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# ON A STRUCTURAL PROPERTY OF THE GROUPS OF ALTERNATING LINKS

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**1. Introduction.** In this paper, we will prove, as a consequence of the main theorem,

THEOREM A. (See Corollary 2.6). *The group of an alternating knot, for which the leading coefficient of the knot polynomial is a prime power, is residually finite and solvable.*

This enlarges considerably the class of knots with groups known to be residually finite [**21**; **24**; **12**; **13**; **14**], and provides a partial answer to a question raised by R. H. Fox [**7**], L. P. Neuwirth [**21**], and others. Theorem A follows from arguments in [**12**] and from

THEOREM B. (See Theorems 2.2 and 2.5). *The commutator subgroup of the group of an alternating knot, for which the leading coefficient of the knot polynomial is a power of the prime $p$, is residually a finite $q$-group for all primes $q \neq p$.*

Theorems A and B are special cases of somewhat more technical results stating the same conclusions for links having the determinant of the principal minors of the linking matrix equal to $\pm 1$ and belonging to certain classes of links including the class of alternating links. We prove these using Brown and Crowell's analysis of the structure of the augmentation subgroup of a link group [**5**]. In terms of their decomposition of the augmentation subgroup of a strongly indecomposable link into an iterated generalized free product, we have the following structure theorem, from which we derive the title of the paper. It is a special case of our main result, Theorem 2.5.

THEOREM C. *The augmentation subgroup of an alternating link is an iterated generalized free product of free groups $X_i$ in which the amalgamated subgroups $H_i$ and $K_i$ are free factors of respective normal subgroups with abelian quotient and of index equal to the leading coefficient of the reduced Alexander polynomial.*

It can be shown that all known examples of residually finite knot groups have commutator subgroups which are residually a finite $q$-group for almost all primes $q$. For example, fibred, or Neuwirth, knots are characterized by having commutator subgroup a finitely generated free group, and therefore, by a well-known theorem of K. Iwasawa [**10**], their commutator subgroups are residually a finite $q$-group for all primes $q$. (An alternating knot, with leading

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coefficient $\pm 1$ is fibred [**18**], and therefore belongs to this class). Other examples include the groups of Whitehead doubles of the trivial knot [**12**], and all two-bridge knot groups [**13**]. (Unfortunately, this last paper is seriously marred by misprints and minor errors. However we believe the result and method of proof are correct, and expect to present a second proof, from a slightly different point of view, in the future). The class also includes all knot groups from the classical knot tables [**14**]. Although a product knot need not have this property, it is shown in [**3**] that the group of a product knot is residually finite if and only if each factor has a residually finite group. If the group of each factor has the above property, so does the group of the product knot, since by H. Schubert's factorization theorem [**22**], the commutator subgroup of the product knot is a free product of the commutator subgroups of the factors, and it follows from a theorem of K. Gruenberg [**8**], that the commutator subgroup of such a product knot is residually a finite $q$-group for almost all primes $q$. In 1968, P. Stebe [**24**] proved that the groups of hose knots are residually finite by showing they are $\Pi_c$. (A group is $\Pi_c$ if for each two elements $g_1$ and $g_2$ in the group, either $g_1 = g_2{}^t$ or else there is a normal subgroup $N$ of finite index such that $g_1 \not\equiv g_2{}^t \pmod N$ for each integer $t$.) However these knots are, in fact, fibred knots according to J. Milnor [**15**], and so these knots also belong to the above class.

We also use slight generalizations of theorems in [**12**] and [**14**]. These results, which sometimes suffice to imply that the augmentation subgroup is residually a finite $p$-group when the leading polynomial coefficient is a composite number, are given in Section 2, along with group theoretical notations and conventions. In Section 3 we present technical results on certain types of matrices. We include the proof of a theorem of C. Bankwitz [**1**] for completeness. These results are used in the main proofs in the later sections to convert information about the leading polynomial coefficient into information about the cosets of appropriate subgroups. In Sections 6 and 7, this information, together with information from the dual graph of the link $l$, is converted via Reidemeister-Schreier rewriting into the requisite information about free bases for the subgroups to complete the proof of the main theorem. In Section 4 we give definitions and results concerning certain classes of strongly indecomposable links and their algebraically unknotted minimal spanning surfaces. Finally, Section 5 contains two reductions. The first determines the generality of our methods, while the second simplifies the ensuing proofs and is very closely related to the arguments on the residual finiteness of knot groups given in [**12**] and [**13**].

**2. Definitions and statements of main results.** In this paper all knots or links are tame and oriented. We will denote the number of components of a link by $\mu$, but will not always distinguish clearly between knots (links with one component) and links of more than one component. We denote the reduced Alexander polynomial of a link by $\Delta(t)$. If the link is a knot, this is just the knot polynomial.

Let $l$ be a link in the oriented 3-sphere $S^3$ and let $G = \pi_1(S^3 - l)$. If we define a homomorphism from $G$ to the additive group of integers, $Z$, by sending the homotopy class of any loop to the sum of the linking numbers of this loop with the various components $l_i$ of $l$, then the kernel of this mapping is called the *augmentation subgroup* of $l$. If $l$ is a knot, then the augmentation subgroup is just the commutator subgroup. $G$ is an extension of the augmentation subgroup by a free cyclic group.

The link $l$ will be called *strongly indecomposable* if it has an orientable spanning surface of maximal Euler characteristic which is connected. (For example every knot is strongly indecomposable.) By Theorem 2.1 of [**5**], the augmentation subgroup $E$ of a strongly indecomposable link $l$ with connected spanning surface $S$ of minimal genus $g$ is an iterated generalized free product of the form

$$(2.1) \qquad \ldots \underset{F}{*} X_{-1} \underset{F}{*} X_0 \underset{F}{*} X_1 \underset{F}{*} \ldots .$$

Here each $X_i$ is isomorphic to $X$, the fundamental group of the three-manifold "$S^3$-split-along-$S$," and the amalgamation from $X_i$ to $X_{i+1}$ represents the identification of the appropriate copies $K_i \subseteq X_i$ and $H_{i+1} \subseteq X_{i+1}$ of the inclusion-induced images (to either side) of $\pi_1(S)$, the group of the spanning surface, into the group $X$. As indicated $F = \pi_1(S)$ is free (of rank $2g + \mu - 1$). Also $X \cong \pi_1(S^3 - S)$, and we will restrict our attention to links for which $S$ is *algebraically unknotted*, that is for which $X$ is free (of rank $2g + \mu - 1$).

If $C$ is a class of groups, we say that the group $G$ is *residually-$C$* provided for each element $1 \neq x$ in $G$ there exists a group $H$ in $C$ and a homomorphism $\phi$ from $G$ onto $H$ such that $1 \neq \phi(x)$. Given a group $G$, we denote the terms of the lower central series of $G$ by $\gamma_1 G = G$, $\gamma_2 G = G'$, $\gamma_3 G, \ldots$. Then a group $G$ is *parafree* (in the variety of all groups) if $G$ is residually nilpotent and $G$ has the same sequence of quotients $G/\gamma_2 G$, $G/\gamma_3 G, \ldots$ by the terms of its lower central series as some free group $F$. If $G/G'$ is free abelian of rank $r$, then we also say that $G$ is parafree of *rank* $r$. (See [**2**]).

With these concepts we turn to consideration of a link group $G$ with $H$, $K$, $X$, and augmentation subgroup $E$ as given by (2.1). If the link $l$ has components $k_1, \ldots, k_\mu$, then the *linking matrix*, $L = ||z_{ij}||$, is the $\mu \times \mu$ matrix defined by $z_{ij} = lk(k_i, k_j)$, for $i \neq j$, and $z_{ii} = -z_{i1} - \ldots - z_{ii-1} - z_{ii+1} - \ldots - z_{i\mu}$. (Note that this matrix is *not* equal to the link matrix used in later sections.) We denote by $L^*$ any principal minor, $\hat{L}(i|i)$, of $L$ (obtained by deleting the $i$th row and column from $L$). Writing $X^n$ for the subgroup of $E$ generated by $n$ consecutive factors $X_i$, and observing the complete analogy between the decomposition (2.1) and Neuwirth's decomposition of the commutator subgroup of a knot group, we can parrot the proof of Theorem 3.2 in [**14**] to obtain the following

THEOREM 2.2. *Let the link $l$ have the group $G$ with $H$, $K$, $X$, and augmentation subgroup $E$ as in* (2.1). *Suppose that $H$ and $K$ are free factors respectively of the* *(free) subgroups $M$ and $N$ of $X$. If $|X : N| = |X : M| = p^m$, for some prime $p$, and* $\det L^* = \pm 1$, *then $E$ is an ascending union of the parafree groups $X^n$, and residually a finite $q$-group for any prime $q \neq p$.*

Under the same conditions as Theorem 2.2, we have the following two corollaries.

COROLLARY 2.3. [**14**] *Any two-generator subgroup of $E$ is free.*

COROLLARY 2.4. [**12**, p. 224]. *$G$ is residually a finite solvable group.*

These results motivate our main result, namely the following.

THEOREM 2.5. *Let $l$ be a pseudo-alternating link* (see Section 4) *in $S^3$. Then*

(1) *$l$ has an orientable connected spanning surface $S$ of maximal Euler characteristic that is algebraically unknotted, and therefore $X \cong \pi_1(S^3 - S)$ is free of rank $2g + \mu - 1$.*

(2) $|X : H \cdot X'| = |X : K \cdot X'| = \Delta(0)$, *for $H$ and $K$ as following* (2.1).

(3) *$H$ and $K$ are respective free factors of $H \cdot X'$ and $K \cdot X'$.*

In particular Theorem 2.5 applies to any pseudo-alternating knot (see Section 4) and therefore to any alternating (link or) knot.

Corollary 2.4 and Theorem 2.5 imply

COROLLARY 2.6. *The group of a pseudo-alternating link is residually a finite solvable group if $\Delta(0)$ is a prime power and the linking matrix minor $L^*$ has determinant* $\pm 1$.

The proof of Theorem 2.5.1 and Theorem 2.5.2 will be given in Section 4, and that of Theorem 2.5.3 in Section 7.

**3. Matrix properties.** In this section we will prove several basic properties of matrices of particular types. These results will be used frequently in the remainder of the paper. We begin with some definitions.

Let $M = ||a_{ij}||$ be an $n \times m$ real matrix with $n \leqq m$. $M$ is said to be of *special type* (*on the rows*) if (1) $a_{ii} > 0$, for all $i$, (2) $a_{ij} \leqq 0$, for $i \neq j$, and (3) $\sum_{j=1}^{n} a_{ij} \geqq 0$, foı all $i = 1, \ldots, n$. We will sometimes write $\epsilon_i(M)$ for the row sum $\sum_{j=1}^{n} a_{ij}$. A matrix of *special type on the columns* is defined similarly, when $n \geqq m$.

Given an $n \times m$ matrix $M$, we denote by $M(i_1 \ldots i_k | j_i \ldots j_d)$ the minor of $M$ consisting of the $i_1$-, $\ldots$, $i_k$-th rows and the $j_1$-, $\ldots$, $j_d$-th columns of $M$. Similarly, $\hat{M}(i_1 \ldots i_k | j_1 \ldots j_d)$ denotes the complementary minor obtained by striking out the $i_1$-, $\ldots$, $i_r$-th rows and the $j_1$-, $\ldots$, $j_d$-th columns of $M$. By a *principal minor* of $M$ we mean a matrix $M(i_1 \ldots i_k | i_1 \ldots i_k)$, and if $1 \leqq k \leqq n - 1$, it will be called a *proper* principal minor.

PROPOSITION 3.1. *Let $M$ be an $n \times m$ real matrix of special type on the rows.*

*Then*

(1) $\det M(1\ldots s|1\ldots s) \geqq 0$, *for* $1 \leqq s \leqq n$,

(2) $(-1)^{s+k+1} \det M(1\ldots s|1\ldots \hat{k} \ldots s\, i) \geqq 0$, *for* $1 \leqq k \leqq s$, *and* $s+1 \leqq i \leqq n$,

(3) $\det M(1\ldots s|1\ldots s) \geqq \sum_{i=s+1}^{n} |\det M(1\ldots s|1\ldots \hat{k} \ldots s\, i)|$,

We remark that by interchanging rows and columns simultaneously, any principal minor of $M$ can be brought into the form described in (3.1.1). Therefore (3.1.1) actually claims that the determinant of any principal minor of $M$ is non-negative. Similar remarks apply to (3.1.2) and (3.1.3).

*Proof.* Since (3.1.1) is an immediate consequence of (3.1.3), we need only prove (3.1.2) and (3.1.3). By simultaneously interchanging rows and columns, if necessary, it suffices to prove (3.1.2) and (3.1.3) for $k = s$.

The proof will proceed by induction on $s$. For $s = 1$, everything is obvious. Suppose (3.1.2) and (3.1.3) hold for $s-1$. To prove (3.1.2) we compute $\det M(1\ldots s|1\ldots s-1\, i)$ by expanding along the $s$-th row, obtaining

$$\begin{aligned}\det M(1\ldots s|1\ldots s-1\, i) &= (-1)^{2s}(a_{si}) \det M(1\ldots s-1|1\ldots s-1)\\ &\quad + \textstyle\sum_{j=1}^{s-1} (-1)^{s+j}(a_{sj}) \det M(1\ldots s-1|1\ldots \hat{j} \ldots s-1\, i).\end{aligned} \tag{3.2}$$

By the inductive assumptions both $(-1)^{s+j} \det M(1\ldots s-1|1\ldots \hat{j} \ldots s-1\, i)$ and $\det M(1\ldots s-1|1\ldots s-1)$ are non-negative, implying that all the summands on the right-hand side of (3.2) are non-positive. Therefore $(-1) \det M(1\ldots s-1\, i) \geqq 0$. This proves (3.1.2) in the case $k = s$.

Next, in order to prove (3.1.3), we expand $\det M(1\ldots s|1\ldots s)$ along the $s$-th row, and then compute both sides of (3.1.3). Using (3.2), we obtain

$$\begin{aligned} D &= \det M(1\ldots s|1\ldots s) - |\textstyle\sum_{i=s+1}^{n} \det M(1\ldots s|1\ldots s-1\, i)|\\ &= \det M(1\ldots s|1\ldots s) + \textstyle\sum_{i=s+1}^{n} \det M(1\ldots s|1\ldots s-1\, i)\\ &= \textstyle\sum_{j=1}^{s} (-1)^{s+j}(a_{sj}) \det M(1\ldots s-1|1\ldots \hat{j} \ldots s)\\ &\quad + \textstyle\sum_{i=s+1}^{n} \{ \sum_{j=1}^{s-1} (-1)^{s+j}(a_{sj}) \det M(1\ldots s|1\ldots \hat{j} \ldots s-1\, i)\\ &\qquad + (-1)^{2s}(a_{si}) \det M(1\ldots s-1|1\ldots s-1)\}\\ &= \textstyle\sum_{j=1}^{s-1} (-1)^{s+j}(a_{sj}) \{\det M(1\ldots s-1|1\ldots \hat{j} \ldots s)\\ &\qquad + \textstyle\sum_{i=s+1}^{n} \det M(1\ldots s-1|1\ldots \hat{j} \ldots s-1\, i)\}\\ &\qquad + \det M(1\ldots, s-1|1\ldots s-1) \cdot \textstyle\sum_{t=s}^{n} a_{st}. \end{aligned} \tag{3.3}$$

Since $\epsilon_s(M) \geqq 0$, we have $\sum_{i=s}^{n} a_{si} \geqq \sum_{j=1}^{s-1} (-a_{sj})$; and hence (3.3) becomes

$$\begin{aligned} D &\geqq \textstyle\sum_{j=1}^{s-1} (-a_{sj}) \det M(1\ldots s-1|1\ldots s-1)\\ &\quad + \textstyle\sum_{j=1}^{s-1} (-1)^{s+j}(a_{sj}) \{\det M(1\ldots s-1|1\ldots \hat{j} \ldots s)\\ &\quad + \textstyle\sum_{i=s+1}^{n} \det M(1\ldots s-1|1\ldots \hat{j} \ldots s-1\, i)\}\\ &= \textstyle\sum_{j=1}^{s-1} (-a_{sj}) \{\det M(1\ldots s-1|1\ldots s-1)\\ &\qquad + (-1)^{s+j+1} \textstyle\sum_{i=s}^{n} \det M(1\ldots s-1|1\ldots \hat{j} \ldots s-1\, i)\}. \end{aligned} \tag{3.4}$$

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Now by the inductive assumption, det $M(1 \ldots s-1|1 \ldots s-1) \geqq \sum_{i=s}^{n} |\det M(1 \ldots s-1|1 \ldots \hat{j} \ldots s-1\ i)|$, and consequently each of the bracketed expressions in the last equation in (3.4) is non-negative. Therefore $D \geqq 0$. This proves (3.1.3) and completes the proof of Proposition 3.1.

As a special case, when $s = n - 1$ and $m = n$, we get the following

COROLLARY 3.5. *Let $M$ be an $n \times n$ real matrix of special type on the rows. Then*

(1) $(-1)^{i+j} \det \hat{M}(i|j) \geqq 0$,

(2) $\det \hat{M}(i|i) \geqq |\det \hat{M}(i|j)|$, *for* $j = 1, \ldots, n, j \neq i$, *and*

(3) $\det M = \sum_{j=1}^{n} a_{ij} |\det \hat{M}(i|j)|$.

In general we cannot replace $\geqq$ by strict inequality in (3.1.1). In terms of the following definitions we will give below a sufficient condition for any principal minor of $M$ to be non-singular.

An $n \times m$ matrix $M$ is said to be of *positive type* (*on the rows*) if (1) $M$ is of special type (on the rows), and (2) at least one of the row sums $\epsilon_i(M)$ is non-zero. Moreover, if every principal minor of $M$ is of positive type, then we say $M$ has *strictly positive type*. A matrix of *strictly positive type on the columns* will be defined in the same manner. Obviously, any principal minor of a matrix of strictly positive type on the rows (columns) is also of strictly positive type on the rows (columns).

PROPOSITION 3.6. *Let $M$ be an $n \times n$ matrix of strictly positive type on the rows. Then* $\det M(i_1 \ldots i_k|i_1 \ldots i_k) > 0$, *for* $1 \leqq k \leqq n$.

*Proof*. Since $M(i_1 \ldots i_k|i_1 \ldots i_k)$ is of strictly positive type on the rows, it suffices to show that $\det M(1 \ldots n|1 \ldots n) > 0$. The proof will be by induction on $n$. The cases $n = 1$ and $n = 2$ are obvious. Therefore suppose that the proposition is true for any $(n-1) \times (n-1)$ matrix of strictly positive type.

We may assume without loss of generality that $a_{11} + \ldots + a_{1n} > 0$. Also, if all $a_{i1} = 0$, for $i \neq 1$, then $\det M = a_{11} \cdot \det \hat{M}(1|1) > 0$, because $a_{11} > 0$ and $\det \hat{M}(1|1) > 0$ by induction. Therefore we can assume that at least $a_{21}$, say, is not zero. Since $a_{11} \neq 0$, we can eliminate all but the first entry from the first row of $M$ so that $\det M = a_{11} \cdot \det N$, where $N$ is an $(n-1) \times (n-1)$ matrix $||b_{ij}||$ defined as

$$b_{ij} = a_{i+1j+1} - \frac{a_{1j+1} \cdot a_{i+11}}{a_{11}}, \quad \text{for } i, j = 1, \ldots, n-1.$$

To prove the proposition, it now suffices to show that $N$ is of strictly positive type on the rows.

First, since $a_{11} > |a_{1i+1}|$ and $a_{i+1i+1} \geqq |a_{i+11}|$, it follows that $b_{ii} > 0$, for all $i$, and also that $b_{ij} \leqq 0$, for all $i \neq j$. (Note that $a_{ij} \leqq 0$, for $i \neq j$). Further-

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more, a simple computation yields

$$\begin{aligned}\sum_{j=1}^{n-1} b_{ij} &= \sum_{j=1}^{n-1} \left\{ a_{i+1j+1} - \frac{a_{i+11} \cdot a_{1j+1}}{a_{11}} \right\} \\ &= \sum_{j=1}^{n-1} a_{i+1j+1} - \frac{a_{i+11}}{a_{11}} \sum_{j=1}^{n-1} a_{1j+1} \\ &= \epsilon_{i+1}(M) - \frac{a_{i+11}}{a_{11}} \epsilon_1(M).\end{aligned}$$

Since $\epsilon_1(M) > 0$ and $a_{21} < 0$, we have in particular

$$\sum_{j=1}^{n-1} b_{1j} = \epsilon_2(M) - \frac{a_{21}}{a_{11}} \cdot \epsilon_1(M) > 0.$$

Therefore $N$ is of positive type on the rows.

Finally we must prove that every principal minor of $N$ is of positive type on the rows. Assume not. Then there is a principal minor $N_0$ of $N$ that is not of positive type, and for simplicity we can assume that $N_0 = N(2 \ldots m|2 \ldots m)$, for $2 \leqq m \leqq n - 1$.

Then $\sum_{i=2}^{m} b_{2i} = 0, \ldots, \sum_{i=2}^{m} b_{mi} = 0$, where

$$\sum_{i=2}^{m} b_{ji} = \sum_{i=2}^{m} \{a_{j+1i+1} - (a_{j+11}/a_{11})\, a_{1i+1}\},$$

for $j = 2, \ldots, m$. Since $a_{11} + \ldots + a_{1n} > 0$, it follows that if $a_{j+11} \neq 0$, then $-(a_{j+11}/a_{11}) \sum_{i=2}^{m} a_{1i+1} > -(a_{j+11}/a_{11})(-a_{11}) = a_{j+11}$, which in turn implies that $0 = \sum_{i=2}^{m} b_{ji} > \sum_{i=2}^{m} a_{j+1i+1} + a_{j+11} \geqq 0$, a contradiction. Thus $a_{j+11} = 0$, so we obtain $a_{j+13} + \ldots + a_{j+1m+1} = 0$, for $j = 2, \ldots, m$. Equivalently, $M(3 \ldots m + 1|3 \ldots m + 1)$ is not of positive type, contradicting our hypotheses. This proves Proposition 3.6.

We still need to strengthen Proposition 3.1 under certain conditions. Suppose that we are given an $n \times n$ matrix $M$ of positive type on the rows *and* the columns. Then we can enlarge $M$ to an $(n + 1) \times (n + 1)$ matrix $M_0$ by adding one row and one column to $M$, where the $(n + 1, k)$ entry of $M_0$ is $-\sum_{i=1}^{n} a_{ik}$; the $(k, n + 1)$ entry is $-\sum_{i=1}^{n} a_{ki}$; and the $(n + 1, n + 1)$ entry is $\sum_{k=1}^{n} \sum_{i=1}^{n} a_{ik}$. It is evident that $M_0$ is also of special type on the rows and columns, but is never of positive type, since the sum of the entries on each row and column is zero. Therefore, $\det M_0 = 0$. This matrix $M_0$ will be called the *augmented matrix* of $M$. Obviously, $\det \hat{M}_0(i|i) = \det M$, for any $i$.

A matrix of special type on the rows and columns is said to be *irreducible* if none of the principal minors $\hat{M}_0(i|i)$ of the augmented matrix $M_0$ of $M$ can be transformed into a block triangular matrix by only interchanging rows and columns simultaneously.

PROPOSITION 3.7. *Let $M$ be a matrix of strictly positive type on the rows and columns. Let $M_0$ be the augmented matrix of $M$. Then for any $i$, $\hat{M}_0(i|i)$ is of strictly positive type on the rows and columns.*

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*Proof*. It suffices to prove that $\hat{M}_0(1|1)$ is of strictly positive type. Assume, to the contrary, that $\hat{M}_0(1|1)$ is not of strictly positive type on the rows. Then there is a principal minor $N$ of $\hat{M}_0(1|1)$ that is not of positive type. Since $M$ is of positive type, $N$ must contain the $(n+1)$-*st* row and column. Therefore, without loss of generality, we may assume that

$$N = M_0(m\; m+1 \ldots n+1|m\; m+1 \ldots n+1).$$

By the assumption, $\sum_{j=m}^{n+1} a_{mj} = \sum_{j=m}^{n+1} a_{m+1j} = \ldots = \sum_{j=m}^{n+1} a_{n+1j} = 0$. On the other hand, by the definition of $M_0$, $\sum_{j=1}^{n+1} a_{kj} = 0$, and therefore $\sum_{j=1}^{m-1} a_{mj} = \ldots = \sum_{j=1}^{m-1} a_{n+1j} = 0$. Then $a_{ij} \leqq 0$, for $i \neq j$, implies $a_{m1} = a_{m2} = \ldots = a_{mm-1} = 0, \ldots, a_{n+11} = \ldots = a_{n+1m-1} = 0$, *i.e.*, that

$$M_0(m \ldots n+1|1 \ldots m-1) = 0.$$

Since $M_0$ is the augmented matrix of $M$, it also has zero column sums, and therefore $\sum_{i=1}^{m-1} a_{i1} = \ldots = \sum_{i=1}^{m-1} a_{im-1} = 0$. This shows that

$$M(1 \ldots m-1|1 \ldots m-1)$$

is not of positive type on the columns, contradicting the hypothesis that $M$ is of strictly positive type on the columns. It follows symmetrically that $N$ is of strictly positive type on the columns, and this proves Proposition 3.7.

**4. Pseudo-alternating links and $*$-products of links.** A link $l$ in $S^3$ is *alternating* if it has a diagram, or regular projection, which is connected and alternating. Following [**19**], a link $l$ with an alternating diagram $L$ can be decomposed, utilizing Seifert circuits, into alternating links $l_i$ having so-called *special* diagrams $L_i$ (containing no Seifert circuits of the second type) so that the $L_i$ decompose $L$. Then $l$ is called a $*$-*product* of the (*special alternating links*) $l_i$, and this is denoted symbolically by $l = l_1 * \ldots * l_m$ with $L = L_1 * \ldots * L_m$.

As was shown in [**19**], we can associate to any diagram $L$ of a link $l$ in $S^3$, a square matrix $M$, called the *link matrix* of $l$ (associated with $L$). Then, if $l = l_1 * \ldots * l_m$, the matrix of $l$ is a block matrix $M = ||M_{ij}||$, $1 \leqq i, j \leqq m$, where $M_{ii}$ is the link matrix of $l_i$. An *L-principal minor*, $M^*$, of $M$ is defined as the principal minor of $M$ obtained from $M$ by deleting $m$ rows and $m$ columns containing a diagonal element in each $M_{ii}$, $1 \leqq i \leqq m$. We will say the link matrix $M$ is *irreducible* if each $L$-principal minor $L^*$ of $L$ is irreducible. In order to analyze the link matrix of an alternating link, we must first consider the link matrices of special alternating links. Such matrices were studied in [**17**], where the following proposition was proven:

PROPOSITION 4.1. *Let $M$ be the link matrix of a special alternating link $l$ with reduced Alexander polynomial $\Delta(t)$. Then $M$ (or $-M$) has the following properties:*

(1) *The L-principal minor $M^*$ (or $-M^*$) of $M$ (or $-M$) is of strictly positive type on the rows and columns,*

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(2) *If $l$ is a prime link, then $M^*$ cannot be brought into block diagonal form only by simultaneously interchanging rows and columns, and*

(3) det $M^* = \pm\Delta(0)$.

PROPOSITION 4.2. *Let the alternating link $l$ have $*$-decomposition $l = l_1 * \ldots * l_n$. Let $\Delta(t)$ and $\Delta_i(t)$ denote the reduced Alexander polynomials of $l$ and $l_i$ respectively. Then $\pm\Delta(0) = \Delta_1(0) \ldots \Delta_n(0)$.*

For a proof of this proposition, see [**17**] or [**19**]. Note that $\Delta_i(t) \neq 0$, since the $l_i$ are special alternating [**6**], and therefore $\Delta(0) \neq 0$.

Next, we need to establish that any alternating link spans an algebraically unknotted, connected surface of maximal Euler characteristic. A very original construction by Seifert, given in [**23**], actually produces an appropriate surface for an alternating link. The proof is given in [**6**] or in [**16**]. However, for use in later sections, we will give an alternative construction of such a spanning surface for an alternating link from a slightly different point of view.

Let $S$ be a connected surface in $S^3$, consisting (as illustrated in Figure 4.1) of a finite number of disks $D_1, \ldots, D_t$ and a finite number of bands, each of which connects two disks in such a way that

(1) each band is twisted only once in the same direction,

(2) the *spine* of $S$ is a graph in the plane, *i.e.*, $S$ has a planar graph as a deformation retract,

(3) the bands are pairwise disjoint, and

(4) $S$ is orientable.

Then the boundary of $S$ is a special alternating link $l$, and $S$ will be called a *primitive flat* (*Seifert*) *surface* for $l$. For example, every special alternating link spans a primitive flat surface. If $S$ contains only two disks, then $S$ was called a *primitive s-surface* in [**18**].

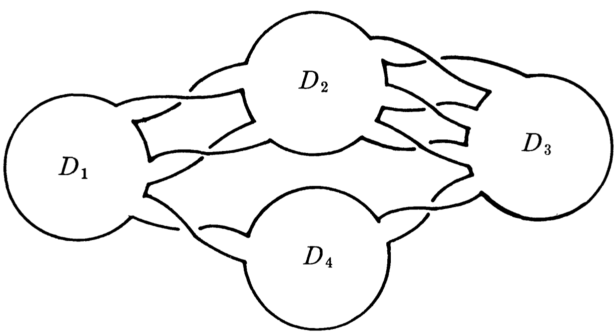


FIGURE 4.1. A primitive flat surface.

We now consider two primitive flat surfaces $S_1$ and $S_2$ in $S^3$, spanning respective links $l_1$ and $l_2$. We take disks $D_{i1}$ and $D_{i2}$ from $S_1$ and $S_2$ respectively, and identify them so that the resulting orientable surface $S = S_1 \cup S_2$ spans a link and that $S - S_1$ and $S - S_2$ are *separated*, i.e., so that there exists a 2-sphere $T$ in $S^3$ such that $T \cap S = D_{i_1} (= D_{i_2})$ and each component of $S^3 - T$

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contains points of $S - D_{i_1}$. Then $S$, or any surface obtained by a finite iteration of this construction, will be called a *generalized flat surface*, and the link spanned by $S$ will be called a *pseudo-alternating* link. As an immediate consequence of the definitions we have

PROPOSITION 4.3. *Any alternating link is pseudo-alternating.*

The converse to the above proposition is false. For example, any torus link of type $(m, n)$, $|m|$, $|n| \geqq 3$, is pseudo-alternating but not alternating.

Also the conclusions of Propositions 4.1 and 4.2 hold for pseudo-alternating links as well, essentially by the same proofs. However, the "factors" of an arbitrary pseudo-alternating link cannot, in general, be recovered from the diagram using Seifert circuits, and we do not know whether every pseudo-alternating link is a $*$-product.

PROPOSITION 4.4. *Let $l$ be a pseudo-alternating link with generalized flat spanning surface $S$. Then $S$ has maximal Euler characteristic.*

PROPOSITION 4.5. *Let $S$ be a generalized flat surface. Then $S$ is algebraically unknotted, i.e., $\pi_1(S^3 - S)$ is free (of finite rank).*

Propositions 4.4 and 4.5 were essentially proven in [**6**; **16**; and **20**], and we omit the details.

Combining the previous propositions with the formula for the rank of the group of a spanning surface of maximal Euler characteristic, we obtain the following proposition, which is exactly Theorem 2.5.1.

PROPOSITION 4.6. *Any pseudo-alternating link spans a (generalized flat) surface $S$ of maximal Euler characteristic with $\pi_1(S^3 - S) = F_{2g+\mu-1}$, a free group of rank $2g + \mu - 1$, where $g$ is the genus of $S$ and $\mu$ is the number of components of the link.*

If $l$ is the boundary of the generalized flat surface $S$ constructed from primitive flat surfaces $S_i$, $1 \leqq i \leqq m$, spanning special alternating links $l_i$, then it also follows, just as for alternating links, that the link matrix $M$ for $l$ is a block matrix $M = ||M_{ij}||$, $1 \leqq i, j \leqq m$, where $M_{ii}$ is the link matrix of $l_i$, and one of each pair $M_{ij}$ and $M_{ji}$ is the zero matrix. It follows that by simultaneously interchanging rows and columns (of blocks), $M$ can be reduced to block triangular form. Furthermore, it is easy to show that if $M_{ii}$ is not irreducible, then the primitive flat surface $S_i$ will be a generalized flat surface obtained from two smaller primitive flat surfaces. This allows us to assume, for the remainder of the paper, that any primitive flat surface spans a link whose link matrix is irreducible.

Much of the algebraic information connected with a special alternating link $l$ can be computed from a spine for the primitive flat surface $S$ spanning $l$. This spine can be obtained from the diagram $L$ of the link and is sometimes called the *dual graph* of $L$ or $l$. Following, for example, [**19**], we give a more detailed description. Suppose $L$ divides $S^2$ into a finite number of regions $r_1, \ldots, r_{n+1}$.

Now each of these regions belongs to one of two classes $\alpha$ and $\beta$, where we say $r_i$ belongs to the *$\beta$-class*, or $r_i$ is a *$\beta$-region*, if the boundary of $r_i$ constitutes an oriented circuit in $L$ under the orientation induced by $l$. (In this case the boundary of $r_i$ is a Seifert circuit of the first type.) Otherwise we say $r_i$ is an *$\alpha$-region* (or belongs to the *$\alpha$-class*). It is not hard to check that no two regions of the same type have a side in common and that the number of sides on each $\alpha$-region is even. Now we take a point, called the centre, from the interior of each $\alpha$-region, and draw a simple arc through each double point of $L$ connecting the centres of the two $\alpha$-regions meeting there. The resulting 1-complex is called the *graph* of $L$ or $l$, and is planar and connected (since $L$ is connected). We will make more use of the *dual graph*, which we will denote by $\mathfrak{G}$, obtained similarly by connecting the centres of adjacent $\beta$-regions through each double point. Of course, $\mathfrak{G}$ is also planar, connected, has edges in one-to-one correspondence with those of the graph of the link, and is dual to the graph of the link.

Conversely, any planar graph $\mathfrak{G}$ of even valency is the dual graph of a special alternating link, for we can replace each vertex by a small disk and each edge by a band in the plane connecting the two corresponding disks. If each band is then twisted once in the same direction without causing any "complication" in the resulting surface $S$, this surface will have spine $\mathfrak{G}$, be orientable, and span a special alternating link $l$. Evidently $S$ is a primitive flat surface for $l$. Later, for the purposes of simplicity, we will choose a preferred sense of twisting on these bands so that, among other things, the link matrix $M^*$ (as opposed to $-M^*$) is of special positive type.

**5. Two reductions.** In order to prove Theorem 2.5.3, we will spend the next two sections specifying the position of the inclusion-induced images, $H$ and $K$, of the group of a spanning surface in the group of the complement of the surface. We begin, in this section, by reducing the complexity of the link diagrams to be considered.

Let $l$ be a strongly indecomposable link in $S^3$, and consider, for each connected, algebraically unknotted spanning surface of maximal Euler characteristic, the corresponding decomposition of the augmentation subgroup, as given by (2.1). If, for some decomposition, $X$ contains subgroups $M$ and $N$ with (1) $X \geqq M \geqq H \cdot X'$ and $X \geqq N \geqq K \cdot X'$, and (2) $H$ and $K$ are contained respectively in $M$ and $N$ as free factors, then we will say $l$ (or the group $G$ of $l$) has the *free factor property* (*with respect to $M$ and $N$*). For example, each fibred, strongly indecomposable link has the free factor property (since we do not exclude improper free factors).

We observe the following straight-forward consequence of the Reidemeister-Schreier rewriting process.

LEMMA 5.1. *Let $F = G_1 * G_2$ be a free group and $H$ be a free factor of a subgroup of finite index in $G_1$. Then $H * G_2$ is a free factor of a subgroup of $F$ (of the same index).*

THEOREM 5.2. *Let $l_1$ and $l_2$ have the free factor property with respect to subgroups of respective index $n_1$ and $n_2$. Let these links span respective generalized flat surfaces $S_1$ and $S_2$ giving rise to the appropriate decompositions and containing the respective disks $D_1$ and $D_2$. If we identify these disks so that the resulting surface $S$ spans a link $l$ and so that $S - S_1$ and $S - S_2$ are separated, then $l$ has the free factor property with respect to subgroups of index $n_1 \cdot n_2$.*

*Proof.* We choose a basepoint $p$ on the identified disk, and generators $\alpha_1, \ldots, \alpha_r$ for $\pi_1(S_1; p)$; $\beta_1, \ldots, \beta_s$ for $\pi_1(S_2; p)$; and $A_1, \ldots, A_r$ and $B_1, \ldots, B_s$ for $X_1 = \pi_1(S^3 - S_1)$ and $X_2 = \pi_1(S^3 - S_2)$ respectively. We suppose that we have chosen the indices so that $S_1$ attaches to the sharp side of $S_2$, and we consider first the various inclusion-induced homomorphisms to the sharp side, $i_1^\#$, $i_2^\#$, and $i^\# : \pi_1(S) \rightarrow X = X_1 * X_2$, where $i_1^\#$, $i_2^\#$ are monomorphisms by definition and $i^\#$ is also a monomorphism by Proposition 4.4. Now

$$i^\#(\alpha_j) = i_1^\#(\alpha_j) \quad \text{for } j = 1, \ldots, r,$$

while

$$i^\#(\beta_k) = V_k(A_j) \cdot i_2^\#(\beta_k) \cdot U_k(A_j), \quad \text{for } k = 1, \ldots, s,$$

where $V_k$ and $U_k$ are words on $A_1, \ldots, A_r$. (See Figure 5.1). If $H_1 = \text{gp}\ \langle i_1^\#(\alpha_j);\ 1 \leqq j \leqq r \rangle$ and $H_2 = \text{gp}\ \langle i_2^\#(\beta_k);\ 1 \leqq k \leqq s \rangle$ are respectively free factors in $M_1$ and $M_2$, then, by Lemma 5.1, $X_1 * M_2$ and therefore $X_1 * H_2$ are free factors in a subgroup of finite index in $X$. In fact this subgroup is ker $\phi$, where $\phi : X \rightarrow X_2/M_2$ homomorphically. Now applying Whitehead's $T$-automorphisms (see, e.g., [**11**, p. 166]) to the free group ker $\phi$, we affect a free

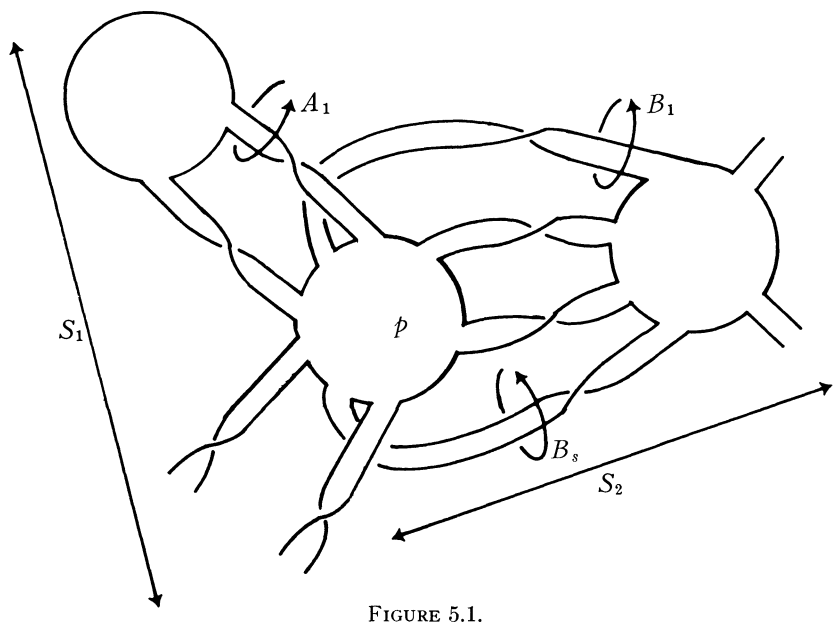


FIGURE 5.1.

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substitution of $i^{\#}(\beta_k)$ for $i_2^{\#}(\beta_k)$, $1 \leqq k \leqq s$, in a free basis for ker $\phi$. Therefore $X_1 * \text{gp}\ \langle i^{\#}(\beta_k); 1 \leqq k \leqq s \rangle$ is a free factor of ker $\phi$. Since $H_1$ is a free factor of a subgroup $M_1$ of finite index in $X_1$, applying Lemma 5.1 again, we have $H = H_1 * \text{gp}\ \langle i^{\#}(\beta_k); 1 \leqq k \leqq s \rangle$ is a free factor of a subgroup of finite index in ker $\phi$ (and in $X$).

The augument to the flat side is symmetric and the remark on indices is obvious, so this completes the proof of Theorem 5.2.

COROLLARY 5.3. *If the link $l$ spans a generalized flat surface constructed from primitive flat surfaces spanning links $l_i$ satisfying the free factor property for subgroups of index $n_i$, then $l$ satisfies the free factor property for subgroups of index the product of the $n_i$. In particular, this applies to any pseudo-alternating link $l$.*

The above results imply that we can restrict our attention to special alternating links with irreducible link matrix when proving Theorem 2.5.3. The next proposition, which allows us to simplify the link diagrams under consideration by removing certain bands connecting given pairs of disks, could be deduced from Proposition 5.2 and the observation that torus links have the free factor property. However, we include a proof, which may be of independent interest.

PROPOSITION 5.4. *Let $\mathfrak{G}_0$ be a planar graph containing adjacent vertices $v_1$ and $v_2$. Let the planar graph $\mathfrak{G}$ be obtained from $\mathfrak{G}_0$ by adjoining a path consisting of $2k - 1$ consecutive edges which join $v_1$ to $v_2$ and which lie in a region adjacent to the edge connecting $v_1$ and $v_2$. Let $l_0$ and $l$ be the alternating links associated with $\mathfrak{G}_0$ and $\mathfrak{G}$, so that the surface $S$ for $l$ is obtained from the surface $S_0$ for $l_0$ by adding a band. Then $l_0$ is special alternating and satisfies the free factor property with respect to subgroups of index $n$ if and only if $l$ is special alternating and satisfies the free factor property with respect to subgroups of index $k \cdot n$.*

*Proof.* Choose orientations, indices, distinguished meridians, etc., so that the appropriate portions of the spanning surface are given by Figure 5.2 and Figure 5.3. Choose $v_1$ as basepoint for the fundamental group of each surface.

We can assume that $\mathfrak{G}_0 - \{e_1\}$ is connected, implying that the regions on either side of $e_1$ are distinct. Otherwise $l_0$ is a product link in the sense of Schubert [**22**] or Hashizume [**9**]. It is easy to see that this case need not be considered, and it also follows immediately that $l_0$ is special alternating if and only if $l$ is.

Choose a maximal tree $\mathfrak{T}$ in $\mathfrak{G}_0 - \{e_1\}$ and a free basis $\alpha_1, \ldots, \alpha_c$ for $\pi_1(S_0)$ in one-to-one correspondence with the edges of $\mathfrak{G}_0 - \mathfrak{T}$ so that $\alpha_1$ corresponds to $e_1$. Choose a dual basis $A_1, \ldots, A_c$ for $\pi_1(S^3 - S_0)$ so that $A_i$ and $\alpha_i$ have linking number $+1$. We can now obtain a free basis for $\pi_1(S)$ by adding a generator corresponding to the added band. In order to simplify the images of $\pi_1(S)$ in $\pi_1(S^3 - S)$, we choose a basis for $\pi_1(S^3 - S)$ to consist of the $A_i$,

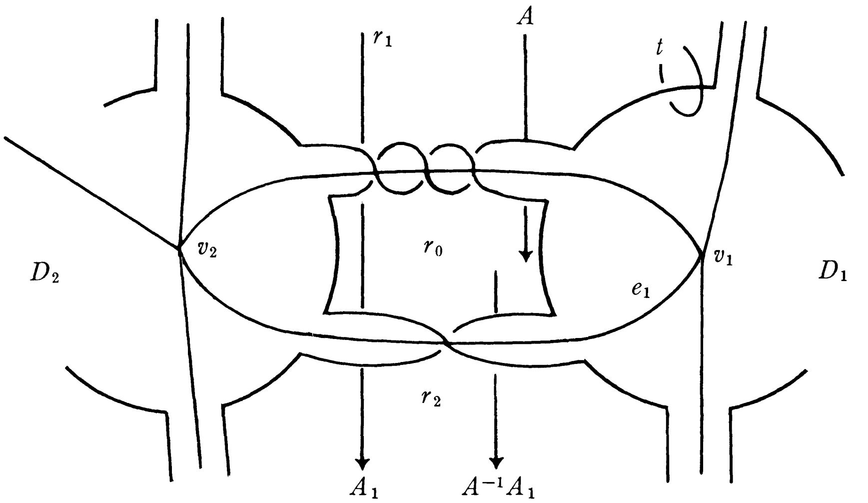


FIGURE 5.2. A portion of the surface $S_0$.

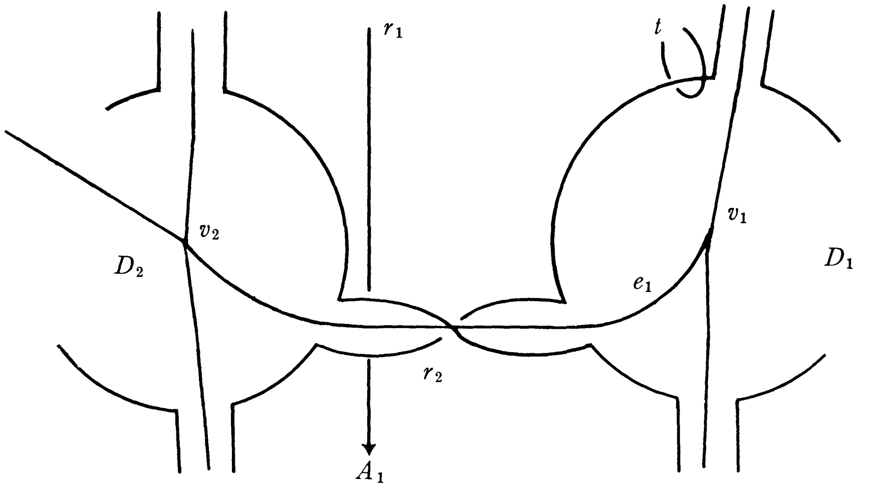


FIGURE 5.3. A portion of the surface $S$.

$2 \leqq i \leqq c$, together with $A$ and $A_1$ as indicated in Figure 5.3. Then the inclusions $i^{\#}$ and $i^{b}$ mapping $\pi_1(S_0)$ and $\pi_1(S)$ into $H_0$, $K_0 \subseteq \pi_1(S^3 - S_0)$ and $H, K \subseteq \pi_1(S^3 - S)$ are as follows:

(5.5) For $S_0$:

$$U_1(A_j) \xleftarrow{b} \alpha_1 \xrightarrow{\#} W_1(A_j)$$
$$U_i(A_j) \leftarrow \alpha_i \rightarrow W_i(A_j),\ 2 \leqq i \leqq c, \text{ and}$$

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(5.6) For $S$:

$$A^k \cdot U_1(A_j) \leftarrow \alpha \rightarrow A^{k-1} \cdot W_1(A_j)$$
$$U_1(A_j) \leftarrow \alpha_1 \rightarrow A^{-1} \cdot W_1(A_j)$$
$$U_i(A_j) \leftarrow \alpha_i \rightarrow W_i(A_j),\ 2 \leqq i \leqq c,$$

where the words $U_i$ and $W_i$, $1 \leqq i \leqq c$, are the same for each surface and do not contain the generator $A$. The proof of the proposition now proceeds just as the proof of Theorem 5.2, using Lemma 5.1 and the Whitehead $T$-automorphisms.

We remark that, in the notation of the above proof, $\mathfrak{G}_0 - \{e_1\}$ is disconnected if and only if $l$ contains a prime factor which is a (2,2)-torus link or a non-fibred $(2k, 2)$-torus link, for $k \geqq 2$.

Now let $l$ be a prime special alternating link from whose diagram we have removed as many bands as possible using Proposition 5.4. In particular we can remove one of any pair of bands corresponding to the boundary of a 2-gon in the complement of the link dual graph $\mathfrak{G}^*$ without affecting the free factor property.

**6. Coset representatives and $RS$-rewriting.** In this section we will choose free bases for $\pi_1(S)$ and $\pi_1(S^3 - S)$ which will enable us to relate the primitive flat surface $S$ and the link matrix $M$ for special alternating links. We will use these to define coset representative systems for $X/H \cdot X'$ and $X/K \cdot X'$ needed for the rewriting processes which compute the inclusion-induced images $H$ and $K$ of $\pi_1(S)$ in $H \cdot X'$ and $K \cdot X'$.

Let $l$ be a special alternating link with primitive flat spanning surface $S$ and dual graph $\mathfrak{G}$. Let $v_0$ be a vertex of $\mathfrak{G}$. The group $\pi_1(S, x_0)$ is a free group of rank $2g + \mu - 1$, and it is freely generated by the classes $\alpha_1, \ldots, \alpha_n$ of simple closed curves corresponding to the boundaries of the bounded regions $r_1, \ldots, r_n$ in the complement of $\mathfrak{G}$. We will assume that the $\alpha_i$ are oriented counter-clockwise.

We assume that $S$ is embedded in $R \times R \times [-\frac{1}{2}, \frac{1}{2}] \subseteq R \times R \times R$, and constructed from $\mathfrak{G}$ which is contained in $R \times R \times \{0\}$. (We select a point at infinity $\infty$ from $S^3$ and consider a Cartesian coordinate system $R \times R \times R = S^3 - \infty$.) We take $p_0 = (0, 0, 1)$ as basepoint for $\pi_1(S^3 - S) \cong X$. Let $c_i$ be a point from the interior of the region $r_i$, $1 \leqq i \leqq n + 1$, and define the oriented closed curve $a_i = f_{n+1} \cup g_{n+1} \cup g_i^{-1} \cup f_i^{-1}$, where $f_j$ is a directed arc joining $p_0$ to $c_j$, $g_j$ is an arc joining $c_j$ to the point $(0, 0, -1)$ in $S^3$, and $f_j^{-1}$ and $g_j^{-1}$ denote the arcs $f_j$ and $g_j$ with reverse direction. Then it is obvious that the $a_i$ form a set of free generators for $\pi_1(S^3 - S, p_0) \cong X$ which are dual to the $\alpha_i$. The $a_i$ will be called a set of *standard generators* for $X$.

We now turn to consideration of the precise form of the free generators $i^\#(\alpha_j)$ and $i^b(\alpha_j)$ for $H$ and $K$ respectively. Suppose the boundary of $r_i$ consists of $2k$ edges $e_1, \ldots, e_{2k}$. Denote by $R(e_i)$ and $L(e_i)$ the regions lying to the right and left side respectively of the edge $e_i$ (with respect to the counter-clockwise orientation). Then we denote the closed path $f_s \cup g_s \cup g_t^{-1} \cup f_t^{-1}$ by $\overrightarrow{R(s)L(t)}$, so that $\overrightarrow{R(s)L(t)}$ and $a_s^{-1}a_t$ represent the same element of $X$. In order that we consider link matrices of positive type (instead of negative) we will consider only the case where the sense of twisting of the bands of $S$ is such that the lefthand boundary component crosses over the righthand component at each crossing. Then it follows from our definitions that

$$i^{\#}(\alpha_j) = P_j \cdot \overrightarrow{R(e_1)L(e_2)} \ldots \overrightarrow{R(e_{2k-1})L(e_{2k})} \cdot P_j^{-1},$$

where $P_j$ is the image under $i^{\#}$ of a path of even length in $\mathfrak{G}$ from $v_0$ to some "first" vertex $v_j$ on the boundary of $r_j$. For later use we will define $\Lambda_j$ by $i^{\#}(\alpha_j) = P_j \cdot \Lambda_j \cdot P_j^{-1}$. We also note that $R(e_{2s-1})$, $s = 1, \ldots, k$, is always a region adjacent to $r_j$, while $L(e_{2s})$ is always $r_j$. In particular, $\Lambda_j$ has the form $(a_{s_1}{}^{-1}a_j) \ldots (a_{s_k}{}^{-1}a_j)$.

Consider now the equations

$$\text{(6.1)} \qquad i^{\#}(\alpha_j) = a_1{}^{v_{j1}} \cdot \ldots \cdot a_n{}^{v_{jn}} \pmod{X'}, \quad \text{for } j = 1, \ldots, n.$$

Define $V^{\#}$ to be the exponent matrix of the system (6.1). Then $V^{\#}$ is an $n \times n$ integer matrix, and we observe

PROPOSITION 6.2. *$V^{\#}$ is an $L$-principal minor of the link matrix $M$ of $l$ associated with the above diagram. Furthermore, if $V^{b}$ denotes the matrix of exponents for $i^{b}(\alpha_j)$ (mod $X'$), then $V^{b} = (V^{\#})^{t}$.*

From this proposition and the definition of the reduced Alexander polynomial, we have immediately (see Propositions 4.1 and 4.2)

COROLLARY 6.3. *For pseudo-alternating links $l$,* $\det V^{\#} = |\Delta(0)| = [X : H \cdot X']$

The proof of Proposition 6.2 follows from the definition of the link matrix and the fact that $l$ is special alternating, and will be omitted.

Let $A_1, \ldots, A_n$ be some fixed permutation of the $a_j$, and suppose $V^{\#}$ is row equivalent (after simultaneously exchanging rows and columns) to

$$\text{(6.4)} \qquad U = \begin{array}{c} \begin{array}{cccccc} A_1 & \cdots & & A_{q+1} & \cdots & A_n \end{array} \\ \left[\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & (-\lambda_{ij}) & \\ & & 1 & & & \\ \hline & & & p_{11} & & (p_{ij}) \\ & 0 & & & \ddots & \\ & & & 0 & & p_{mm} \end{array}\right] \end{array},$$

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where $||-\lambda_{ij}||$ is a $q \times m$ matrix ($q$ may be equal to 0), and we may assume $p_{ii} > 1$, $0 \leqq p_{ij} < p_{jj}$ for $j > i$, and $0 \leqq \lambda_{ij} < p_{jj}$.

Then $X/H \cdot X'$ is an abelian group defined by the following relations:

$$(6.5)\quad \left\{\begin{array}{lcl} A_1 & = & A_{q+1}{}^{\lambda_{11}} \cdot \ldots \cdot A_{q+m}{}^{\lambda_{1m}} \\ & \vdots & \\ A_q & = & A_{q+1}{}^{\lambda_{q1}} \cdot \ldots \cdot A_{q+m}{}^{\lambda_{qm}} \\ 1 & = & A_{q+1}{}^{p_{11}} \cdot \ldots \cdot A_{q+m}{}^{p_{1m}} \\ 1 & = & A_{q+2}{}^{p_{22}} \cdot \ldots \cdot A_{q+m}{}^{p_{2m}} \\ & \vdots & \\ 1 & = & A_{q+m}{}^{p_{mm}} \end{array}\right.$$

Since $p_{ij} < p_{jj}$ and $|X : H \cdot X'| = p_{11} \cdot \ldots \cdot p_{mm} = \det V^{\#}$, we can select as right coset representatives of $X$ modulo $H \cdot X'$ the elements

$$(6.6)\qquad \{A_{q+1}{}^{s_1} \cdot \ldots \cdot A_{q+m}{}^{s_m};\quad 0 \leqq s_j \leqq p_{jj} - 1\}.$$

Let $\phi$ be the coset representation function for $X$ modulo $H \cdot X'$ defined by (6.5) and (6.6). If we use the cosets (6.6) and the coset representation function $\phi$ to define a Reidemeister-Schreier rewriting process $\tau$ for $H \cdot X'$, then $H \cdot X'$ will be freely generated by the non-trivial symbols among those of the form

$$(6.7)\qquad A(j; s_1, \ldots, s_m) = A_{q+1}{}^{s_1} \cdot \ldots \cdot A_{q+m}{}^{s_m} \cdot A_j \cdot [A_{q+1}{}^{t_1} \cdot \ldots \cdot A_{q+m}{}^{t_m}]^{-1}.$$

where the exponents $t_k$ are determined by the equations (6.5). Note that only $p_{11} \cdot \ldots \cdot p_{mm}(q + m - 1) + 1$ of the generators in (6.7) are non-trivial since each $A(q + j; s_1 \ldots, s_j, 0, \ldots, 0) \approx 1$, for $0 \leqq s_j < p_{jj} - 1$.

We have already chosen generators for $H$ of the form $i^{\#}(\alpha_i) = P_i \cdot \Lambda_i \cdot P_i^{-1}$, $i = 1, \ldots, n = q + m$, where $P_i$ denotes some path joining the basepoint for $\pi_1(S)$ to some vertex on the boundary of $r_i$, $\Lambda_i$ is of the form $\Lambda_i = (A_{k_1}{}^{-1}A_i)(A_{k_2}{}^{-1}A_i) \cdot \ldots \cdot (A_{k_e}{}^{-1}A_i)$, and $e = a_{ii}$. We apply the Reidemeister-Schreier rewriting $\tau$ to $i^{\#}(\alpha_i)$, obtaining

$$\tau \circ i^{\#}(\alpha_i) = \tau(P_i) \cdot [A(k_1, s_1, \ldots)^{-1}A(i; t_1, \ldots) \cdot \ldots] \cdot \tau(P_i)^{-1},$$

and we claim

PROPOSITION 6.8. *Any generator $A(i; t_1, \ldots, t_m)$ appears at most once in $\tau(\Lambda_i)$, that is, all generators occuring in $\tau(\Lambda_i)$ are distinct. (Of course any trivial generators appearing will represent the same group element.)*

*Proof.* Because of the similarities of the arguments, we will prove Proposition 6.8 only for $i = 1$. Assume that $A(1; t_1, \ldots, t_m)$ occurs twice in $\tau(\Lambda_1) = (A_{k_1}{}^{-1}A_1) \cdot \ldots \cdot (A_{k_e}{}^{-1}A_1)$, say at the $p$-th and $q$-th term. (Note that $A_{n+1}$ is

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trivial, so some factors may not involve a non-trivial $A_{k_i}^{-1}$, but consist only of $A_1$.) Since $\phi(w)$ represents the right coset containing $w$, we have therefore assumed $\phi(P_1 \cdot A_{k_1}^{-1}A_1 \cdot \ldots \cdot A_{k_p}^{-1}) = \phi(P_1 \cdot A_{k_1}^{-1}A_1 \cdot \ldots \cdot A_{k_q}^{-1})$, or equivalently

$$\phi(A_1 \cdot A_{k_{p+1}}^{-1}A_1 \cdot \ldots \cdot A_{k_q}^{-1}) = 1. \tag{6.9}$$

If we consider (6.9) as a relation in $X/H \cdot X'$, i.e., as a consequence of the relations (6.5), and replace the relation $i^\#(\alpha_1) = 1$ by (6.9), then the resulting exponent matrix $U$ is of special type on the rows (implying $\det U \geqq 0$) and satisfies $\det U \equiv 0$ (mod $D = \det V^\# = \det M^*$). On the other hand, if we replace $i^\#(\alpha_1) = 1$ by its product with the inverse of (6.9), the resulting exponent matrix $W$ is still of special type on the rows, so $\det V^\# - \det U = \det W \geqq 0$. Thus $\det V^\# \geqq \det U$, and either $\det U = D$ or $\det U = 0$. If $\det U = D$, then $\det W = 0$. We wish to obtain a contradiction from $\det U = 0$ or $\det W = 0$. The arguments are identical, but for notational simplicity we will consider only the case $\det U = 0$. Suppose that $U$ is obtained from $V^\#$ by replacing the first row $(c_{11}, \ldots, c_{1n})$ of $V^\#$ by $(b_1, \ldots, b_n)$, with $b_1 > 0$, $b_i \leqq 0$ for $i \neq 1$, and $b_1 + \ldots + b_n \geqq 0$. Then $U$ is of special type on the rows, $\hat{U}(1|x) = \hat{V}^\#(1|x)$ for each $x$, and in particular $\hat{U}(1|1) = \hat{V}^\#(1|1)$, so $\hat{U}(1|1)$ is of strictly positive type. We will show $V^\#$ is not irreducible, contradicting our assumptions.

We compute $\det U$ by expanding along the first row to obtain (see Corollary 3.5.3)

$$\begin{aligned} 0 &= \det U = b_1 \cdot \det \hat{U}(1|1) + b_2|\det \hat{U}(1|2)| + \ldots + b_n|\det \hat{U}(1|n)| \\ &= (-b_2 - \ldots - b_n + \epsilon_1(U))\det \hat{U}(1|1) + b_2|\det \hat{U}(1|2)| \\ &\qquad\qquad + \ldots + b_n|\det \hat{U}(1|n)| \\ &= -b_2\{\det \hat{U}(1|1) - |\det \hat{U}(1|2)|\} - \ldots \\ &\qquad - b_n\{\det \hat{U}(1|1) - |\det \hat{U}(1|n)\} + \epsilon_1(U) \det \hat{U}(1|1). \end{aligned} \tag{6.10}$$

Since $\det \hat{U}(1|1) \geqq |\det \hat{U}(1|i)|$, $i \geqq 2$, and since $\det \hat{U}(1|1) > 0$ by Corollary 3.5.2 and Proposition 3.6, it follows from (6.10) that $\epsilon_1(U) = b_1 + \ldots + b_n = 0$. Therefore, $b_1 > 0$ yields at least one negative $b_i$, say, $b_2 < 0$. Then (6.10) implies that $\det \hat{U}(1|1) = |\det \hat{U}(1|2)|$. Since $|\det \hat{U}(1|2)| = -\det \hat{U}(1|2)$ by Corollary 3.5.1, we have $\det \hat{U}(1|1) + \det \hat{U}(1|2) = 0$, implying that the following $(n-1) \times (n-1)$ matrix $P$ is singular:

$$P = \begin{bmatrix} c_{21} + c_{22} & c_{23} & \cdot & \cdot & \cdot & c_{2n} \\ c_{31} + c_{32} & c_{33} & \cdot & \cdot & \cdot & c_{3n} \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ c_{n1} + c_{n2} & c_{n3} & \cdot & \cdot & \cdot & c_{nn} \end{bmatrix}$$

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Since $P$ is of special type, $P$ cannot be of strictly positive type on the rows, for otherwise det $P > 0$. Therefore $P$ contains a principal minor $P_0$ which is not of positive type on the rows. We note that $P_0$ must contain the first column, or else $P_0$ would be a principal minor of $\hat{U}(12|12) = \hat{V}^{\#}(12|12)$, which is of strictly positive type, implying $P_0$ would be of positive type. Now we may assume without loss of generality that $P_0 = P(12\ldots k|12\ldots k)$, $1 \leqq k \leqq n-1$. Then

$$(6.11) \quad 0 = \epsilon_j(P_0) = (c_{j1} + c_{j2}) + c_{j3} + \ldots + c_{jk}, \quad 2 \leqq j \leqq k+1.$$

Since $\epsilon_2(U) \geqq 0, \ldots, \epsilon_{k+1}(U) \geqq 0$, (6.11) implies that $\epsilon_2(U) = 0, \ldots, \epsilon_{k+1}(U) = 0$, and further that $c_{2,k+2} = \ldots = c_{2,n} = 0, \ldots, c_{k+1k+2} = \ldots = c_{k+2,n} = 0$. This means that $U(2\ldots k+1|k+2\ldots n) = 0$ and that $V^{\#}(2\ldots k+1|k+2\ldots n) = 0$, since $\hat{U}(1|1) = \hat{V}^{\#}(1|1)$. Therefore $V^{\#}$ is not irreducible, since $V^{\#}$ is a principle minor of the link matrix $M$ with $\hat{M}(1|1)$ not irreducible. This contradiction completes the proof of Proposition 6.8.

**7. The proof of Theorem 2.5.3.** We are now in a position to prove the final proposition establishing the free factor property for pseudo-alternating links. In view of the definitions and reductions of the last two sections, it suffices to show

PROPOSITION 7.1. *Let $l$ be a special alternating link. Let $M = ||c_{ij}||$ be the link matrix of $l$ assoicated with the diagram $L$. Suppose that $M$ is irreducible. Then the group of $l$ has the free factor property.*

*Proof.* We will consider only the images to the #-side, as the proof of the other case is similar.

Suppose we choose a vextex $v_0$ on $\mathfrak{G}$ as basepoint for $\pi_1(S)$; basis $\alpha_1, \ldots, \alpha_n$ for $\pi_1(S, v_0)$; basis $A_1, \ldots, A_n$ for $X$; and coset representatives as given by (6.5) with representative function $\phi$; all as in Section 6. We can then assign to each vertex $v_i$ of $\mathfrak{G}$ a unique, well-defined coset representative, namely $\phi \circ i^{\#}(P_i)$, for some path $P_i$ from $v_0$ to $v_i$ in $\mathfrak{G}$. This allows us to assign to any directed edge in $\mathfrak{G}$ a unique, possibly trivial, $RS$-generator $A(j;*)^{\epsilon}$ from (6.7), where $\epsilon = \pm 1$ is determined by the orientation of the edge. It follows that whenever the given edge occurs in a loop in $\mathfrak{G}$, the corresponding occurrence of $A_j$ in the image of the loop under $i^{\#}$ rewrites to $A(j;*)^{\pm\epsilon}$ under $\tau$. We will call the edge *non-trivial* if its corresponding $RS$-generator is non-trivial.

Consider now the collection of those edges in the graph of the link which are dual to non-trivial edges in the dual graph $\mathfrak{G}$. This collection forms a subgraph of the graph which is connected and which includes the centre of each $\alpha$-region, for if not, in either case, there is a non-trivial loop in $\mathfrak{G}$ with trivial image under $i^{\#}$. Let $\mathfrak{T}^*$ be a maximal tree in this subgraph. We choose as a root for $\mathfrak{T}^*$, the vertex $x_0$ of the region $r_{n+1}$, and assign to each vertex of $\mathfrak{T}^*$ (and therefore to each $\alpha$-region) a non-negative integer, or *level*, namely the minimal number of edges of $\mathfrak{T}^*$ separating this vertex from $x_0$. We can assume that the bounded regions $r_j$ have been indexed so that $i < k$ implies level $(r_i) \geqq$ level $(r_k)$. Then each region $r_j$ will be connected via an edge of $\mathfrak{T}^*$ to a unique region of one lower level, and we substitute $i^\# (\alpha_j)$ for the non-trivial $RS$-generator $A(i_j; *_j)$ which corresponds to the edge of $\mathfrak{G}$ dual to this connecting edge. To see that this is a free substitution, we note that by Proposition 6.8 this generator can appear at most once in $\tau(\Lambda_j)$, while by the above construction it does not appear at all in $\tau(\Lambda_k)$ for $k < j$. If we choose the paths $P_i$, $1 \leqq i \leqq n$, in the complement of $\mathfrak{T}^*$, then in fact the generator $A(i_j; *_j)$ will not appear in any $\tau(P_i)$, nor in $\tau(\alpha_k)$ for $k < j$. It follows that these free substitutions can be performed on the regions in order, completing the proof of Proposition 7.1 and Theorem 2.5.3.

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