CYCLIC SURGERY ON SATELLITE KNOTS

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1. Introduction. In [**9**] L. Moser classified all manifolds obtained by Dehn surgery on torus knots. In particular she proved the following (see also [**8**, Chapter IV]).

THEOREM 1 [**9**]. *Nontrivial surgery with slope m/n on a nontrivial torus knot $T(p,q)$ gives a manifold with cyclic fundamental group iff $m = npq \pm 1$ and the manifold obtained is the lens space $L(m, nq^2)$.*

J. Bailey and D. Rolfsen [**1**] gave the first example of Dehn surgery on a nontorus knot that produces a lens space. They showed that -23 surgery on the $(11,2)$-cable on the trefoil knot gives the lens space $L(23,7)$. Later R. Fintushel and R. Stern [**4**] constructed lens spaces by surgery on a variety of nontorus knots. In particular they proved the following (see also [**7**, Theorem 7.5]).

THEOREM 2 [**4**]. *Nontrivial surgery with slope m/n on a nontrivial cable knot $C_{r,s}$ on a nontrivial torus knot $T(p,q)$ gives a manifold with cyclic fundamental group iff $s = 2$, $r = 2pq \pm 1$, $m/n = 4pq \pm 1$ and the manifold is the lens space $L(4pq \pm 1, 4q^2)$.*

We prove the following.

MAIN THEOREM. *Nontrivial Dehn surgery with slope m/n on a satellite knot K gives a manifold with cyclic fundamental group iff K is a cable $C_{r,s}$ on a torus knot $T(p,q)$ with $s = 2$, $r = 2pq \pm 1$, $m/n = 4pq \pm 1$ and the manifold is the lens space $L(4pq \pm 1, 4q^2)$.*

To prove the main theorem we will apply the following theorems proved by Gabai. Recall that a knot K in a solid torus $D^2 \times S^1$ is a *n-bridge braid* if K can be isotoped to be a braid in $D^2 \times S^1$ which lies in $\partial D^2 \times S^1$ except for n bridges.

THEOREM 3 [**5**, Theorem 1.1.1]. *Let K be a knot in a solid torus with nonzero wrapping number. If nontrivial surgery on K gives a solid torus, then K is either a 0 or 1-bridge braid.*

THEOREM 4 [**6**, Lemma 3.2]. *Let K be a knot in a solid torus. If K is a 1-bridge braid, then only the surgery with slope $\pm(t + j\omega)\omega \pm b$ or $\pm(t + j\omega)\omega \pm b \pm 1$ on K can possibly give a solid torus, where ω is the winding number of K in the solid torus, $t + j\omega$ is the twist number of K with $0 < t < \omega - 1$ and with j being some integer, b is the bridge width of K with $0 < b < \omega - 1$.*

Similar results to those in the main theorem were independently obtained by S. Wang [**11**], Y. Wu [**12**] and S. Bleiler–R. Litherland [**2**].

2. Preliminaries. We work in the PL category.

Let $K \subset S^3$ be a satellite knot. Let K^* be a nontrival companion knot of K. Let $N^* = \overline{K^* \times D^2} \subset S^3$ be a solid torus neighbourhood of K^* in S^3 with $K \subset \text{int}(N^*)$ and let $M^* = \overline{S^3 - N^*}$. Let μ^* and λ^* be a meridian and a preferred longitude of $\partial N^* = \partial M^*$ respectively, that is, $H_1(\partial N^*) = H_1(\partial M^*) = Z[\mu^*] \oplus Z[\lambda^*]$, $[\mu^*] = 0$ in $H_1(N^*) = Z[\lambda^*]$ and $[\lambda^*] = 0$ in $H_1(M^*) = Z[\mu^*]$.

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Suppose $[K]=\omega[\lambda^*]$ in $H_1(N^*)$. We may assume that $\omega \geq 0$ by choosing a proper orientation for K. Then $\omega \geq 0$ is the winding number of K in N^*.

Let $N=K\times D^2 \subset \operatorname{int}(N^*)$ be a solid torus neighbourhood of K in N^* and let $M=\overline{S^3-N}$ and $M_0=\overline{N^*-N}$. Let μ and λ be a meridian and a preferred longitude of $\partial N=\partial M$ respectively, that is, $H_1(\partial N)=H_1(\partial M)=Z[\mu]\oplus Z[\lambda]$, $[\mu]=0$ in $H_1(N)=Z[\lambda]$ and $[\lambda]=0$ in $H_1(M)=Z[\mu]$. Then $H_1(M_0)=Z[\mu]\oplus Z[\lambda^*]$, $[\lambda]=\omega[\lambda^*]$ in $H_1(M_0)$ and $[\mu^*]=\omega[\mu]$ in $H_1(M_0)$ (by choosing proper orientations for μ and λ).

Let $M(m/n)$ and $M_0(m/n)$ be the manifolds obtained from Dehn surgery on K with nontrivial slope m/n. *From now on we assume that $\pi_1(M(m/n))$ is cyclic.* Since any satellite knot is not a torus knot, we may assume that $n=1$ by [**3**, Corollary 1].

Elementary homological arguments prove the following.

Lemma 1 [**7**, Lemma 3.3(ii)]. $\ker(H_1(\partial M_0(m))\to H_1(M_0(m)))$ *is the cyclic subgroup of* $H_1(\partial M_0(m))$ *generated by*

$$\begin{cases} \dfrac{m}{(\omega,m)}[\mu^*]+\dfrac{\omega^2}{(\omega,m)}[\lambda^*] & \text{if} \quad \omega\neq 0,\\[2ex] [\mu^*] & \text{if} \quad \omega=0.\end{cases}$$

3. Proof of the main theorem.

Lemma 2. *$M_0(m)$ is a solid torus.*

Proof. We first show that $M_0(m)$ is irreducible. Suppose that, on the contrary, $M_0(m)$ is reducible. Then by [**10**, Corollary 4.4], K is a cable $C_{r,s}$ on K^* and the slope used is that of the cabling annulus, that is, $m=rs$. Then by [**7**, Corollary 7.3], $M(m)\cong M^*(r/s)\# L(s,r)$. Hence $\pi_1(M(m))\cong\pi_1(M^*(r/s))*\pi_1(L(s,r))$. Since $K=C_{r,s}$ can not be a trivial cable on K^*, $|s|>1$. If K^* is a torus knot, then $\pi_1(M^*(r/s))\neq 1$, since torus knots satisfy Property P; if K^* is not a torus knot, then by [**3**, Corollary 1], $\pi_1(M^*(r/s))\neq 1$. Hence $\pi_1(M(m))$ is a free product of two nontrivial groups, contradicting the assumption that $\pi_1(M(m))$ is cyclic. Hence $M_0(m)$ is irreducible.

Since $\pi_1(M(m))$ is cyclic, $\partial M_0(m)$ is a compressible torus in $M(m)$. Let $B^2\subset M(m)$ be a compressing 2-cell for $\partial M_0(m)$. Since K^* is nontrivial, $B^2\subset M_0(m)$. Performing 2-surgery on $\partial M_0(m)$ using B^2, we get a 2-sphere which must bound a 3-cell in $M_0(m)$. Hence $M_0(m)$ is a solid torus. ∎

By Lemma 2 and Theorem 3, K is a 0 or 1-bridge braid in N^*. Hence $\omega\neq 0$ and $\omega\neq 1$ by the definition of satellite knot.

Let B^2 be a proper meridian 2-cell of $M_0(m)$. Then $[\partial B^2]$ is a primitive element of $H_1(\partial M_0(m))$ and $[\partial B^2]\in\ker(H_1(\partial M_0(m))\to H_1(M_0(m)))$. By Lemma 1,

$$[\partial B^2]=\begin{cases} \pm\left(\dfrac{m}{(\omega,m)}[\mu^*]+\dfrac{\omega^2}{(\omega,m)}[\lambda^*]\right) & \text{if} \quad \omega\neq 0,\\[2ex] \pm[\mu^*] & \text{if} \quad \omega=0,\end{cases}$$

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in $H_1(\partial M_0(m))$. Hence

$$M(m)=\begin{cases} M^*\left(\dfrac{m}{\omega^2}\right) & \text{if} \quad \omega \neq 0, \\ M^*\left(\dfrac{\pm 1}{0}\right) & \text{if} \quad \omega = 0. \end{cases}$$

Since $\omega \neq 0$,

$$M(m)=M^*\left(\frac{m}{\omega^2}\right)=M^*\left(\frac{m/(\omega^2,m)}{\omega^2/(\omega^2,m)}\right),$$

and thus

$$Z_{|m|}=H_1(M(m))=H_1\left(M^*\left(\frac{m/(\omega^2,m)}{\omega^2/(\omega^2,m)}\right)\right)=Z_{|m|/(\omega^2,m)}.$$

Hence $(\omega^2, m)=1$.

LEMMA 3. *K^* is a torus knot.*

Proof. Suppose that K^* is not a torus knot. Then by [**3**, Corollary 1], $\omega^2=1$ and thus $\omega=1$, contradicting $\omega\neq 1$. ■

LEMMA 4. *K is a cable knot on K^*.*

Proof. By Lemma 3, $K^*=T(p,q)$, a torus knot. By Theorem 1, $\pi_1(M(m))=\pi_1(M^*(m/\omega^2))$ can possibly be cyclic only when m is equal to

$$\omega^2 pq \pm 1. \tag{$*$}$$

Suppose that K is not a cabled knot. Then K is a 1-bridge braid in N^*. By Theorem 4, $M_0(m)$ can possibly be a solid torus only when m is equal to

$$\pm(t+j\omega)\omega \pm b \quad \text{or} \quad \pm(t+j\omega)\omega\pm b\pm 1. \tag{$**$}$$

Now it is enough to show that no value from ($*$) can be equal to any value from ($**$). We need to show that $|\omega^2pq+1\pm(t+j\omega)\omega\pm b|>0$, $|\omega^2pq-1\pm(t+j\omega)\omega\pm b|>0$, $|\omega^2pq+1\pm(t+j\omega)\omega\pm b\pm 1|>0$ and $|\omega^2pq-1\pm(t+j\omega)\omega\pm b\pm 1|>0$. We verify the first inequality. The rest of the inequalities can be verified similarly.

If $|pq\pm j|\neq 0$, then $|\omega^2pq+1\pm(t+j\omega)\omega\pm b|=|(pq\pm j)\omega^2\pm t\omega\pm b+1|>|pq\pm j|\omega^2-t\omega-b-1\geq\omega^2-(\omega-2)\omega-(\omega-2)-1=\omega+1>0$; if $|pq\pm j|=0$, then $|\omega^2pq+1\pm(t+j\omega)\omega\pm b|=|\pm t\omega\pm b+1|\geq t\omega+b-1>0$. ■

Now the main theorem follows from Lemma 3, Lemma 4 and Theorem 2. ■

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