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Universal Series on a Riemann Surface

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Abstract. Every holomorphic function on a compact subset of a Riemann surface can be uniformly approximated by partial sums of a given series of functions. Those functions behave locally like the classical fundamental solutions of the Cauchy–Riemann operator in the plane.

1 Introduction

In [8], Stefanopoulos shows the existence of a series of fundamental solutions of the Cauchy-Riemann operator that are universal on subsets of $\mathbb{C}$. More precisely, he obtains the following theorems.

Theorem 1.1 *Let K be a compact subset of $\mathbb{C}$ with connected complement, and let $\{s_n\}_n$ be a countable set in $\mathbb{C}\setminus K$ with an accumulation point there. Then there exists a sequence $\{c_n\}_n$ in $\mathbb{C}$ with the property that, given $f \in \mathcal{O}(K)$, there exists an increasing sequence $\{n_k\}_k$ in $\mathbb{N}$ such that*

$$\lim_{k\to\infty} \sup_{z\in K}\Big| f(z) - \sum_{j=1}^{n_k} c_j \frac{1}{z-s_j}\Big| = 0.$$

Moreover, the set of such sequences $\{c_n\}_n$ is G_δ and dense in $\mathbb{C}^{\mathbb{N}}$, endowed with the cartesian topology, and contains a dense vector subspace of $\mathbb{C}^{\mathbb{N}}$, except for the zero sequence.

Theorem 1.2 *For $a \in \mathbb{R}$ let $\sigma = [a,\infty)$ and set $\mathbb{C}_\sigma = \mathbb{C}\setminus\sigma$. Let $\{a_n\}_n$ be a countable subset of σ with an accumulation point in σ. Then there exists a sequence $\{c_n\}_n$ in $\mathbb{C}$ with the property that, given $f \in \mathcal{O}(\mathbb{C}_\sigma)$, there exists an increasing sequence $\{n_k\}_k$ in $\mathbb{N}$ such that for any compact set $K \subset \mathbb{C}_\sigma$,*

$$\lim_{k\to\infty} \sup_{z\in K}\Big| f(z) - \sum_{j=1}^{n_k} c_j \frac{1}{z-a_j}\Big| = 0.$$

Moreover, the set of such sequences $\{c_n\}_n$ is G_δ and dense in $\mathbb{C}^{\mathbb{N}}$, endowed with the cartesian topology, and contains a dense vector subspace of $\mathbb{C}^{\mathbb{N}}$, except for the zero sequence.

The aim of this paper is to generalize those ideas to the case of non-compact Riemann surfaces. One of the main tools used by Stefanopoulos to prove those theorems was an abstract characterization of universality that we shall now present (see [8, Theorem 2.1], or [4, Proposition 7] along with [7, Theorem 1.2]).

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Let X be a complex vector space endowed with a metric d that is compatible with the vector space operations and invariant under translation. Given a sequence $x = \{x_k\}_k \subset X$, define $U(x)$ to be the set of sequences of complex numbers $\{a_k\}_k$ such that the partial sums $\sum_{k=1}^{n} a_k x_k$, $n \in \mathbb{N}$ are dense in X.

Theorem 1.3 *The following assertions are equivalent:*

- $U(x) \neq \varnothing$*;*
- *span*$\{x_n, x_{n+1}, \dots\}$ *is dense in* X *for all* $n \in \mathbb{N}$*;*
- $U(x)$ *is a dense* G_δ *set in* $\mathbb{C}^{\mathbb{N}}$ *and contains a dense subspace of* $\mathbb{C}^{\mathbb{N}}$*, except for the zero sequence.*

The other crucial result in [8] is a theorem on approximation by fundamental solutions of a differential operator, which we recall here in the particular case that is of interest to us (see [9]).

Theorem 1.4 *Let* $K \subset \mathbb{C}$ *be a compact subset and* $\sigma \subset \mathbb{C} \setminus K$ *with an accumulation point there. Then span*$\{\frac{1}{z-y} : y \in \sigma\}$ *is dense in* $\mathcal{O}(K)$*.*

The main result of this paper, proved in Section 4, is a version of the above theorem that holds in the case of open Riemann surfaces. Using it, we then proceed to establish theorems analogous to 1.1 and 1.2.

2 A Cauchy-type Integral Formula

The goal of this section is to derive an integral formula that will prove to be an invaluable tool later on. From now on, by a Riemann surface M we shall mean a connected complex manifold without boundary of dimension 1. We shall be mainly interested in the case where M is open (non-compact). We shall therefore make use of the following theorem by Gunning and Narasimhan (see [5]). For an open set $\Omega \subset M$, denote by $\mathcal{O}(\Omega)$ the set of all holomorphic functions on Ω.

Theorem 2.1 *Let* M *be an open Riemann surface. There exists* $\Phi \in \mathcal{O}(M)$*, which is a local homeomorphism.*

Define the univalence radius of Φ at $y \in M$ as $r_y = \sup A_y$, where A_y is the set of all $r > 0$ such that $\{|\zeta - \Phi(y)| < r\}$ is the biholomorphic image by Φ of a neighbourhood of y. Denoting by $B(a, r)$ the open disc in $\mathbb{C}$ with center $a \in \mathbb{C}$ and radius $r > 0$, for each $y \in M$ choose s_y such that $0 < s_y < r_y$ and set $U_y = \Phi^{-1}(B(\Phi(y), s_y))$, the closure of which is compact. The collection $\{U_y\}_{y\in M}$ is thus an open cover of M.

Lemma 2.2 *For each* $y \in M$*, let* f_y *be a meromorphic function defined on* $\Phi(U_y)$ *such that* $f_{y_1} = f_{y_2}$ *on* $\Phi(U_{y_1}) \cap \Phi(U_{y_2})$*. Then there exists a unique meromorphic* $(1,0)$*-form* ω *on* M *such that* $(\Phi^{-1})^*\omega = f_y d\zeta$ *on* $\Phi(U_y)$*.*

Proof Notice that on $\Phi(U_{y_1}) \cap \Phi(U_{y_2})$, we have

$$(\Phi|_{U_{y_2}} \circ (\Phi|_{U_{y_1}})^{-1})^*(f_{y_2} d\zeta) = (Id)^*(f_{y_2} d\zeta) = f_{y_2} d\zeta = f_{y_1} d\zeta.$$

We can therefore define ω as $(\Phi|_{U_y})^*(f_y d\zeta)$ on U_y. ∎

Since we shall use the previous result in another form, we state it here, its proof being similar to the one above.

Lemma 2.3 *For each $y_1, y_2 \in M$, let f_{y_1,y_2} be a meromorphic function defined on $\Phi(U_{y_1})\times\Phi(U_{y_2})$ such that $f_{y_1,y_2} = f_{y_3,y_4}$ on $\Phi(U_{y_1})\times\Phi(U_{y_2})\cap\Phi(U_{y_3})\times\Phi(U_{y_4})$. Then there exists a unique meromorphic $(1,0)$-form ω on $M\times M$ such that $(\Phi^{-1}\times\Phi^{-1})^*\omega = f_{y_1,y_2}d\zeta_1$ on $\Phi(U_{y_1})\times\Phi(U_{y_2})$, where ζ_1, ζ_2 are the coordinates on $\mathbb{C}\times\mathbb{C}$.*

Recall that given an open cover $\{V_i\}_i$ of M, a Mittag–Leffler distribution is a collection of meromorphic functions f_i defined on V_i such that $f_i - f_j \in \mathcal{O}(V_i \cap V_j)$ for all i, j. We say that such a distribution has a solution if there exists a meromorphic function f on M such that $f - f_i \in \mathcal{O}(V_i)$ for all i. According to [6, Theorem 5.5.1], we have the following.

Theorem 2.4 *If M is a Stein manifold, every Mittag–Leffler distribution has a solution.*

Moreover, [2, Corollary 26.8] gives the following.

Theorem 2.5 *Every open Riemann surface is Stein.*

Let $\{U_y = \Phi^{-1}(B(\Phi(y), s_y))\}_y$ be the usual open cover of M, and let $\{V_\alpha\}$ be an open cover of $M \times M \setminus \{(p,p) : p \in M\}$, that is, an open cover of $M \times M$ without its diagonal, where each V_α is a subset of $M \times M \setminus \{(p,p) : p \in M\}$ that can be expressed as a product of open subsets of M on which Φ is biholomorphic. Then $\{U_y \times U_y, V_\alpha\}_{y,\alpha}$ is an open cover of $M \times M$. Set $f_y = \frac{1}{\Phi(p)-\Phi(q)}$, which is meromorphic on $U_y \times U_y$ and $f_\alpha = 0$ on V_α. It is easily checked, using the fact that Φ is injective, that this gives a Mittag–Leffler distribution. Now, we know that M is Stein by the previous theorem and thus that $M \times M$ is also Stein, see [3]. Hence, by Theorem 2.4, there exists a meromorphic function $C(p,q)$ defined on $M \times M$ which is a solution to our Mittag–Leffler problem. This function C, which has the same local behaviour as a translated fundamental solution of the Cauchy–Riemann operator in the plane, is the candidate to replace the functions of the type $\frac{1}{z-a}$ in the statements of our theorems corresponding to 1.1 and 1.2.

Lemma 2.6 *Let f_1, f_2 be meromorphic functions on M which are both solutions of the Mittag-Leffler problem $\{(V_i, g_i)\}$, where each V_i is a chart. Then, $f_1 - f_2 \in \mathcal{O}(M)$.*

By considering $g_y(p,q) = f_y(q,p) = -f_y(p,q)$ on $U_y \times U_y$ and $g_\alpha = 0$ on V_α, we see that $C(q,p)$ and $-C(p,q)$ are both solutions of this new distribution. By Lemma 2.6, we get that $C(p,q) = -C(q,p) + h(p,q)$ with $h \in \mathcal{O}(M \times M)$. This relation will be of great use later. Now, by Lemma 2.3, there exists a $(1,0)$-form, which we shall denote by $\gamma(p,q)$, that can locally be expressed as

$$C \circ (\Phi^{-1} \times \Phi^{-1}) d\zeta.$$

It is then clear that $\gamma(p,q)$ is holomorphic away from the diagonal $\{(p,p) : p \in M\}$, by construction of $C(p,q)$. Using $\gamma(p,q)$, the formula we seek is within our reach.

Theorem 2.7 *Let $f \in C_c^\infty(M)$, $y \in M$ and $U = \Phi^{-1}(B(\Phi(y), s_y))$. For $0 < \epsilon < 1$, define $U_\epsilon = \{p \in U : |\Phi(p) - \Phi(y)| < \epsilon s_y\}$ and set $M_\epsilon = M \setminus U_\epsilon$. Then,*

$$-2\pi i f(y) = \lim_{\epsilon \to 0} \int_{M_\epsilon} \gamma(\,\cdot\,, y) \wedge \overline{\partial} f.$$

Proof First note that $\gamma(\,\cdot\,, y)$ is holomorphic on M_ϵ. Moreover, since $\gamma(\,\cdot\,, y) \wedge f$ is of type $(1, 0)$ and M is of complex dimension 1, we have that $\partial(\gamma(\,\cdot\,, y) \wedge f) = 0$ on M_ϵ. Hence, we have

$$\begin{aligned} d(\gamma(\,\cdot\,, y) \wedge f) &= \partial(\gamma(\,\cdot\,, y) \wedge f) + \overline{\partial}(\gamma(\,\cdot\,, y) \wedge f) = \overline{\partial}(\gamma(\,\cdot\,, y) \wedge f) \\ &= \overline{\partial}\gamma(\,\cdot\,, y) \wedge f + \gamma(\,\cdot\,, y) \wedge \overline{\partial} f = \gamma(\,\cdot\,, y) \wedge \overline{\partial} f \end{aligned}$$

on M_ϵ.

Note that M_ϵ is a manifold with boundary, so by Stokes' theorem, we get

$$\int_{\partial M_\epsilon} \gamma(\,\cdot\,, y) \wedge f = \int_{M_\epsilon} d(\gamma(\,\cdot\,, y) \wedge f) = \int_{M_\epsilon} \gamma(\,\cdot\,, y) \wedge \overline{\partial} f.$$

On the other hand, on $\Phi(U)$ we have

$$(\Phi|_U^{-1})^*(\gamma(\,\cdot\,, y)) = \big(C(\,\cdot\,, y) \circ \Phi|_U^{-1}\big)\, d\zeta = \Big(\frac{1}{\zeta - \Phi(y)}\Big)\, d\zeta + g(\zeta, y) d\zeta,$$

where g is a holomorphic function of ζ, hence

$$\begin{aligned} \lim_{\epsilon \to 0} \int_{\partial U_\epsilon} \gamma(\,\cdot\,, y) \wedge f &= \lim_{\epsilon \to 0} \int_{|\zeta - \Phi(y)| = \epsilon s_y} \Big(\frac{(f \circ \Phi^{-1})(\zeta)}{\zeta - \Phi(y)} + g(\zeta, y)(f \circ \Phi^{-1})(\zeta) \Big)\, d\zeta \\ &= \lim_{\epsilon \to 0} \int_0^{2\pi} (f \circ \Phi^{-1})(\Phi(y) + \epsilon s_y e^{i\theta}) i (1 + \epsilon s_y e^{i\theta} g(\zeta, y)) d\theta \\ &= 2\pi i (f \circ \Phi^{-1})(\Phi(y)) = 2\pi i f(y). \end{aligned}$$

Finally, we obtain that

$$\begin{aligned} \lim_{\epsilon \to 0} \int_{\partial M_\epsilon} \gamma(\,\cdot\,, y) \wedge f &= -\lim_{\epsilon \to 0} \int_{\partial U_\epsilon} \gamma(\,\cdot\,, y) \wedge f = -2\pi i f(y) \\ &= \lim_{\epsilon \to 0} \int_{M_\epsilon} \gamma(\,\cdot\,, y) \wedge \overline{\partial} f. \end{aligned}$$

■

3 Statement of the Main Result

Using the notation introduced in the previous section, we have the following.

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Theorem 3.1 *Let M be an open Riemann surface, $K \subset M$ a compact subset and $\sigma \subset (M \setminus K)$ with an accumulation point in each component of $M \setminus K$. Then $\Sigma = span\{C(\cdot, y) : y \in \sigma\}$ is dense in $\mathcal{O}(K)$. Moreover, for all $f \in \mathcal{O}(K)$, there exists $\{f_n\} \subset \Sigma$ such that $f_n \to f$ uniformly on K and for all $p \in K$ and for all chart (V, ψ) containing p,*

$$\frac{\partial^\alpha}{\partial x^\alpha}(f_n \circ \psi^{-1})(\psi(p)) \to \frac{\partial^\alpha}{\partial x^\alpha}(f \circ \psi^{-1})(\psi(p)),$$

for all multi-index α.

The proof, exposed in the next section, uses a classical argument to show density, namely the Hahn–Banach Theorem, along with an application of the Cauchy integral formula established in Theorem 2.7.

4 Approximation by Fundamental Solutions of the Cauchy–Riemann Operator

We begin with a lemma that provides us with a way to approximate an open Riemann surface by a sequence of compact subsets; see [2, Corollary 23.6]. Recall first that given a compact subset $K \subset M$, $h_M(K)$ stands for the union of K with all the relatively compact components of its complement.

Lemma 4.1 *Let M be an open Riemann surface. There exists a sequence $\{K_j\}$ of compact subsets of M such that:*

- *$\bigcup_j K_j = M$;*
- *$K_j \subset int(K_{j+1})$;*
- *If $K \subset M$ is compact, then there exists $j \in \mathbb{N}$ such that $K \subset K_j$*
- *$h_M(K_j) = K_j$.*

Given an open set $\Omega \subset M$, let $\{K_j\}$ be such an exhaustion of Ω. If $f, g \in C(\Omega)$, define

$$d(f, g) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{\sup_{K_j} |f - g|}{1 + \sup_{K_j} |f - g|}.$$

Endow $C(M)$ with this metric. An element $G \in (C(M))'$ is a linear functional on $C(M)$ that is continuous with respect to the topology induced by d. We shall set out a few lemmas before fulfilling our promise to prove the main result. The first one is obvious.

Lemma 4.2 *Let M, N be Riemann surfaces, $G \in (C(M))'$ and $f \in C(M \times N)$. Then $G(f(\cdot, q))$ is continuous in q.*

We also need the following, which is not very surprising.

Lemma 4.3 *Let $K \subset M$ be a compact subset, $G \in (C(M))'$ with compact support contained in the interior of K and $g \in C^\infty(K \times (M \setminus K))$ holomorphic in the second variable. Let $\rho \in C_c^\infty(M)$ such that $0 \le \rho \le 1$, $\rho = 1$ on $\operatorname{supp}(G)$ and $\operatorname{supp}(\rho) \subset K$. Extending $\rho(\cdot)g(\cdot, q)$ by zero outside the support of ρ, we have that $G[\rho(\cdot)g(\cdot, q)]$ is holomorphic in the second variable for $q \notin K$.*

It is also easy to prove that if $h \in \mathcal{O}(M \times M)$, then $G[h(p, \cdot)]$ is holomorphic on M. We now recall the definition and a result on the compactification of a Riemann surface. See [1].

Definition 4.4 Let M be a Riemann surface and N be a topological space. Let $\Psi\colon M \to N$ be a homeomorphism onto $\Psi(M) \subset N$. We say that Ψ is a *compactification* of M if

- N is compact;
- $\Psi(M) \subset N$ is open;
- $\Psi(M)$ is dense in N.

We set $\beta = N \setminus \Psi(M)$.

Theorem 4.5 *Let M be a Riemann surface. There exists a unique compactification $\Psi\colon M \to \overline{M}$ of M such that*

- *$\overline{M}$ is a locally connected Hausdorff space;*
- *β is totally disconnected;*
- *β is non-separating in $\overline{M}$: for each open connected subset $G \subset \overline{M}$, $G \setminus \beta$ is connected.*

The following fact about complementary connected components of compact subsets of M is well known.

Lemma 4.6 *Let M be an open Riemann surface. If $K \subset M$ is compact, then $M \setminus K$ has only finitely many non relatively compact connected components.*

Proof Let $\{D_\alpha\}_{\alpha \in A}$ be the non relatively compact connected components of $M \setminus K$. Let $\Psi\colon M \to \overline{M}$ be the unique compactification of M of Theorem 4.5. Since $\overline{M}$ is compact and locally connected, there exist finitely many open connected sets $V_1, \dots, V_N$ of $\overline{M}$ such that $\beta \subset \bigcup_{i=1}^N V_i$ and $\Psi(K) \cap V_i = \varnothing$ for all i. Suppose that $\Psi(D_\alpha) \subset \overline{M} \setminus \bigcup_{i=1}^N V_i$. Seeing as $\overline{M}$ is compact, $\Psi(D_\alpha)$ must be relatively compact in $\overline{M}$. However, since $\overline{M} \setminus \bigcup_{i=1}^N V_i$ is closed, we have $\overline{\Psi(D_\alpha)} \subset \overline{M} \setminus \bigcup_{i=1}^N V_i \subset \Psi(M)$. But then we get that D_α is relatively compact in M, because Ψ is a homeomorphism on $\Psi(M)$, an open subset of $\overline{M}$. This contradiction shows that for each $\alpha \in A$, there exists i_α such that $\Psi(D_\alpha) \cap (V_{i_\alpha} \setminus \beta) \neq \varnothing$. Moreover, $V_i \setminus \beta \subset \Psi(M) \setminus \Psi(K)$ is connected for each i, since β is non-separating in $\overline{M}$, and so $\Psi^{-1}(V_i \setminus \beta) \subset M \setminus K$ is connected also. Hence, two connected components $D_{\alpha_1}, D_{\alpha_2}$ meeting $\Psi^{-1}(V_i \setminus \beta)$ must belong to the same connected component, which in turn implies that $|A| \leq N$. ■

The next step is a bit technical, but of the utmost importance to us, as it provides a kind of integral representation of the compactly supported continuous linear functionals on $C(M)$.

Lemma 4.7 *Let $G \in (C(M))'$ with support contained in $K \subset M$ compact, W a relatively compact neighbourhood of K and $f \in C_c^\infty(M) \cap \mathcal{O}(\overline{W})$. If $\theta \in C_c^\infty(M)$ is such that $\theta = 1$ on K and* $\operatorname{supp}(\theta) \subset W$*, then*

$$G\Big(\theta(y) \int_{M \setminus \overline{W}} \gamma(\cdot, y) \wedge \overline{\partial} f\Big) = -\int_{M \setminus \overline{W}} \alpha \wedge \overline{\partial} f$$

for some form α.

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Proof First note that G will act on functions of the y variable, a fact we shall emphasize with the notation G_y. Also, the function

$$y \mapsto \theta(y) \int_{M \setminus \overline{W}} \gamma(\,\cdot\,, y) \wedge \overline{\partial} f$$

is continuous on M if extended by zero outside the support of θ. Taking $\{U_i\}$ an open cover of $M \setminus \overline{W}$ such that:

- U_i is locally finite for all i;
- $\overline{U_i} \cap \operatorname{supp}(\theta) = \varnothing$ for all i;
- for all i there exists $y_i \in M$ and $s_{y_i} \in A_{y_i}$ such that $U_i = \Phi^{-1}(B(\Phi(y_i), s_{y_i}))$,

and $\{\rho_i\}$ a partition of unity subordinate to $\{U_i\}$, we get

$$\begin{aligned}
&G_y\Big(\theta(y) \int_{M\setminus\overline{W}} \gamma(\,\cdot\,, y) \wedge \overline{\partial} f\Big) \\
&\quad = G_y\Big(\sum_{i \in I} \int_{\Phi(U_i)} (\Phi^{-1})^*(\rho_i(p)\theta(y)C(p,y)) \frac{\partial(f \circ \Phi^{-1})}{\partial \overline{\zeta}} d\zeta d\overline{\zeta}\Big) \\
&\quad = \sum_{i \in I} G_y\Big(\int_{\Phi(U_i)} (\Phi^{-1})^*(\rho_i(p)\theta(y)C(p,y)) \frac{\partial(f \circ \Phi^{-1})}{\partial \overline{\zeta}} d\zeta d\overline{\zeta}\Big)
\end{aligned}$$

where the last equality holds since the sum is finite. f is holomorphic on a neighbourhood of $\overline{W}$ and is of compact support in M, thus $\overline{\partial} f$ is of compact support in $M \setminus \overline{W}$, and it suffices to integrate on the said support. By the choice of U_i, notice that $\overline{\Phi(U_i)}$ is compact and that Φ is injective on a neighbourhood V_i of $\overline{U_i}$. We can thus integrate on $\Phi(\overline{U_i}) = \overline{\Phi(U_i)}$ rather than on $\Phi(U_i)$, which will not change the value of the integral, since the boundary of that set is obviously of measure zero. We obtain

$$\begin{aligned}
&G\Big(\theta(y) \int_{M\setminus\overline{W}} \gamma(\,\cdot\,, y) \wedge \overline{\partial} f\Big) \\
&\quad = \sum_{i \in I} G_y\Big(\int_{\overline{\Phi(U_i)}} (\Phi^{-1})^*(\rho_i(p)\theta(y)C(p,y)) \frac{\partial(f \circ \Phi^{-1})}{\partial \overline{\zeta}} d\zeta d\overline{\zeta}\Big) \\
&\quad = \sum_{i \in I} \int_{\overline{\Phi(U_i)}} (\Phi^{-1})^*(\rho_i(p)) G_y\left[(\Phi^{-1})^*(\theta(y)C(p,y))\right] \frac{\partial(f \circ \Phi^{-1})}{\partial \overline{\zeta}} d\zeta d\overline{\zeta},
\end{aligned}$$

where, once again, we extend $(\Phi^{-1})^*(\theta(y)C(p,y))$ by zero outside the support of θ in order for it to be continuous in y on M. The Riemann sums of the integral in the middle expression converge uniformly in y on compact subsets of M to the said integral, so the fact that G is continuous and linear yields the last equality. Note that this last integral actually makes sense because of Lemma 4.2. Using the relation given by Lemma 2.6 between $C(p, y)$ and $C(y, p)$, we can write

$$\begin{aligned}
&G\Big(\theta(y)\int_{M\setminus\overline{W}}\gamma(\,\cdot\,,y)\wedge\overline{\partial}f\Big)\\
&\quad=\sum_{i\in I}\int_{\overline{\Phi(U_i)}}(\Phi^{-1})^*(\rho_i(p))G_y\left[(\Phi^{-1})^*(-\theta(y)C(y,p))\right]\frac{\partial(f\circ\Phi^{-1})}{\partial\overline{\zeta}}d\zeta d\overline{\zeta}\\
&\qquad+\sum_{i\in I}\int_{\overline{\Phi(U_i)}}(\Phi^{-1})^*(\rho_i(p))G_y\left[(\Phi^{-1})^*(\theta(y)h(p,y))\right]\frac{\partial(f\circ\Phi^{-1})}{\partial\overline{\zeta}}d\zeta d\overline{\zeta}\\
&\quad=\sum_{i\in I}\int_{\Phi(U_i)}(\Phi^{-1})^*(\rho_i(p))G_y\left[(\Phi^{-1})^*(-\theta(y)C(y,p))\right]\frac{\partial(f\circ\Phi^{-1})}{\partial\overline{\zeta}}d\zeta d\overline{\zeta}\\
&\qquad+\sum_{i\in I}\int_{\Phi(U_i)}(\Phi^{-1})^*(\rho_i(p))G_y\left[(\Phi^{-1})^*(h(p,y))\right]\frac{\partial(f\circ\Phi^{-1})}{\partial\overline{\zeta}}d\zeta d\overline{\zeta}
\end{aligned}$$

since $h \in \mathcal{O}(M \times M)$, and thus

$$G_y\left[(\Phi^{-1})^*(\theta(y)h(p,y))\right] = G_y\left[(\Phi^{-1})^*(h(p,y))\right],$$

because $\theta = 1$ on a neighborhood of supp G.

Now, by Lemma 2.2 applied to $M \setminus \overline{W}$, we obtain a form α that can be expressed locally as

$$G_y\left[(\Phi^{-1})^*(\theta(y)C(y,p))\right]d\zeta = G_y\left[\theta(y)C(y,\Phi^{-1}(\zeta))\right]d\zeta$$

on $\Phi(U_i)$. On the other hand, $G_y\left[(\Phi^{-1})^*h(p,y)\right]$ is holomorphic on M in the first variable by the remark made below Lemma 4.3, so applying Lemma 2.2 once again yields a holomorphic $(1,0)$-form β such that

$$G\Big(\theta(y)\int_{M\setminus\overline{W}}\gamma(\,\cdot\,,y)\wedge\overline{\partial}f\Big) = -\int_{M\setminus\overline{W}}\alpha\wedge\overline{\partial}f + \int_{M\setminus\overline{W}}\beta\wedge\overline{\partial}f.$$

Since f has compact support, we invoke Stokes' theorem to write

$$\begin{aligned}
0 &= \int_M d(\beta\wedge f) = \int_M \overline{\partial}(\beta\wedge f) = \int_M \overline{\partial}\beta\wedge f + \int_M \beta\wedge\overline{\partial}f\\
&= \int_{M\setminus\overline{W}}\beta\wedge\overline{\partial}f + \int_{\overline{W}}\beta\wedge\overline{\partial}f = \int_{M\setminus\overline{W}}\beta\wedge\overline{\partial}f
\end{aligned}$$

and thus

$$G\Big(\theta(y)\int_{M\setminus\overline{W}}\gamma(\,\cdot\,,y)\wedge\overline{\partial}f\Big) = -\int_{M\setminus\overline{W}}\alpha\wedge\overline{\partial}f. \qquad\blacksquare$$

Since $\mathcal{O}(K) \subset C(K)$, we may endow $\mathcal{O}(K)$ with the topology induced by that of $C(K)$. Before proving our main result in its full generality, we must first establish a special case.

Theorem 4.8 *Let M be an open Riemann surface, and let $K \subset M$ be a compact subset such that $M \setminus K$ has finitely many connected components. Let $\sigma \subset (M \setminus K)$ with an accumulation point in each component of $M \setminus K$. Then $\Sigma = span\{C(\cdot, y) : y \in \sigma\}$ is dense in $\mathcal{O}(K)$.*

Proof Since $M \setminus K$ only has finitely many connected components, we can easily find $\sigma' \subset \sigma$ with a limit point in each of these connected components such that the distance between σ' and K is positive. We shall actually show that $\Sigma' = span\{C(\cdot, y) : y \in \sigma'\} \subset \Sigma$ is dense in $\mathcal{O}(K)$.

Let $g \in (C(K))'$ and suppose $g|_{\Sigma'} = 0$, in other words suppose $g(C(\cdot, y)) = 0$ for all $y \in \sigma'$. By the Hahn–Banach Theorem, it suffices to show that we have $g(\psi) = 0$ for all $\psi \in \mathcal{O}(K)$. To do so, we shall want to use our Cauchy-type integral formula, so define G, a non-zero element of $(C(M))'$ supported by K, by setting $G(f) = g(f|_K)$.

Now fix an arbitrary $\psi \in \mathcal{O}(K)$, which is holomorphic on an open neighbourhood Z of K by definition. Since the distance between K and σ' is positive, we can find W a relatively compact open neighbourhood of K such that $\sigma' \cap \overline{W} = \varnothing$ and $\overline{W} \subset Z$. Moreover, we can choose W small enough so that σ' has an accumulation point in every connected component of $M \setminus \overline{W}$. Let $\rho \in C_c^\infty(M)$ such that $0 \leq \rho \leq 1$, $\rho = 1$ on a neighbourhood of K and $\operatorname{supp}(\rho) \subset W$. Notice that $C(p, y) \in \mathcal{O}(W \times (M \setminus \overline{W}))$, and extending by zero outside of $\operatorname{supp}(\rho)$, we also have $\rho(p)C(p, y) \in C^\infty(M \times (M \setminus \overline{W}))$. Since $\rho(p)C(p, y)$ is holomorphic in the second variable for $y \notin \overline{W}$, we can apply Lemma 4.3 to conclude that $G_p(\rho(p)C(p, y))$ is also holomorphic for $y \notin \overline{W}$. Now, $G_p(\rho(p)C(p, y)) = g(C(\cdot, y)) = 0$ for $y \in \sigma' \subset \sigma$, and since σ' accumulates in each connected component of $M \setminus \overline{W}$, this means that $G_p(\rho(p)C(p, y)) = 0$ for all $y \in M \setminus \overline{W}$.

Let $\rho_1 \in C_c^\infty(M)$ such that $0 \leq \rho_1 \leq 1$, $\rho_1 = 1$ on a neighbourhood of $\overline{W}$ and $\operatorname{supp}(\rho_1) \subset Z$. Then $\overline{\partial}(\rho_1\psi) = 0$ on $\overline{W}$. Since $\rho_1\psi \in C_c^\infty(M)$, using our Cauchy formula yields

$$\begin{aligned} g(\psi) = G(\rho_1\psi) &= G_y\Big(-\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{M_\epsilon}\gamma(\cdot, y)\wedge\overline{\partial}(\rho_1\psi)\Big) \\ &= G_y\Big(-\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{M_\epsilon\setminus\overline{W}}\gamma(\cdot, y)\wedge\overline{\partial}(\rho_1\psi)\Big) \\ &= G_y\Big(-\lim_{\epsilon\to 0}\frac{\rho(y)}{2\pi i}\int_{M_\epsilon\setminus\overline{W}}\gamma(\cdot, y)\wedge\overline{\partial}(\rho_1\psi)\Big), \end{aligned}$$

where the last inequality holds because $\rho = 1$ on a neighbourhood of $\operatorname{supp}(G)$. But $\operatorname{supp}(\rho) \subset W$, and it is clear that $\gamma(\cdot, y) \wedge \overline{\partial}(\rho_1\psi)$ is smooth over $M \setminus \overline{W}$ if $y \in W$, so extending by zero outside $\operatorname{supp}(\rho)$, we have

$$\begin{aligned} g(\psi) &= G_y\Big(-\lim_{\epsilon\to 0}\frac{\rho(y)}{2\pi i}\int_{M_\epsilon\setminus\overline{W}}\gamma(\cdot, y)\wedge\overline{\partial}(\rho_1\psi)\Big) \\ &= G_y\Big(-\frac{\rho(y)}{2\pi i}\int_{M\setminus\overline{W}}\gamma(\cdot, y)\wedge\overline{\partial}(\rho_1\psi)\Big). \end{aligned}$$

Using Lemma 4.7, we find

$$g(\psi) = \frac{1}{2\pi i}\int_{M\setminus\overline{W}} \alpha \wedge \overline{\partial}(\rho_1\psi).$$

Now let $p \in M \setminus \overline{W}$. By the proof of Lemma 4.7, there exists $V \subset M \setminus \overline{W}$ a relatively compact open set containing p such that Φ is injective on a neighbourhood of V and

$$(\Phi^{-1})^*\alpha = G_y\left[(\Phi^{-1})^*(\rho(y)C(y,p))\right]d\zeta = G_y\left[\rho(y)C(y,\Phi^{-1}(\zeta))\right]d\zeta$$

on $\Phi(V)$. Since $G_y\left[\rho(y)C(y,p)\right]$ vanishes on $M\setminus\overline{W} \supset V$, we see that $\operatorname{supp}(\alpha) \subset \overline{W}$. Therefore,

$$g(\psi) = \frac{1}{2\pi i}\int_{M\setminus\overline{W}} \alpha \wedge \overline{\partial}(\rho_1\psi) = 0. \qquad \blacksquare$$

Rejoice, for we can now prove Theorem 3.1.

Proof Set $M \setminus K = \bigcup_{\alpha\in A} B_\alpha \cup D_1 \cup \cdots \cup D_N$, where each B_α is a relatively compact connected component and each D_i is a non relatively compact connected component (see Lemma 4.6). K being compact implies that $h_M(K) = K \cup \bigcup_{\alpha\in A} B_\alpha$ is also compact (see [2, Theorem 23.5]), hence bounded. Because the B_α are disjoint, either there are finitely many B_α and then the result follows from Theorem 4.8, or else for each $R > 0$ there are only finitely many B_α whose inner radius

$$r_\alpha := \sup_{x\in B_\alpha} \sup\{r \geq 0 : B(x,r) \subset B_\alpha\}$$

is greater than or equal to R. In that case, set $K_j = K \cup \bigcup_{r_\alpha \leq 1/j} B_\alpha$, which has the property that $M \setminus K_j$ only has finitely many connected components. We can thus apply Theorem 4.8 to K_j with $\sigma_j := \sigma \cap (M \setminus K_j)$ to get that $\Sigma_j = span\{C(\,\cdot\,,y) : y \in \sigma_j\}$, and hence Σ, is dense in $\mathcal{O}(K_j)$.

Fortunately, there is a way to recover $\mathcal{O}(K)$ from the $\mathcal{O}(K_j)$. Indeed, notice that if $f \in \mathcal{O}(K)$, there exists an open set $U \supset K$ such that $f \in \mathcal{O}(U)$ and

$$\delta_f = \inf_{x\in\partial U, y\in K} d(x,y) > 0.$$

But then $f \in \mathcal{O}(K_j)$ for all j such that $2/j \leq \delta_f$, so $\mathcal{O}(K) = \bigcup_{j=1}^{\infty} \mathcal{O}(K_j)$, and Σ is dense in $\mathcal{O}(K)$.

To prove the remaining statement, let $f \in \mathcal{O}(K)$. We just showed that there exists $j_0 \in \mathbb{N}$ such that $f \in \mathcal{O}(K_{j_0})$. As in the proof of Theorem 4.8, replace σ_{j_0} with $\sigma'_{j_0} \subset \sigma_{j_0}$ such that $d(\sigma'_{j_0}, K_{j_0}) > 0$ to get $\{f_n\} \in \Sigma'_{j_0}$ such that $f_n \to f$ uniformly on $K_{j_0} \supset K$. But since $d(\sigma'_{j_0}, K_{j_0}) > 0$, there is a relatively compact open set V such that $K \subset V$, $f \in \mathcal{O}(V)$ and $f_n \in \mathcal{O}(V)$ for all $n \in \mathbb{N}$. Fix $p \in K$, and let (U,ψ) be a chart containing p with $U \subset V$. By the classical Cauchy formula, we know that for every multi-index α,

$$\frac{\partial^\alpha}{\partial x^\alpha}(f_n \circ \psi^{-1}) \to \frac{\partial^\alpha}{\partial x^\alpha}(f \circ \psi^{-1})$$

uniformly on compact subsets of $\psi(U)$, and thus on $\psi(p)$. $\blacksquare$

This allows us to establish the existence of a series of "fundamental solutions" that is universal on a compact subset of M.

Corollary 4.9 *Let M be an open Riemann surface, $K \subset M$ a compact subset, and $\{a_j\}_j \subset M \setminus K$ with an accumulation point in each connected component of $M \setminus K$. Then there exists a sequence $\{b_j\}_j$ in $\mathbb{C}$ with the property that, given $f \in \mathcal{O}(K)$, there exists an increasing sequence $\{n_k\}_k$ in $\mathbb{N}$ such that*

$$\lim_{k\to\infty} \sup_{z\in K} |f(z) - \sum_{j=1}^{n_k} b_j C(z, a_j)| = 0.$$

Moreover, the set of such sequences $\{b_j\}_j$ is G_δ and dense in $\mathbb{C}^{\mathbb{N}}$, endowed with the cartesian topology, and contains a dense vector subspace of $\mathbb{C}^{\mathbb{N}}$, except for the zero sequence.

Proof Apply Theorem 3.1 to $\{a_j\}_{j\geq J}$ and use Theorem 1.3. ∎

5 Universality on Open Subsets

By working a little harder, it is also possible to generalize Theorem 1.2 for open subsets of Riemann surfaces. Call an exhaustion *regular* if it has the four properties of Lemma 4.1. Now, let $\Omega \subset M$ be an open subset, and let $\{K_k\}, \{L_l\}$ be two regular exhaustions of Ω. For each $k \in \mathbb{N}$ (respectively $l \in \mathbb{N}$) , let P_k (respectively Q_l) be a connected component of $\Omega \setminus K_k$ (respectively $\Omega \setminus L_l$). Two sequences of pairs $\{K_k, P_k\}_{k\in\mathbb{N}}$ and $\{L_l, Q_l\}_{l\in\mathbb{N}}$ such that $P_{k+1} \subset P_k$ for all $k \in \mathbb{N}$ and $Q_{l+1} \subset Q_l$ for all $l \in \mathbb{N}$ are said to be equivalent if and only if for all $k \in \mathbb{N}$ there exists $l_k \in \mathbb{N}$ such that $Q_{l_k} \subset P_k$ and for all $l \in \mathbb{N}$ there exists $k_l \in \mathbb{N}$ such that $P_{k_l} \subset Q_l$. Such an equivalence class will be called an *end* of Ω.

Definition 5.1 An end $\mathcal{E}$ of Ω meets $S \subset M \setminus \Omega$ if there is a point $s \in S$ such that for all choices of $\{K_k, P_k\}_{k\in\mathbb{N}}$ in $\mathcal{E}$ and for all $k \in \mathbb{N}$, we have $s \in \tilde{P}_k$, where $\tilde{P}_k$ is the connected component of $M \setminus K_k$ containing P_k. Similarly, a connected component P of $\Omega \setminus K$ meets $S \subset M \setminus \Omega$ if there is a point $s \in S$ such that $s \in \tilde{P}$.

The next result links the ends of Ω with the complementary components of arbitrary compact subsets of Ω.

Lemma 5.2 *Let $\Omega \subset M$ be an open set, let $S \subset M \setminus \Omega$ be any set, and let $K \subset \Omega$ be a compact subset such that $h_\Omega(K) = K$. If each end of Ω meets S, then each connected component of $\Omega \setminus K$ meets S.*

Proof Let P be a connected component of $\Omega \setminus K$ such that $\tilde{P}$ is disjoint from S, and consider a regular exhaustion $\{K_j\}$ of Ω. By definition of a regular exhaustion, there exists $J_0 \in \mathbb{N}$ such that $K \subset K_j$ for all $j \geq J_0$. Hence, we can find P_{J_0} a connected component of $\Omega \setminus K_{J_0}$ such that $P_{J_0} \subset P$, for otherwise we would have $P \subset K_{J_0}$, which in turn would mean that P is relatively compact in Ω, contrary to the hypothesis that $h_\Omega(K) = K$. We shall now show by induction that it is possible to choose a sequence $\{P_j\}_{j\geq J_0}$ of connected components of $\Omega \setminus K_j$ in such a way that $P_{j+1} \subset P_j$ for all

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$j \geq J_0$. P_{J_0} is already chosen, so suppose that P_j is defined for $j \geq J_0$. First note that P_j is not contained in K_{j+1}, otherwise it would have to be relatively compact, as above. Now, let U be a connected component of $P_j \cap (\Omega \setminus K_{j+1}) \neq \varnothing$ and define $P_{j+1} \subset \Omega \setminus K_{j+1}$ as the connected component containing U. Also define F to be the connected component of $\Omega \setminus K_j$ containing P_{j+1}. Since P_{j+1} contains U, we have $P_{j+1} \cap P_j \neq \varnothing$ and thus $F = P_j$, which shows that $P_{j+1} \subset P_j$. But then $\{K_j, P_j\}_{j \geq J_0}$ defines an end of Ω which does not meet S. ■

A result analogous to [8, Lemma 4.1] now follows easily from Lemma 5.2.

Theorem 5.3 *Let M be an open Riemann surface, $\Omega \subset M$ an open subset and $\{a_j\}_j \subset M \setminus \Omega$ a countable set such that, if we denote by A the set of limit points of $\{a_j\}$, then each end of Ω meets A. Then, given $f \in \mathcal{O}(\Omega)$, $\epsilon > 0$ and $N \in \mathbb{N}$, there exist $n \in \mathbb{N}$ and $b_1, \dots, b_n \in \mathbb{C}$ such that $d(f, \sum_{j=1}^{n} b_j C(\cdot, a_{j+N})) < \epsilon$.*

Proof By Lemmas 4.1 and 5.2, choose $\{K_k\}$ a regular exhaustion such that each connected component of $\Omega \setminus K_k$ meets A. It is easy to show that for every $k \in \mathbb{N}$, each connected component of $M \setminus K_k$ contains a connected component of $\Omega \setminus K_k$, and thus contains an element of A. Therefore, pick $m \in \mathbb{N}$ large enough so that $\sum_{k=m+1}^{\infty} \frac{1}{2^k} < \epsilon/2$ and apply Theorem 3.1 to $\{a_j\}_{j>N}$ and K_m to obtain complex numbers $b_1, \dots, b_n$ such that

$$\sup_{K_m} \Big| f(z) - \sum_{j=1}^{n} b_j C(z, a_{N+j}) \Big| < \frac{\epsilon}{2m}.$$

Setting $\theta_N^n(z) = \sum_{j=1}^{n} b_j C(z, a_{N+j})$ and using the fact that $K_l \subset K_m$ for all $l \leq m$, we get

$$d\Big(f, \sum_{j=1}^{n} b_j C(\cdot, a_{N+j}) \Big) = \sum_{k=1}^{m} \frac{1}{2^k} \frac{\sup_{K_k} |f - \theta_N^n|}{1 + \sup_{K_k} |f - \theta_N^n|} + \sum_{k=m+1}^{\infty} \frac{1}{2^k} \frac{\sup_{K_k} |f - \theta_N^n|}{1 + \sup_{K_k} |f - \theta_N^n|} < m \sup_{K_m} |f - \theta_N^n| + \frac{\epsilon}{2} < \epsilon. \quad ■$$

Finally, we find a generalization of Theorem 1.2.

Corollary 5.4 *Let M, Ω and $\{a_j\}_j$ be as in the preceding theorem. Then, there exists a sequence $\{b_j\}_j$ in $\mathbb{C}$ with the property that, given $f \in \mathcal{O}(\Omega)$, there exists an increasing sequence $\{n_k\}_k$ in $\mathbb{N}$ such that*

$$\lim_{k \to \infty} d\Big(f, \sum_{j=1}^{n_k} b_j C(\cdot, a_j) \Big) = 0.$$

Moreover, the set of such sequences $\{b_j\}_j$ is G_δ and dense in $\mathbb{C}^{\mathbb{N}}$, endowed with the cartesian topology, and contains a dense vector subspace of $\mathbb{C}^{\mathbb{N}}$, except for the zero sequence.

Proof Use Theorems 5.3 and 1.3. ■

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