. Pure Appl. Math.* **4**(3) (2003), 56.
16. R. C. Brown and J. Cook, Continuous invertibility of minimal Sturm–Liouville operators in Lebesgue spaces, *Proc. R. Soc. Edinb.* A **136** (2006), 53–70.
17. R. C. Brown and D. B. Hinton, Two separation criteria for second order ordinary or partial differential operators, *Math. Bohem.* **124**(2) (1999), 273–292.
18. R. C. Brown, D. B. Hinton and M. F. Shaw, Some separation criteria and inequalities associated with linear second order differential operators, in *Function spaces and applications*, pp. 7–35 (Chemical Rubber Company, Boca Raton, FL, 2000).
19. A. Bulabaev and L. Shuster, Summability with a weight of solutions of the Sturm–Liouville equation in L_p, in *Functional analysis, differential equations and their applications*, Volume 11–16 (Kazakh Gos. Univ., Alma-Ata, 1987).
20. A. Bulabaev, M. Otelbaev and L. Shuster, Properties of Green's function of a Sturm–Liouville operator and their applications, *Diff. Eqns* **25**(7) (1989), 773–779.
21. N. Chernyavskaya, Conditions for correct solvability of a simplest singular boundary value boundary problem, *Math. Nachr.* **243** (2002), 5–18.
22. N. Chernyavskaya and L. Shuster, Estimates for the Green function of a general Sturm–Liouville operator and their applications, *Proc. Am. Math. Soc.* **127**(5) (1999), 1413–1426.

23. N. Chernyavskaya and L. Shuster, Weight summability of solutions of the Sturm–Liouville equation, *J. Diff. Eqns* **151**(2) (1999), 456–473.
24. N. Chernyavskaya and L. Shuster, Conditions for correct solvability of a simplest singular boundary value problem of general form, I, *Z. Analysis Anwend.* **7**(1) (2000), 65–84.
25. N. Chernyavskaya and L. Shuster, Regularity of the inverse problem for a Sturm–Liouville equation in $L_p(R)$, *Meth. Applic. Analysis* **25** (2006), 205–235.
26. N. Chernyavskaya and L. Shuster, A criterion for correct solvability in $L_p(R)$ of a general Sturm–Liouville equation, *J. Lond. Math. Soc.* **80**(2) (2009), 99–120.
27. N. Chernyavskaya and L. Shuster, A criterion for compactness in $L_p(R)$ of the resolvent of the maximal Sturm–Liouville operator of general form, preprint (arXiv:0912.0359, 2009).
28. N. Chernyavskaya and L. Shuster, Weight estimates for solutions of linear singular differential equations of the first order and the Everitt–Giertz problem, *Diff. Integ. Eqns* **25**(5) (2012), 467–504.
29. N. Chernyavskaya, N. El-Natanov and L. Shuster, Weighted estimates for solutions of a Sturm-Louiville equation in the space $L_1(R)$, in *Proceedings of the fifth international conference on mathematical modeling and computer simulation of material technologies, MMT-2008: Ariel University Center of Samaria, Ariel, Israel, September 8–12*, Volume 2, pp. 121–123 (Ariel University Center of Samaria, 2008).
30. N. Chernyavskaya, N. El-Natanov and L. Shuster, Weighted estimates for solutions of a Sturm–Louiville equation in the space $L_1(R)$, *Proc. R. Soc. Edinb.* A **141** (2011), 1175–1206.
31. R. Courant, *Partial differential equations* (Wiley, 1962).
32. E. B. Davies and E. M. Harrell, Conformally flat Riemannian metrices, Schrödinger operators and semi-classical approximation, *J. Diff. Eqns* **66**(2) (1987), 165–188.
33. D. S. Dzhumabaev and R. A. Medetbekova, Separability of a second-order linear differential operator, *Izv. Akad. Nauk Kazakh. SSR Ser. Fiz.-Mat.* **5** (1983), 21–26.
34. W. D. Evans and A. Zettl, Dirichlet and separation results for Schrödinger-type operators, *Proc. R. Soc. Edinb.* A **80** (1978), 151–162.
35. W. N. Everitt and M. Giertz, Some properties of the domains of certain differential operators, *Proc. Lond. Math. Soc.* **23**(3) (1971), 301–324.
36. W. N. Everitt and M. Giertz, Some inequalities associated with certain differential operators, *Math. Z.* **126** (1972), 308–326.
37. W. N. Everitt and M. Giertz, On some properties of the domains of the powers of formally self-adjoint differential expressions, *Proc. Lond. Math. Soc.* **24**(3) (1972), 756–768.
38. W. N. Everitt and M. Giertz, An example concerning the separation properties of differential operators, *Proc. R. Soc. Edinb.* A **71** (1972), 159–165.
39. W. N. Everitt and M. Giertz, On limit point and separation criteria for linear differential expressions, in *Proceedings of Equadiff III, 3rd Czechoslovak conference on differential equations and their applications, Brno, Czechoslovakia, August 28–September 1, 1972*, pp. 31–41 (Czechoslovak Academy of Sciences, 1972).
40. W. N. Everitt and M. Giertz, A Dirichlet type result for ordinary differential operators, *Math. Ann.* **203** (1973), 119–128.
41. W. N. Everitt and M. Giertz, Inequalities and separation for certain ordinary differential operators, *Proc. Lond. Math. Soc.* **28**(3) (1974), 352–372.
42. W. N. Everitt and M. Giertz, On certain ordinary differential expressions and assoicated integral inequalities, in *New developments in differential equations*, North-Holland Mathematics Studies, Volume 21, pp. 161–174 (North-Holland, Amsterdam, 1976).
43. W. N. Everitt and M. Giertz, Inequalities and separation for Schrödinger type operators in $L_2(R^n)$, *Proc. R. Soc. Edinb.* A **79**(3) (1978), 257–265.

44. W. N. Everitt, M. Giertz and J. Weidmann, Some remarks on a separation and limit-point criterion of second order ordinary differential expressions, *Math. Ann.* **200** (1973), 335–346.
45. E. Z. Grinshpun and M. Otelbaev, Smoothness of solutions of a nonlinear Sturm–Liouville equation in $L_1(-\infty,\infty)$, *Izv. Akad. Nauk Kazakh. SSR Ser. Fiz.-Mat.* **5** (1984), 26–29.
46. A. L. Izmaylov, Smoothness of solutions of differential equations and separability differential equations, PhD thesis, Alma-Ata (1978).
47. A. L. Izmaylov and M. Otelbaev, Summability with weight of the solution of a differential equation in an unbounded domain, *Izv. Akad. Nauk Kazakh. SSR Ser. Fiz.-Mat.* **1** (1977), 36–40.
48. A. Kufner and L. E. Persson, *Weighted inequalities of Hardy type* (World Scientific, 2003).
49. M. Lukachev and L. Shuster, On uniqueness of the solution of a linear differential equation without boundary conditions, *Funct. Diff. Eqns* **14** (2007), 337–346.
50. K. Mynbaev and M. Otelbaev, Spectrum, coercivity, boundary value problem for differential equations and related problems of functional analysis, in *Applied problems of mathematical physics and functional analysis*, Volumes 167–168, pp. 81–88 (Nauka, Moscow, 1985).
51. K. Mynbaev and M. Otelbaev, *Weighted function spaces and the spectrum of differential operators* (Nauka, Moscow, 1988).
52. R. Oinarov, Separability of the Schrödinger operator in the space of summable functions, *Dokl. Akad. Nauk SSSR* **285**(5) (1985), 1062–1064.
53. R. Oinarov, Properties of Sturm–Liouville operator in L_p, *Izv. Akad. Nauk Kazakh. SSR* **1** (1990), 43–47.
54. M. Otelbaev, The summability with weight of the solution of a Sturm–Liouville equation, *Mat. Zametki* **6**(6) (1974), 969–980.
55. M. Otelbaev, The separation of elliptic operators, *Dokl. Akad. Nauk SSR Ser. Fiz.-Mat.* **234**(3) (1977), 540–543.
56. M. Otelbaev, The smoothness of the solution of differential equations, *Izv. Akad. Nauk. Kazakh SSR* **5** (1977), 45–48.
57. M. Otelbaev, A criterion for the resolvent of a Sturm–Liouville operator to be a kernel, *Math. Notes* **25** (1979), 296–297.
58. M. Otelbaev, Coercive estimates and separability theorems for elliptic equations in $L_p(R)$, in *Studies in the theory of differentiable functions of several variables and its applications, IX*, Proceedings of the Steklov Institute of Mathematics, Volume 161, pp. 195–217 (American Mathematical Society, Providence, RI, 1984).
59. D. Ž. Raĭmbekov, Smoothness of the solution in L_2 singular equation, *Izv. Akad. Nauk Kazak. SSR Ser. Fiz.-Mat.* **3** (1974), 78–83.
60. M. Sapenov and L. A. Shuster, On the summability with weight of the solutions of binomial differential equations, *Izv. Akad. Nauk Kazakh. SSR Ser. Fiz.-Mat.* **1** (1987), 38–42.
61. M. Sapenov and L. A. Shuster, Estimates of Green's function and a theorem on separability of a Sturm–Liouville operator in L_p (1985), manuscript available at VINITI, No. 8257-B85.
62. A. Z. Togochuev, Summability of the solution of a differential equation of odd order with weight, *Izv. Akad. Nauk Kazakh. SSR Ser. Fiz.-Mat.* **5** (1985), 55–58.
63. A. Zettl, Separation for differential operators and the L^p-spaces, *Proc. Am. Math. Soc.* **55**(1) (1976), 44–46.