A CLASS OF THREE-GENERATOR, THREE-RELATION, FINITE GROUPS

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Mennicke (**2**) has given a class of three-generator, three-relation finite groups. In this paper we present a further class of three-generator, three-relation groups which we show are finite.

The groups presented are defined as:

$$G_1(\alpha,\beta,\gamma) = \{a,b,c \mid c^{-1}ac = a^{\alpha}, cbc^{-1} = b^{\beta}, c^{\gamma} = a^{-1}b^{-1}ab\},$$
$$G_2(\alpha,\beta,\gamma) = \{a,b,c \mid c^{-1}ac = a^{\alpha}, c^{-1}bc = b^{\beta}, c^{\gamma} = a^{-1}b^{-1}ab\},$$

with $\alpha^{|\gamma|} \neq 1$, $\beta^{|\gamma|} \neq 1$, $\gamma \neq 0$.

We prove the following result.

THEOREM 1. *Each of the groups presented is a finite soluble group.*

We state the following theorem proved by Macdonald (**1**).

THEOREM 2. $G_1(\alpha, \beta, 1)$ *is a finite nilpotent group.*

1. In this section we make some elementary remarks.

Suppose that in each case, c has finite order; then $G_2(\alpha, \beta, \gamma)$ is a factor group of $G_1(\alpha, \delta, \gamma)$ for suitable δ and it follows from Theorem 2 that the normal subgroup $N_i(\alpha, \beta, \gamma)$ of $G_i(\alpha, \beta, \gamma)$ generated by a and b is a finite nilpotent group, and since $G_i'(\alpha, \beta, \gamma)$ is a subgroup of $N_i(\alpha, \beta, \gamma)$, we have $G_i'(\alpha, \beta, \gamma)$ is a finite nilpotent group, whence $G_i(\alpha, \beta, \gamma)$ is a finite soluble group. Furthermore, finiteness of the order of c follows if we show that c is of finite order in all cases with γ equal to ± 1, since c^{γ} of finite order implies c of finite order; and since $G_2(\alpha, \beta, -\gamma) \cong G_2(\beta, \alpha, \gamma)$, the theorem will be proved if we show that c has finite order in $G_1(\alpha, \beta, -1)$ and $G_2(\alpha, \beta, 1)$.

The groups $G_i(0, \beta, \gamma)$ and $G_i(\alpha, 0, \gamma)$ are easily treated, for then the groups are finite metacyclic.

If we add relations implying that $G_i(\alpha, \beta, \gamma)$ is a commutative group, we see that all groups other than $G_i(2, 2, \pm 1)$ have order greater than 1.

Theorem 1 when proved together with Theorem 2 imply that $G_i(2, 2, \pm 1)$ is trivial.

2. Finiteness of the order of c in $G_1(\alpha, \beta, -1)$. Note that

$$G_1(\alpha, \beta, -1) \cong G_1(\beta, \alpha, -1),$$

thus we may assume that $\alpha \geqq \beta$, giving three possible cases:

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Case (i): $\alpha > 1, \beta > 1$,
Case (ii): $\alpha < 0, \beta < 0$,
Case (iii): $\alpha > 1, \beta < 0$.

We will prove Case (i) in detail, the other two cases being essentially similar.

The defining relations for $G_1(\alpha, \beta, -1)$ are:

$$c^{-1}ac = a^{\alpha}, \tag{2.1}$$

$$cbc^{-1} = b^{\beta}, \tag{2.2}$$

$$b^{-1}ab = ac^{-1}, \tag{2.3}$$

or

$$a^{-1}ba = bc. \tag{2.4}$$

From (2.1) and (2.3) we have, for $\omega > 0$,

$$b^{-1}a^{\omega}b = (ac^{-1})^{\omega} = a^{1+\alpha+\dots+\alpha^{\omega-1}}c^{-\omega}; \tag{2.5}$$

similarly, (2.2) and (2.4) yield, for $\omega > 0$,

$$a^{-1}b^{\omega}a = (bc)^{\omega} = b^{1+\beta+\dots+\beta^{\omega-1}}c^{\omega}. \tag{2.6}$$

The relation to which calculation will be applied is

$$c(a^{-\alpha}b^{-1}a^{\alpha}b)c^{-1} = a^{-1}b^{-\beta}ab^{\beta}. \tag{2.7}$$

With $\alpha > 1, \beta > 1$ we have:

$$\begin{aligned} c(a^{-\alpha}b^{-1}a^{\alpha}b)c^{-1} &= ca^{-\alpha}(b^{-1}a^{\alpha}b)c^{-1} \\ &= ca^{-\alpha}a^{1+\alpha+\dots+\alpha^{\alpha-1}}c^{-\alpha-1} \quad \text{by (2.5)} \\ &= ca^{1+\alpha^2+\alpha^3+\dots+\alpha^{\alpha-1}}c^{-\alpha-1} \end{aligned} \tag{2.8}$$

which, together with

$$\begin{aligned} a^{-1}b^{-\beta}ab^{\beta} &= (a^{-1}b^{-\beta}a)b^{\beta} \\ &= c^{-\beta}b^{-1-\beta-\dots-\beta^{\beta-1}}b^{\beta} \quad \text{by (2.6)} \\ &= c^{-\beta}b^{-1-\beta^2-\beta^3-\dots-\beta^{\beta-1}} \end{aligned} \tag{2.9}$$

and (2.7), yields

$$a^{1+\alpha^2+\dots+\alpha^{\alpha-1}}c^{-\alpha-1} = c^{-\beta-1}b^{-1-\beta^2-\dots-\beta^{\beta-1}}, \tag{2.10}$$

conjugation of (2.10) by c yields

$$cac^{-1}a^{\alpha+\alpha^2+\dots+\alpha^{\alpha-2}}c^{-\alpha-1} = c^{-\beta-1}b^{-\beta-\beta^3-\dots-\beta^{\beta}}. \tag{2.11}$$

Elimination of $c^{\beta+1}$ from (2.10) and (2.11) yields

$$a^{1-\alpha+\alpha^2-\alpha^{\alpha}}c^{-\alpha-2} = c^{-\alpha-2}b^{1-\beta+\beta^2-\beta^{\beta}} \tag{2.12}$$

whereas elimination of $c^{\alpha+1}$ from (2.10) and (2.11) yields

$$a^{1-\alpha+\alpha^2-\alpha^{\alpha}}c^{-\beta-2} = c^{-\beta-2}b^{1-\beta+\beta^2-\beta^{\beta}}. \tag{2.13}$$

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Combining (2.12) and (2.13), we have

$$(2.14)\qquad c^{\alpha-\beta}a^{1-\alpha+\alpha^2-\alpha^\alpha}c^{\beta-\alpha} = a^{1-\alpha+\alpha^2-\alpha^\alpha}$$

whence (if $\alpha \neq \beta$) (2.1) yields a of finite order, then (2.5) yields c of finite order. In the case $\alpha = \beta > 1$, the relation (2.10) yields

$$(2.15)\qquad a^\delta c^{-\alpha-1} = c^{-\alpha-1}b^{-\delta}, \quad \text{where } \delta = 1 + \alpha^2 + \alpha^3 + \ldots + \alpha^{\alpha-1},$$

and since we have

$$(2.16)\qquad b^{-\delta} = c^{-\alpha-1}b^{-\alpha^{\alpha+1}\cdot\delta}c^{\alpha+1} = (c^{-\alpha-1}b^{-\delta}c^{\alpha+1})^{\alpha^{\alpha+1}} = a^{\delta\cdot\alpha^{\alpha+1}}$$

it follows that $b^{-\delta}$ in the centre of $G_1(\alpha, \beta, -1)$, hence b^δ and c commute, whereby (2.2) yields b of finite order, whence (2.6) yields c of finite order.

With $\alpha < 0$, $\beta < 0$, the relation (2.10) becomes

$$(2.17)\qquad a^{1+\alpha+\ldots+\alpha^{-\alpha-1}+\alpha^{1-\alpha}}c^{\alpha-1} = c^{\beta-1}b^{-1-\beta-\ldots-\beta^{-\beta-1}-\beta^{1-\beta}};$$

relations (2.12) and (2.13) become

$$(2.18)\qquad a^{1-\alpha^{-\alpha}+\alpha^{1-\alpha}-\alpha^{2-\alpha}}c^{\alpha-2} = c^{\alpha-2}b^{1-\beta^{-\beta}+\beta^{1-\beta}-\beta^{2-\beta}}$$

and

$$(2.19)\qquad a^{1-\alpha^{-\alpha}+\alpha^{1-\alpha}-\alpha^{2-\alpha}}c^{\beta-2} = c^{\beta-2}b^{1-\beta^{-\beta}+\beta^{1-\beta}-\beta^{2-\beta}},$$

which together yield c of finite order.

For the case $\alpha > 1$, $\beta < 0$, (2.10) becomes

$$(2.20)\qquad a^{1+\alpha^2+\alpha^3+\ldots+\alpha^{\alpha-1}}c^{-\alpha+\beta-1} = c^{-1}b^{1+\beta+\ldots+\beta^{-\beta-1}+\beta^{1-\beta}}$$

and as previously, c is of finite order.

3. Finiteness of the order of c in $G_2(\alpha, \beta, 1)$. Note that

$$G_2(-1, \beta, \gamma) \cong G_1(-1, \beta, \gamma) \quad \text{and} \quad G_2(\alpha, -1, \gamma) \cong G_1(\alpha, -1, \gamma).$$

Thus we may consider $|\alpha| > 1$, $|\beta| > 1$. The defining relations are:

$$(3.1)\qquad c^{-1}ac = a^\alpha,$$

$$(3.2)\qquad c^{-1}bc = b^\beta,$$

$$(3.3)\qquad b^{-1}ab = ac,$$

and the following relations hold:

$$(3.4)\qquad c^{-2}(a^2)c^2 = (a^2)^{\alpha^2}, \quad c^{-2}bc^2 = b^{\beta^2}, \quad c^2 = b^{-1}(a^2)b(a^2)^{-r},$$

where $r = (\alpha^2 + \alpha)/2$.

Let G be the group defined by (3.4), i.e.

$$G = \{a, b, c \mid c^{-1}ac = a^\alpha, c^{-1}bc = b^\beta, c = b^{-1}aba^{-\gamma}\},$$

where $\alpha > 1, \beta > 1, \gamma \geqq 1$. Then if we show that G is a finite group, Theorem 1 will follow.

4. Finiteness of G. The defining relations are:

(4.1) $$c^{-1}ac = a^{\alpha}, \quad \alpha > 1,$$

(4.2) $$c^{-1}bc = b^{\beta}, \quad \beta > 1,$$

(4.3) $$b^{-1}ab = ca^{\gamma}, \quad \gamma > 0.$$

We first prove the following result.

LEMMA. *For $n > 0$ and $m \geqq 0$ we have*

(4.4) $$a^m b^n = c^r b^s a^t$$

for suitable integers r, s, and t, each depending on m and n, where $t > \gamma m$ for $m > 1$.

Proof. Induction on n. For $n = 1$, (4.3) yields

(4.5) $$b^{-1}a^m b = c^m a^{\gamma(1+\alpha+\ldots+\alpha^{m-1})} \quad \text{for } m > 0$$

whence $a^m b = bc^m a^{\gamma(1+\alpha+\ldots+\alpha^{m-1})} = c^m b^{\beta^m} a^{\gamma(1+\alpha+\ldots+\alpha^{m-1})}$ and

$$\gamma(1 + \alpha + \ldots + \alpha^{m-1}) > \gamma m$$

for $m > 1$. Suppose that $a^m b^n = c^r b^s a^t$; then

$$\begin{aligned} a^m b^{n+1} &= c^r b^s a^t b \\ &= c^r b^s b c^t a^{\gamma(1+\alpha+\ldots+\alpha^{t-1})} \quad \text{by (4.5)} \\ &= c^{r+t} b^{(s+1)\beta^t} a^{\gamma(1+\alpha+\ldots+\alpha^{t-1})} \end{aligned}$$

and $\gamma(1 + \alpha + \ldots + \alpha^{t-1}) > \gamma m$ if $t > \gamma m$.

Now from (4.3) we have

(4.6) $$c = b^{-1}aba^{-\gamma} = b^{-\beta}a^{\alpha}b^{\beta}a^{-\gamma\alpha}$$

which with (4.4) yields

(4.7) $$c = b^{-\beta}c^r b^s a^{t-\gamma\alpha} \quad \text{with } t > \gamma\alpha$$

or

(4.8) $$c^p b^q a^m = 1$$

for suitable p, q, and m with $m \neq 0$. Conjugation by c yields

(4.9) $$a^{m(\alpha-1)} \in \{b\}$$

whereby $a^{m(\alpha-1)}$ commutes with c and (4.1) yields

(4.10) $$a^{m(\alpha-1)^2} = 1, \quad m \neq 0.$$

Substitution in (4.5) and (4.2) yields c and b of finite orders whence (4.1), (4.2), and (4.3) show that G is a finite group since every element of G may be written in the form $c^r b^s a^t$ for suitable r, s, and t.

We do not settle the question as to the precise order of each $G_i(\alpha, \beta, \gamma)$. However, if p is a prime which divides the order of G_i, then p divides either $\alpha^{|\gamma|} - 1$, $\beta^{|\gamma|} - 1$ or γ, whereby (**1**) yields an upper bound on the orders.

$G_i(3, 3, \pm 2)$ is a 2-group and $G_i(-2, -2, \pm 3)$ a 3-group.

Some of the $G_i(\alpha, \beta, \gamma)$ are well known as three-generator, four-relation groups, for example, for γ odd we have

$$G_2(2, 2, \gamma) = \{a, b, c \mid ab = ba, c^{-1}ac = a^2, c^{-1}bc = b^2, c^\gamma = 1\}.$$

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References

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