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CENTRES OF RANK-METRIC COMPLETIONS

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In this paper, we are primarily concerned with the behaviour of the centre with respect to the completion process for von Neumann regular rings at the pseudo-metric topology induced by a pseudo-rank function.

Let R be a (von Neumann) regular ring, and N a pseudo-rank function (all terms left undefined here may be found in **[6]**). Then N induces a pseudo-metric topology on R, and the completion of R at this pseudo-metric, $\bar{R}$, is a right and left self-injective regular ring. Let $Z(\,)$ denote the centre of whatever ring is in the brackets. We are interested in the map $Z(R) \rightarrow Z(\bar{R})$.

If R is simple, $Z(R)$ is a field, so is discrete in the topology; yet Goodearl has constructed an example with $Z(R) = \mathbf{R}$ and $Z(\bar{R}) = \mathbf{C}$ [**5**, 2.10]. There is thus no hope of a general density result.

However, we show that in many cases, $Z(\bar{R})$ is algebraic (or "almost algebraic") over $Z(R)$. This holds, for example if (a) R is algebraic over a central field F, or if (b) R is a direct limit as an F-algebra (F a field) of finite products of simple self-injective regular rings R_i where each $Z(R_i)$ is algebraic over F. Result (b) (Theorem 1.5) requires a deep and surprising result of von Neumann **[7]**. We therefore give an independent proof of the following result (also covered by (b)), answering a question of Israel Halperin. Let $\{R_i\}$ be a directed family of regular rings of finite direct sums of finite AW^* factors, and let $R = \lim R_i$ (as $\mathbf{C}$-algebras); then $Z(\bar{R})$ is almost algebraic over $\mathbf{C}$; in particular, if N is extremal (equivalently, $\bar{R}$ is simple), $Z(\bar{R}) = \mathbf{C}$.

We also show that if R is an F-algebra direct limit of semisimple rings each of whose centres is a finite product of copies of F, and $\bar{R}$ is simple, then $Z(\bar{R})$ is canonically equal to F. This improves a result of Goodearl [**5**, 2.7], where the semisimple rings are allowed only to be simple with centre F.

On the other hand, we provide examples of simple algebraic regular F-algebras R with $Z(R) = F$, but where $Z(\bar{R})$ can be an arbitrary finite or countable dimensional extension field of F; we also construct examples where the algebraic closure of F appears as $Z(\bar{R})$. In building these, we also obtain examples of rings similar to those of Farkas and Snider **[4]** and Menal and Raphael **[10]**, simple algebraic algebras which are not regular.

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We also construct examples (necessarily non-algebraic) of simple regular rings R where $Z(\bar{R})$ is of uncountable dimension over $Z(R)$, provided only that the latter is infinite.

1. Completions and algebraicity. There are two basic results here.

THEOREM 1.1. [**5**, 2.4] *Let R be a semisimple artinian ring, and let x be an element of R. There exists z in the centre of R, as well as y in R, such that the right ideal generated by $x - z$ is subisomorphic (as a right module) to that generated by $xy - yx$; i.e.,*

$$(x - z)R \lesssim (xy - yx)R.$$

LEMMA 1.2. *Let a, b be commuting elements of a regular ring S, and let M be a pseudo-rank function of S. If $p = p(x)$ is a polynomial in one variable having all coefficients in S, then*

$$M(p(a) - p(b)) \leqq M(a - b).$$

Remark. The commuting hypothesis is necessary, because one can find $n \times n$ matrices over any field F, such that the rank of $A - B$ is one, but that of $A^n - B^n$ is $n(p(x) = x^n)$.

Proof. Write

$$p(x) = \sum_{j=0}^{m} s_j x^j,$$

with s_j in S. Then

$$p(a) - p(b) = \sum_{j=1}^{m} s_j(a^j - b^j).$$

Since a and b commute, we may find t_j in S with $a^j - b^j = t_j(a - b)$, so

$$p(a) - p(b) = (\sum s_j t_j)(a - b),$$

and thus

$$M(p(a) - p(b)) \leqq M(a - b).$$

LEMMA 1.3. *Let R be a regular ring algebraic over F, with pseudo-rank function N. Let a in $\bar{R}$ be the limit of a Cauchy sequence (with respect to N) $\{a_n\}$ in R, and suppose that each a_n commutes with a. Then there exists a sequence of monic polynomials $\{p_n\}$ in $F[x]$ such that $\{N(p_n(a))\}$ converges to zero.*

Proof. Since each a_n is algebraic over F, there exists a monic polynomial p_n in $F[x]$ such that $p_n(a_n) = 0$. By Lemma 1.2,

$$N(p_n(a)) = N(p_n(a) - p_n(a_n)) \leqq N(a - a_n),$$

which tends to zero.

Von Neumann announced ([Collected Works, "Arithmetic of continuous geometries"] that every simple continuous ring contains elements which are "transcendental" over the centre, that is, elements r such that $p(r)$ is invertible ($N(p(r)) = 1$ at the unique rank function N) for all monic polynomials p with coefficients from the centre [**8**]. Hence not every element in $\bar{R}$ can be approximated by elements of R that commute with it.

If A is a (unital) subring of a commutative ring B, and b is an element of B, we say b is almost algebraic over A (with respect to a rank or pseudo-rank function N) if there is a sequence of monic polymonials p_n with coefficients from A such that $\{N(p_n(b))\} \to 0$. If B is a field, then this notion obviously degenerates to algebraicity.

THEOREM 1.4. *Let R be an algebraic F-algebra and a regular ring, and suppose N is a pseudo-rank function on R. Then every element of $Z(\bar{R})$ is almost algebraic over F. In particular, if $\bar{R}$ is a finite product of simple rings, then $Z(\bar{R})$ is algebraic over F.*

Proof. The first result is an immediate consequence of Lemma 1.3. To see the other one, observe that if $Z(\bar{R})$ is a finite product of fields, and if

$$k = \min\{N(E) \mid E \text{ minimal central idempotent of } \bar{R}\},$$

then for a in $Z(\bar{R})$, $N(a) < k$ implies $a = 0$.

In [Collected Works of von Neumann, Continuous geometries and their arithmetics], von Neumann announced the following amazing result, proved in [7, 1.2.1]:

If R is a non-artinian simple continuous ring with centre F, then the set of F-algebraic elements in R is dense with respect to the metric topology obtained from the unique rank function.

This allows us to prove the following. Without the density of algebraic elements being available, the proof below would yield the same conclusion provided the hypothesis "each R^i is a finite product of simple self-injective regular rings" is strengthened to "each R^i is semisimple artinian" (no change to the algebraicity of $Z(R^i)$ hypothesis).

THEOREM 1.5. *Let R be a direct limit of F-algebras, $R = \lim R^i$, where each R^i is a product of simple right and left self-injective regular rings, and where each $Z(R^i)$ (not necessarily a field) is algebraic over F. Let N be a pseudo-rank function on R. Then the centre of the rank-metric completion, $\bar{R}$, is almost algebraic over F. In particular, if $\bar{R}$ is simple, $Z(\bar{R})$ is algebraic over F (so if $F = \mathbf{C}$ and N is extremal, $Z(\bar{R}) = \mathbf{C}$).*

Proof. Select t in the centre of $\bar{R}$; for given $\epsilon > 0$, choose a in R such that $N(a - t) < \epsilon$. There exists i with a belonging to R^i. Write $R^i = S_1 \times S_2$, where S_1 is a finite product of matrix rings over division rings, and S_2 is a finite product of type II continuous simple rings. Let E be the identity element of S_1, and express $a = aE + a(1 - E)$.

By 1.1, there exists α in $Z(S_1)$, as well as y in S_1 such that

$$(aE - \alpha E)S_1 \lesssim (aEy - yaE)S_1.$$

Since $y = Ey$,

$$a(Ey) - (Ey)a = (aE)y - y(aE);$$

thus $N(aE - \alpha E) < 2\epsilon$. Let p be a monic polynomial in $F[x]$ such that $p(\alpha)E = 0$ (p exists, since $Z(S_1)$ is contained in $Z(R^i)$, and the latter is algebraic over F). By 1.2,

$$N(p(a)E) < 2\epsilon.$$

In S_2, there exists an element b algebraic over $Z(S_2)$, hence algebraic over F, with

$$N(b - a(1 - E)) < \epsilon$$

(apply von Neumann's result to each factor, and express N as a convex linear combination of the individual rank functions). There exists monic q in $F[x]$ with $q(b)(1 - E) = 0$.

Now $N(b + aE - t) < 2\epsilon$. Set $f = pq$ (in $F[x]$). Then

$$\begin{aligned} f(b(1 - E) + aE) &= q(b)(1 - E)p(b) + q(a)p(a)E \\ &= q(a)p(a)E. \end{aligned}$$

Hence $N(f(b + aE)) < 2\epsilon$, so by 1.2,

$$N(f(t)) \leqq N(f(b + aE)) + N(b + aE - t) < 4\epsilon.$$

THEOREM 1.6. *Let R be an F-algebra which can be written as a direct limit of F-algebras (with F-algebra maps), $R = \lim R^i$, where each R^i is semisimple artinian with centre a product of copies of F. If N is an extremal pseudo-rank function on R, then the completion of R at the N-induced topology has (the canonical copy of) F as its centre.*

Proof. Select z in $Z(\bar{R})$. Given $\epsilon > 0$, there exists a in some R^i such that $N(z - a) < \epsilon$ (here "a" should really be replaced by the image of a in R, but for this argument, this would be unnecessarily pedantic; however, in Section 3, more care has to be taken). By 1.1, there exist e in $Z(R^i)$ and y in R^i such that

$$(a - e)R^i \lesssim (ay - ya)R^i.$$

Hence $N(a - e) < 2\epsilon$. Thus $N(z - e) < 3\epsilon$.

The centre of R^i is a finite product of copies of F, so e satisfies a polynomial of the form

$$p(x) = (x - f_1)(x - f_2)\ldots(x - f_k)$$

for some selection of $f_1, \ldots, f_k$ in F. From $p(e) = 0$ and 1.2, we deduce that $N(p(z)) < 3\epsilon$. Since N is extremal, $Z(\bar{R})$ is a field, so if ϵ is chosen less than or equal to $1/3$, $p(z) = 0$. Thus z satisfies one of the irreducible factors of p, and therefore belongs to F.

This improves on [**5**, 2.8], where the algebra R is an infinite tensor product of matrix rings over a field F, tensored (once) with a division ring D whose centre is F.

In particular, 1.6 applies to limits of finite products of matrix rings over a field F. An exciting question would be to decide if for R such a limit with $\bar{R}$ a non-artinian simple ring, is $\bar{R}$ isomorphic to the completion of the infinite tensor product M_2F of 2×2 matrix rings? This would be the precise analogy of the situation with hyperfinite type II_f W^* factors.

2. Rank and inner derivations of finite AW^* **algebras.** Professor I. Halperin has asked, if R is the regular ring of a finite W^* factor of type II, and S is the limit of say $M_{2^n}R \to M_{2^{n+1}}$ (block diagonal embeddings), is the centre of the completion of S in its unique rank metric just $\mathbf{C}$. This follows from 1.4, but we give another proof which avoids using von Neumann's result. This proof gives a result for inner derivations and rank analogous to the usual results with respect to the norm.

We abbreviate $ab - ba$ by $[a, b]$.

LEMMA 2.1 (i). *Let A, B be bounded self-adjoint operators in $B(H)$ such that every element of* spec A *is less than or equal to every element of* spec B. *Then $A \leqq B$.*

(ii) *Let a, b be self-adjoint elements of the regular ring of a finite AW^* algebra, with the same hypotheses on the spectra. Then $a \leqq b$ (that is, there exists d in R such that $b - a = dd^*$).*

Proof. Subtract an appropriate real scalar β from both A and B (respectively, a and b) so that

$$0 = \min\{\alpha | \alpha \in \text{spec } B - \beta \text{ (resp. spec } b - \beta)\}.$$

Then $A - \beta$ $(a - \beta)$ is negative, $B - \beta$ $(b - \beta)$ is positive, and the difference is thus positive.

LEMMA 2.2. *Let R be a unit regular ring, with unique rank function N, and such that $K_0(R)$ is unperforated. Let r be an element of R.*

(a) *If there exists α in $Z(R)$ such that $N(r - \alpha) < \frac{1}{2}$, then there exists an invertible element v in R such that $N([r, v]) = 2N(r - \alpha)$.*

(b) *If* $N(r - \alpha) = \frac{1}{2}$ *for some* α *in* $Z(R)$, *then for all* $\epsilon > 0$, *there exists an invertible element* v *of* R *such that* $N([r, v]) > 1 - \epsilon$.

(c) *If* R *satisfies comparability, then the conclusion of* (a) *holds if* $N(r - \alpha) = \frac{1}{2}$ *for some* α *in the centre of* R.

If R *is *-regular and satisfies* $LP \sim RP$ ([**1**]), *then "invertible" can be replaced by "unitary" in the above.*

Proof. A unit regular ring with unique rank function is simple, so $Z(R)$ is a field. Let q be an idempotent such that the right annihilator of $r - \alpha$ is qR. Then $rq = \alpha q$, and $R(r - \alpha) = R(1 - q)$. In case (a),

$$N(1 - q) < \frac{1}{2} < N(q).$$

By [**3**, 1.4], $1 - q \lesssim q$, so we may find an idempotent e with $eq = qe = e$, and an invertible element v such that $v(1 - q)v^{-1} = e$, $vev^{-1} = 1 - q$, and $v(q - e) = (q - e)v = q - e$ (write

$$R = (q - e)R \oplus (1 - q)R \oplus eR;$$

define an R-module endomorphism of R as the identity on $(q - e)R$, and so that it interchanges $1 - q$ and e). Set $s = vrv^{-1}$. Writing

$$r = r(1 - q) + rq = r(1 - q) + \alpha(q - e) + \alpha e$$

$$s = \alpha(1 - q) + \alpha(q - e) + se,$$

we deduce that $r - s = (r - \alpha)(1 - q) + (\alpha - s)e$. Since $(1 - q)e = 0$, we have that

$$N(r - s) = N((r - \alpha)(1 - q)) + N(\alpha - s)e);$$

but $r - \alpha = (r - \alpha)(1 - q)$, so

$$N((r - \alpha)(1 - q)) = N(r - \alpha).$$

As $(\alpha - s)e$ is conjugate (via v) to $(\alpha - r)(1 - q)$,

$$N((\alpha - s)e) = N(r - \alpha).$$

Thus

$$N(r - vrv^{-1}) = 2N(r - \alpha).$$

As v is invertible,

$$N(r - vrv^{-1}) = N([r, v]).$$

To prove (c), just observe that when comparability holds, $N(1 - q) = \frac{1}{2}$ implies $1 - q$ is equivalent to q, and the same process can be applied as

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in the proof of (a), with $e = q$.

In case (b), we may assume R is not artinian (else (c) applies). Hence we may decompose $1 - q = (1 - q') + f$, where $1 - q'$ and f are orthogonal idempotents, with $0 \leqq N(f) < \epsilon$. Then $N(1 - q') < N(q)$, so $1 - q' \lesssim q$, and the same process as in (a) works.

If R is *-regular with $LP \sim RP$, then all the idempotents occurring above can be replaced by projections, and the equivalences are now implementable by unitaries in place of invertible elements.

PROPOSITION 2.3. *Let M be a type II_f AW^* factor, with regular ring R, and rank function N. Suppose $r = r^*$ is an element of R. Then there exists a unitary w in M such that*

$$N([r, w]) = \min\{1, \inf\{2N(r - \alpha) | \alpha \in \mathbf{R}\}\}.$$

In particular, if $N(r - \alpha) \geqq \frac{1}{2}$ for all real scalars α, then there is a unitary w in M such that $[r, w]$ is invertible in R.

Proof. If $N(r - \alpha) \leqq \frac{1}{2}$ for some scalar (necessarily real) α, then 2.2 applies. We may thus assume that $N(r - \alpha) > \frac{1}{2}$ for all scalars α.

Set $v = (1 + r^2)^{-1/2}$, $t = rv$. Then v, t belong to M, commute, and $r = tv^{-1}$. Let C be any masa containing v and t; we may identify C with $L^\infty(X, u)$, where $u(E) = N(P_E)$ for E a measurable subset of X and P_E its characteristic function. Defining $M(X, u)$ (as in [**2**, 1.2]) as

$$\{f:X \to \mathbf{C}| u - mble, |f(x)| < \infty \text{ a.e.}\}/\{f:X \to \mathbf{C}| f = 0 \text{ a.e.}\},$$

we have that $M(X, u)$ is the completion of $L^\infty(X, u)$ at N, and thus sits inside R. With this identification, r may be viewed as a u-measurable real-valued function that is finite almost everywhere.

Define the real number

$$\alpha = \sup\left\{\beta | u(r^{-1}([-\infty, \beta))) < \frac{1}{2}\right\}.$$

Define the three disjoint sets,

$$K = r^{-1}(\{\alpha\}), J = r^{-1}([-\infty, \alpha)), L = r^{-1}((\alpha, \infty]).$$

Set $u(J) = a$, $u(K) = b$, and $u(L) = c$. Since for all scalars β,

$$u(r^{-1}(\{\beta\})) = 1 - N(r - \beta) < \frac{1}{2},$$

it follows that $a \leqq \frac{1}{2}$, while $a + b \geqq \frac{1}{2}$.

Thus $b + c \geqq \frac{1}{2}$ and $a + c \geqq \frac{1}{2}$ (the latter since $N(r - \alpha) > \frac{1}{2}$). Define the non-negative real numbers, d, e, f, via

$$2d = a + b - c, 2e = a + c - b, 2f = b + c - a.$$

Then $d + e + f = \frac{1}{2}$ and $a = d + e, b = d + f, c = e + f$.

Let P_J, P_K, P_L denote the characteristic functions of the corresponding sets. Decompose orthogonally, all projections lying in C,

$$P_J = P_{J,d} + P_{J,e}, P_K = P_{K,d} + P_{K,f}, P_L = P_{L,e} + P_{L,f},$$

where the subscript d, e, or f gives the value of the corresponding projection at N, that is, $N(P_{J,d}) = N(P_{K,d}) = d$, etc. Then there exists a unitary w in R (hence in M) interchanging projections with the same N-value. Thus:

$$wP_{J,d}w^* = P_{K,d}; \ wP_{K,d}w^* = P_{J,d};$$
$$wP_{J,e}w^* = P_{L,e}; \ wP_{L,e}w^* = P_{J,e};$$
$$wP_{K,f}w^* = P_{L,f}; \ wP_{L,f}w^* = P_{K,f}.$$

Set $s = wrw^*$. We have that

$$r = rP_{J,d} + rP_{J,e} + \alpha P_{K,d} + \alpha P_{K,f} + rP_{L,e} + rP_{L,f} \quad \text{and}$$
$$s = \alpha P_{J,d} + sP_{J,e} + sP_{K,d} + sP_{K,f} + sP_{L,e} + \alpha P_{L,f}.$$

Because of the orthogonality of the six projections, $N(r - s)$ is the sum of the N-values of each of the six terms; moreover since $(\alpha - s)P_{K,d}$ is conjugate via w to $-(r - \alpha)P_{J,d}$ (and similarly with the other two pairs), we have that

$$N(r - s) = 2N((r - \alpha)P_{J,d}) + 2N((r - s)P_{J,e}) + 2N((r - \alpha)P_{L,f}).$$

Now $rP_{J,d}$ commutes with $P_{J,d}$, and r never hits α on the support of $P_{J,d}$ (which is in that of P_J). Hence $(r - \alpha)P_{J,d}$ is invertible in $P_{J,d}M(X, u)$. Thus

$$N((r - \alpha)P_{J,d}) = N(P_{J,d}) = d.$$

Similarly,

$$N((r - \alpha)P_{L,f}) = f.$$

Although $rP_{J,e} = r'$ and $sP_{J,e} = s'$ commute with $P_{J,e}$, they need not commute with each other. However, setting T to be the corner AW^* algebra of M determined by $P_{J,e}$,

$$\text{spec}_T r' \subset [-\infty, \alpha] \text{ and } \text{spec}_T s' \subset [\alpha, \infty]$$

(r', s' are in the regular ring of T). By Lemma 2.2, $r' \leqq s'$ in $P_{J,e}RP_{J,e}$. Suppose $(s' - r')q = 0$ for some projection $q \leqq P_{J,e}$. By subtracting α from r at the outset, we may assume $\alpha = 0$. Then $r' \leqq 0 \leqq s'$. Since $s' - r' \geqq 0$, s', we have that

$$0 = q(s' - r')q \geqq qs'q \geqq 0.$$

Hence $qs'q = 0$; as s' is positive, $s'q = 0$, and thus $r'q = 0$.

On the other hand, as r never hits 0 (formerly α) on J, r' is invertible in $P_{J,e}M(X, u)P_{J,e}$ and hence is invertible in $P_{J,e}RP_{J,e}$. This contradicts $r'q = 0$, unless $q = 0$. Hence $s' - r'$ is a nonzero-divisor in $P_{J,e}RP_{J,e}$, so must be invertible. Therefore

$$N(rP_{J,e} - sP_{J,e}) = N(r' - s') = N(P_{J,e}) = e.$$

Thus

$$N(r - s) = 2d + 2e + 2f = 1.$$

As $r - s = r - wrw^*$, $N([r, w]) = 1$.

The following can possibly be extended to the regular ring of a general finite type AW^* algebra, but there would be no obvious consequences from doing so.

COROLLARY 2.4. *Let T be a possibly infinite cartesian product of copies of regular rings of* II_f *AW^* factors, $T = \prod R_i$. For all $r = r^*$, there exists a central element α of T, as well as a unitary of w of T, such that as principal right T-ideals.*

$$(r - \alpha)T \lesssim [r, w]T.$$

Proof. The hypotheses of 2.3 apply to each R_i. Write $r = (r_i) \in T$. For each i, there exists α_i in $Z(R_i) = \mathbf{C}1_i$, together with a unitary u_i such that

$$N_i([r_i, u_i]) = \min\{1, 2N_i(r_i - \alpha_i)\}.$$

In particular,

$$N_i(r_i - \alpha_i) \lesssim N_i([r_i, u_i]),$$

so

$$(r_i - \alpha_i)R_i \lesssim [r_i, u_i]R_i.$$

The desired result follows from setting $\alpha = (\alpha_i)$, $w = (u_i)$.

In the limit, $\lim R^i$ described below, there is no assumption that the maps preserve any involution; it is only required that they be $\mathbf{C}$-algebra homomorphisms.

THEOREM 2.5. *Let $\{R^i\}$ be a directed family of finite products of regular rings of finite AW^* factors* (*with* **C**-*algebra homomorphisms*). *Set* $R = \lim R^i$. *Let N be an extremal pseudo-rank function on R. Then the natural map from* **C** (*in the centre of R*) *to the centre of the completion of R at N, is an* (*onto*) *isomorphism.*

Proof. Select t in $Z(\bar{R})$. Let $F_i:R^i \to R$ be the canonical map. There exists s in some R^i such that $N(t - F_i(s)) = \delta$ is arbitrarily small (δ to be chosen later). Then if N also denotes the restriction of N to R^i, for all r in R^i we have that

$$N([s, r]) \lesssim 2\delta.$$

Since on the level of R^i, all pseudo-rank functions are invariant under the involution, we have that if $s_1 = \frac{1}{2}(s + s^*)$, $s_2 = \frac{1}{2}i(s - s^*)$ (so $s = s_1 - is_2$), for all r in R^i, each of

$$N([s_1, r]), N([s_2, r])$$

is less than 4δ.

Since s_1 and s_2 are both self-adjoint, 2.4 applies, so there exist: central elements α_1, α_2 of R^i such that

$$(s_j - \alpha_j)R^i \lesssim [s_j, w_j]R^i \quad (j = 1, 2)$$

for some unitaries w_1, w_2 in R^i. Thus

$$N(s_j - \alpha_j) \leqq N([s_j, w_j]) \leqq 4\delta.$$

Set $\alpha = \alpha_1 - i\alpha_2$. Then $N(s - \alpha) < 8\delta$.

Now the centre of each R^i is a finite product of copies of **C**. There thus exists a monic polynomial p in **C**$[x]$ such that $p(\alpha) = 0$. As $[s, \alpha] = 0$, 1.2 applies, so $N(p(s)) \leqq 8\delta$. Another application of 1.2 yields that $N(p(t)) \leqq 9\delta$. If we had chosen $\delta < 1/9$, then $N(p(t))$ would be less than 1. Since the centre of $\bar{R}$ is a field (as N is extremal), $N(p(t)) < 1$ implies that $p(t) = 0$. Hence $Z(\bar{R})$ is algebraic over (the image of) **C**, so must actually be **C**.

3. Examples. In this section, we present several classes of examples. Given a field F and a finite dimensional extension field, K, 3.1 provides a means of constructing a simple algebraic regular F-algebra $R_F(K)$ with unique rank function and centre F, but whose completion has centre K. With K algebraic but of countable dimension over F, essentially the same construction works. In 3.2, a variation is discussed, where examples similar to those occurring in **[4]** and **[10]** (non-regular simple algebraic algebras which are epimorphism-final) are obtained for suitable K, by tensoring certain of the examples in 3.1 with K. The completions of these

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can be regular or not regular, depending on whether an infinite product converges or not. Finally, 3.3 concerns a simple regular ring whose completion has centre a field of uncountable dimension and not algebraic over the centre of the original.

Example 3.1. Let K be a finite dimensional extension field of F. We will construct $R \equiv R_F(K)$, an F-algebra with the following properties:

$R_F(K)$ is simple, regular, and algebraic over F with centre F;

$R_F(K)$ has comparability of idempotents and thus has a unique rank function, N;

The completion of $R_F(K)$ at N has centre K.

This mimics [**5**, 2.10].

The basic map is an F-algebra (but not a K-algebra) map

$$f_{n,m}:M_nK \to M_{mn}K$$

obtained as follows. Let $g:K \to M_dF$ be an embedding, where $d = \dim {}_FK$, and assume $m > d$. Write

$$M_nK = M_nF \otimes_F K,$$

and for z in M_nF, define

$$f_{n,m}(z) = z \oplus z \oplus \ldots \oplus z \quad (m \text{ times}).$$

For α in K, set

$$f_{n,m}(\alpha I_n) = \alpha I_n \oplus \ldots \oplus \alpha I_n \oplus (I_n \otimes g(\alpha))$$

($m - d$ copies of αI_n and one of $I_n \otimes g(\alpha)$). It is readily checked that $f_{n,m}$ is a unital map of F-algebras.

Let $\{n(1), n(2), \ldots\}$ be a sequence of integers exceeding d, such that $\sum 1/n(i) < \infty$. Define $R_F(K)$ to be the limit of

$$M_{n(1)}K \xrightarrow{f_{n(1),n(2)}} M_{n(2)n(1)}K \xrightarrow{f_{n(2)n(1),n(3)}} M_{n(3)n(2)n(1)}K \to \ldots$$

To simplify the notation, let A_k the k-th matrix algebra appearing in the limit (size: $\prod_{i=1}^k n(i)$), and denote by f^k the map $A_k \to A_{k+1}$. So

$$R_F(K) = \lim f^k:A_k \to A_{k+1}.$$

Obviously, $R_F(K)$ is simple and algebraic over F, with comparability of idempotents (since each A_k is simple, algebraic, etc.). Now $Z(A_k) = K$, but the image of any α in $K \setminus F$ at the next level is not central. Thus $Z(R) = F$. Set $s_k:A_k \to R_F(K)$ to be the map into the limit.

The unique rank function on $R \equiv R_F(K)$ is given by

$$N(s_k(r)) = \frac{\text{rank } r}{\prod_1^k n(i)} \quad \text{for } r \text{ in } A_k.$$

Let I^k denote the identity matrix of A_k. We now show that

$$\{s_k(\alpha I^k)\}_{k \in \mathbf{N}}$$

converges with respect to N, to a central element of $\bar{R}$. A simple computation reveals that in A_{k+1},

$$\operatorname{rank}(f^k(\alpha I^k) - \alpha I^{k+1}) \leqq \left(\prod_{i=1}^{k} n(i)\right) \cdot d;$$

so

$$N(s_k(\alpha I^k) - s_{k+1}(\alpha I^{k+1})) \leqq d/n(k+1).$$

Since $\sum_k 1/n(k) < \infty$, the sequence is indeed Cauchy. Let a in $\bar{R}$ denote the limit. For x in $R_F(K)$, say

$$x = s_t(r_t) = s_{t+1}(r_{t+1}) = \dots \quad (r_j \in A_j, j \geqq t),$$

$$N(ax - xa) = \lim \operatorname{rank}(r_t\alpha I^t - \alpha I^t r_t)/\textstyle\prod n(i) = 0,$$

so a commutes with everything in $R_F(K)$, and thus is central in $\bar{R}$.

Since R admits a unique rank function, $\bar{R}$ is simple and thus $Z(\bar{R})$ is a field. Clearly, if p in $F[x]$ is the irreducible polynomial satisfied by α, then $p(a) = 0$. It is readily checked that the assignment $\alpha \to a$ yields an embedding $K \to Z(\bar{R})$.

When K is separable over F, we can conclude that this map is onto and more, as follows.

Set $L = Z(\bar{R})$; this is algebraic over F, by 1.2, and setting E to be the algebraic closure of F, we have the inclusion of fields $F \subset K \subset L \subset E$. We observe that $E \otimes_F K$ is isomorphic to a direct sum of d copies of E. Then $R_E = R_F(K) \otimes_F E$ is a limit of finite direct sums of matrix rings over E, so is algebraic (over F, and thus over E) and an E-algebra, as well as being regular (here is where we use separability). Moreover, since $R_F(K)$ is simple with centre F, R_E is simple with centre E. Finally, because the number of simple direct summands (at each level) in this limit representation is bounded by d, it follows that the $\mathbf{Z}$-rank of $K_0(R_E)$ is at most d.

Since E can be embedded in a (possibly uncountable) tensor product of matrix rings over F (of various sizes), it follows easily that N extends to (at least one) rank function on R_E, which we shall also call N. Complete R_E at this metric. Certainly $L \otimes E = Z(\bar{R}) \otimes E$ is contained in the centre of $\overline{R_E}$. If $[L:F] < \infty$, then $L \otimes E$ is just $[L:F]$ copies of E. Hence $Z(\overline{R_E})$ would contain at least $[L:F]$ minimal idempotents, and so N (on R_E) can be expressed a convex combination (with strictly positive coefficients) of $[L:K]$ distinct extremal pseudo-rank functions on R_E. Since $K_0(\overline{R_E})$ is a dimension group **[3]** of rank at most d, it has at most d pure states and thus $[L:F] \leqq d$, so equality holds, and thus (as $K \subset L$

and $[K:F] = d)$ $K = L$.

If on the other hand, $[L:F] = \infty$, then $L \otimes E$ has infinitely many idempotents, and we would obtain that N is an integral of infinitely many pseudo-rank functions on R_E, contradicting the finiteness of rank $K_0(R_E)$. Thus in the separable case, this argument yields not only that $Z(\bar{R}) = K$, but also that the $\mathbf{Z}$-rank of $K_0(R_E)$ is d.

Via 1.1, one can show directly that $Z(\bar{R}) = K$, separable or not.

Any countable dimensional algebraic field extension can arise as the centre of a suitable $\bar{R}$, by allowing the fields to change in the limit. For example if $K = \cup K_i$, $[K_i:F] < \infty$, construct R as a suitable limit

$$K_1 \to M_{m(1)}K_2 \to M_{m(2)}K_3 \to \dots$$

where the maps are basically as before. This works when $\dim {}_FK$ is countable, and presumably a limit argument can be used to obtain the result for first uncountable $\dim {}_FK$. The algebraic closure can always be obtained, simply by tensoring together (over F) all the $R_F(K)$ (at least when F is of characteristic zero or perfect) as K varies over all the finite dimensional extension fields within a fixed algebraic closure of F.

To obtain an example of an arbitrary algebraic extension field (where however, R is not algebraic over F), start with K an algebraic extension field of F, and let S be the completion of a sufficiently large tensor product of copies of $\lim M_{2^n}F$, so that there is an embedding $g:K \to S$. Then $S \otimes K$ is regular (by [**9**, 2.1 (iii)], $S \otimes K$ is a limit of simple self-injective regular rings). Then an appropriate direct limit of matrix rings over $S \otimes K$ as before, using g, will yield a simple algebra R with centre F, whose completion at its unique rank function (R has compatibility) has centre K.

Example 3.2. Here we construct simple algebraic algebras which are not regular, the "easy" way. Let F be a non-perfect field of characteristic p, and let K be a non-separable (actually, purely inseparable) extension field of the form $K = F(\beta)$, where $\beta^p \in F$. Form $R_F(K)$ as in the first example, and consider $S = R_F(K) \otimes_F K$.

As $K \otimes K/J(K \otimes K) \simeq K$, we see that $K_0(S)$ has rank one, and of course cancellation of projections holds as well. S is simple, as it is the tensor product of a central simple F-algebra with a simple one, and has centre K. On the other hand, S is not regular; the proof given by Menal and Raphael [**10**] for their examples works equally well here ($s_k(I^k \otimes \beta - \beta I^k \otimes 1)$ has no quasi-inverse). The rank of K_0 of their examples is two.

Now recall that $g:K \to M_pF$ is the regular representation; thus S embeds in $R \otimes M_pF \simeq M_pR$, via g, $r \otimes \alpha \to r \otimes g(\alpha)$, and this embedding yields a rank function on S agreeing with N on $R \equiv R_F(K)$; call it N as well.

Part of the definition of $R_F(K)$ entailed that the sequence of integers $\{n(1), n(2), \dots\}$ have their reciprocals summable. Drop this restriction from the definition.

Now we show that the completion of S at N is not regular (but is self-injective) if $\sum 1/n(i) < \infty$, and then we outline a proof that $\bar{S}$ is regular if $\sum 1/n(i) = \infty$.

From the inclusion $R \rightarrow S \rightarrow M_pR$, we obtain the inclusions of completed rings, $\bar{R} \rightarrow \bar{S} \rightarrow M_p\bar{R}$. There is a natural map $\bar{R} \otimes K \rightarrow \bar{S}$. We note that $\bar{R} \otimes K$ is the centralizer of $1 \otimes K$ in $M_p\bar{R}$ [**11**, 5.1]; thus $\bar{R} \otimes K$ is closed in $M_p\bar{R}$. Since $R \otimes K \simeq S$, the map from $R \otimes K$ has dense image in $\bar{S}$, so must be an isomorphism. Rewriting this, we have $\bar{R} \otimes K = \overline{R \otimes K}$.

A result of Menal (appearing in the proof of [**9**, 2.1 (iii)]) yields that $\bar{R} \otimes K$ is self-injective. Another application of [**11**, 5.1] gives us that

$$Z(\bar{R}) \otimes K = \overline{Z(R \otimes K)} = Z(\bar{S}).$$

When $\sum 1/n(i) < \infty$, $Z(\bar{R}) = K$, so $Z(\bar{S}) = K \otimes K$; this admits nilpotents, so $\bar{S}$ is not regular.

On the other hand, if $\sum 1/n(i)$ is infinite, $Z(\bar{R}) = F$ (use 1.1 and examine the behaviour of the elements in the centres of each A_k), so $\bar{S}$ is simple and self-injective, and thus is regular.

Example 3.3. Here we sketch the construction of non-algebraic examples where the centre blows up in an incredible fashion. We shall obtain a simple regular ring R with centre an infinite field F such that the centre of $\bar{R}$ (at some extremal rank function on R) contains a non-principal ultrapower of F, so it is of uncountable dimension over F.

Let T be the completion of lim $M_{2^n}F$. Set S to be the cartesian product of countably many copies of T; there is an embedding $g:S \rightarrow T$. Let $d:T \rightarrow S$ denote the diagonal map $d(t) = (\dots, t, t, \dots)$.

Define the maps (in analogy with those of 3.1)

$$M_nS \rightarrow M_{nm}S, \quad s \rightarrow \text{diag}(s, s, s, \dots, s, dg(s)).$$

Because T is simple, it is easy to check that the limit of these maps, R, is simple and that its centre is F. Moreover, R is unit regular, so admits an extremal rank function. If the usual condition on the $n(i)$ holds, it is also easy to check that there is a map from the centre of $S(= \prod F)$ to the centre of the completion of R. Finally, the extremal rank function can be chosen so that the kernel of the map $\prod F \rightarrow Z(\bar{R})$ is not a principal maximal ideal. Thus the image is a non-principal ultrapower of F, hence is not algebraic over F, and is of uncountable dimension (recalling that F is infinite here). It is probable that the map is onto, but I was not able to prove it.

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