Can. J. Math. Vol. **46** (4), 1994 pp. 734–745

HOMOMORPHISMS ON FUNCTION ALGEBRAS

M. I. GARRIDO, J. GÓMEZ GIL AND J. A. JARAMILLO

ABSTRACT. Let A be an algebra of continuous real functions on a topological space X. We study when every nonzero algebra homomorphism $\varphi: A \longrightarrow R$ is given by evaluation at some point of X. In the case that A is the algebra of rational functions (or real-analytic functions, or C^m-functions) on a Banach space, we provide a positive answer for a wide class of spaces, including separable spaces and super-reflexive spaces (with nonmeasurable cardinal).

Introduction. Let A be an algebra of continuous real functions defined on a topological space X. We shall be concerned here with the problem as to whether every nonzero algebra homomorphism $\varphi: A \longrightarrow \mathbf{R}$ is given by evaluation at some point of X, in the sense that there exists some a in X such that $\varphi(f) = f(a)$ for every f in A. This problem goes back to the work of Michael [22], motivated by the question of automatic continuity of homomorphisms in a symmetric $*$-algebra. More recently, the problem has been considered by different authors, mainly in the case of algebras of smooth functions: algebras of differentiable functions on a Banach space in [2], [14], [16] and [17]; algebras of differentiable functions on a locally convex space in [4], [5], [6] and [7], and algebras of smooth functions in the abstract context of "smooth spaces" in [21].

In this paper we shall be interested both in the general case and in the case of functions on a Banach space. Section 1 is devoted to algebras of continuous functions over an arbitrary topological space X. Some results for quite general algebras are obtained, including a characterization of algebras for which every homomorphism is a point evaluation on X. Some applications are also given. In Section 2 we concentrate on algebras of continuous functions over an arbitrary subset Ω of a Banach space E. Our results here are valid not only for the algebra of C^m-functions, but also for the algebras of real-analytic and rational functions on Ω. In particular, we obtain a positive result for these algebras (every homomorphism is a point evaluation on Ω) when the space E injects linearly into $\ell_p(\Gamma)$, for some $1 < p < \infty$ and some index set Γ (with nonmeasurable cardinal). This condition is satisfied, for instance, if E is separable, or if E is super-reflexive (with nonmeasurable cardinal). For spaces not verifying this condition, such as $c_0(\Gamma)$ with Γ uncountable, the situation is shown to be different: in this case the answer to the problem depends on the geometry of Ω.

Part of this work was carried out while the first and last named authors were visiting Kent State University, and it is a pleasure to thank the University for its hospitality.

The second and third authors were partially supported by DGICYT PB 90-0044.

Received by the editors April 30, 1993.

AMS subject classification: 46E25, 54C40.

https://doi.org/10.4153/CJM-1994-041-3 Published online by Cambridge University Press

Special thanks are due to Richard M. Aron, José G. Llavona and Angeles Prieto for some helpful conversations concerning this paper.

1. **General results.** For a topological space X, let $C(X)$ be the algebra of all continuous real functions defined on X, and let $C^*(X)$ be the subalgebra of all bounded functions in $C(X)$. If A is a subalgebra of $C(X)$, we denote by $\operatorname{Hom} A$ the set of all nonzero multiplicative linear functionals on A. For each $a \in X$, let δ_a be the functional $f \longrightarrow \delta_a(f) = f(a)$ on A; clearly $\delta_a \in \operatorname{Hom} A$. We shall write $\operatorname{Hom} A = X$ when every $\varphi \in \operatorname{Hom} A$ is of the form $\varphi = \delta_a$ for some $a \in X$. Recall that a subalgebra A of $C(X)$ is said to be *inverse-closed* (respectively, closed under bounded inversion) if whenever $f \in A$ and $f(x) \neq 0$ (respectively, $|f(x)| \geq 1$) for every $x \in X$, then $1/f \in A$.

We shall use the following well-known result:

LEMMA 1.1. *Let X be a topological space, let $A \subset C(X)$ be an inverse-closed subalgebra with unit and let $\varphi \in \operatorname{Hom} A$. Then:*

(1) Given $f_1, f_2, \ldots, f_n \in A$, there exists $a \in X$ such that $\varphi(f_j) = f_j(a)$, for $j = 1, 2, \ldots, n$.

(2) If X is compact, there exists $a \in X$ such that $\varphi = \delta_a$.

If X is a completely regular space, let βX be the Stone-Čech compactification of X and, for $f \in C(X)$, let $\hat{f}\colon \beta X \longrightarrow \mathbf{R} \cup \{\infty\}$ denote the continuous extension of f. Note that if f is bounded then $\hat{f}$ is finite. For each $\xi \in \beta X$ we define the algebra

$$A_\xi = \{f \in C(X) : \hat{f}(\xi) \neq \infty\}.$$

PROPOSITION 1.2. *Let X be a completely regular space, let $A \subset C(X)$ be a subalgebra with unit, closed under bounded inversion, and let $\varphi \in \operatorname{Hom} A$. Then there exists $\xi \in \beta X$ such that $A \subset A_\xi$ and $\varphi(f) = \hat{f}(\xi)$ for every $f \in A$.*

PROOF. Define $A^* = A \cap C^*(X)$ and $\hat{A} = \{\hat{f} : f \in A^*\}$. Note that $\hat{A}$ is an inverse-closed subalgebra of $C(\beta X)$. Consider the algebra homomorphism $\hat{\varphi} \in \operatorname{Hom} \hat{A}$ given by $\hat{\varphi}(\hat{f}) = \varphi(f)$, for every $f \in A^*$. From Lemma 1.1 we obtain that there exists $\xi \in \beta X$ such that $\varphi(f) = \hat{f}(\xi)$, for every $f \in A^*$.

We have that $A \subset A_\xi$. Indeed, if there exists $f \in A$ with $\hat{f}(\xi) = \infty$, consider $g = (1 + f^2)^{-1}$. Then $g \in A^*$ and $\varphi(g) = \hat{g}(\xi) = 0$, but this is a contradiction since g is a unit.

We can then consider the algebra homomorphism δ_ξ on A defined by $\delta_\xi(f) = \hat{f}(\xi)$, for every $f \in A$. Now let $f \in A$ such that $\varphi(f) = 0$; then $h = f^2(1 + f^2)^{-1} \in A^*$ and $0 = \varphi(h) = \hat{h}(\xi)$. Thus $\hat{f}(\xi) = 0$. This shows that $\operatorname{Ker} \varphi \subset \operatorname{Ker} \delta_\xi$ and therefore $\varphi = \delta_\xi$ on A.

REMARKS 1.3. (1) In Proposition 1.2, the point $\xi \in \beta X$ is not unique, in general. We can consider as an example the subalgebra $A \subset C(\mathbf{R})$ of all bounded uniformly continuous functions on $\mathbf{R}$. In this case each $\xi \in \beta\mathbf{R}$ defines a homomorphism on A, and it is not difficult to check that two points $\xi, \eta \in \beta\mathbf{R}$ define different homomorphisms on A if, and only if, there exist $C, D \subset \mathbf{R}$ such that $\operatorname{dist}(C, D) > 0$ and $\xi \in \bar{C}^{\beta\mathbf{R}}$, $\eta \in \bar{D}^{\beta\mathbf{R}}$

(see [18] Proposition 9 for an analogous situation). Now let (x_n) and (y_n) be discrete sequences in $\mathbf{R}$ such that $M = \{x_n\}$ and $N = \{y_n\}$ are disjoint closed subsets of $\mathbf{R}$, and $|x_n - y_n| \longrightarrow 0$. Taking convergent subnets $(x_{n_\alpha}) \longrightarrow \xi$ and $(y_{n_\alpha}) \longrightarrow \eta$ in $\beta\mathbf{R}$, we have that $\xi \neq \eta$ but, since $|x_{n_\alpha} - y_{n_\alpha}| \longrightarrow 0$, the points ξ and η define the same homomorphism on A.

(2) We cannot delete the condition "A is closed under bounded inversion" in Proposition 1.2. For instance, if $X = [0, 1]$ and $A \subset C([0, 1])$ is the subalgebra of all polynomial functions on $[0, 1]$, then $\beta X = X$ but every $\xi \in \mathbf{R}$ defines a homomorphism on A.

(3) Let X be a completely regular space and let $A \subset C(X)$ be a subalgebra with unit. If $\varphi \in \operatorname{Hom} A$ is positive (that is, $\varphi(f) \geq 0$ whenever $f \geq 0$) then Proposition 1.2 implies that there exists $\xi \in \beta X$ such that $A \subset A_\xi$ and $\varphi(f) = \hat{f}(\xi)$ for every $f \in A$. Indeed, the algebra $B = \{f/g : f, g \in A; |g| \geq 1\}$ is closed under bounded inversion, and φ can be extended to a homomorphism $\tilde{\varphi} \in \operatorname{Hom}(B)$ by the formula $\tilde{\varphi}(f/g) = \varphi(f)/\varphi(g)$. Now Proposition 1.2 applies to $\tilde{\varphi}$. On the other hand, it also follows from Proposition 1.2 that, if A is closed under bounded inversion, then every $\varphi \in \operatorname{Hom} A$ is positive.

Our next result follows at once from Proposition 1.2.

PROPOSITION 1.4. *Let X be a completely regular space and let $A \subset C(X)$ be a subalgebra with unit, closed under bounded inversion. Suppose that for each $\xi \in \beta X \setminus X$ there exists $f \in A$ such that $\hat{f}(\xi) = \infty$. Then* $\operatorname{Hom} A = X$.

The condition in Proposition 1.4 is quite abstract, but it can be applied directly in many cases. For example, if $A \subset C(\mathbf{R}^n)$ is a unital subalgebra closed under bounded inversion and A contains the projections $\pi_j \colon \mathbf{R}^n \longrightarrow \mathbf{R}$ $(j = 1, 2, \ldots, n)$, then Proposition 1.4 implies that $\operatorname{Hom} A = \mathbf{R}^n$. Indeed, in this case $(\pi_1^2 + \cdots + \pi_n^2)^\wedge(\xi) = \infty$ for every $\xi \in \beta\mathbf{R}^n \setminus \mathbf{R}^n$. In particular, A could be the algebra of all rational functions, or all real-analytic functions, or all C^m-functions $(1 \leq m \leq \infty)$ on $\mathbf{R}^n$. More generally, if X is locally compact, σ-compact and noncompact, there exists $h \in C(X)$ such that

$$(*) \qquad \hat{h}(\xi) = \infty, \quad \text{for every } \xi \in \beta X \setminus X.$$

Now if $A \subset C(X)$ is a unital subalgebra closed under bounded inversion and A contains a function with property $(*)$, then $\operatorname{Hom} A = X$.

On the other hand, Proposition 1.4 certainly applies to algebras which are not inverse-closed, as the following example shows. We recall that, with some technical modifications, an analogous example can be constructed for any realcompact non-pseudocompact space X.

EXAMPLE 1.5. Let X be a locally compact, σ-compact, noncompact space. Consider $g_0 \in C(\beta X)$ such that $\beta X \setminus X = \{\xi \in \beta X : g_0(\xi) = 0\}$. Using the fact that $\beta X \setminus X$ is not a P-space (see [13]) it is possible to find $g_1 \in C(\beta X)$ and $\eta \in \beta X \setminus X$ so that $\eta \in Z = \{\xi \in \beta X \setminus X : g_1(\xi) = 0\}$ but Z is not a neighbourhood of η in $\beta X \setminus X$. Consider

$$g = \frac{1}{g_0^2 + g_1^2}\Big|_X.$$

https://doi.org/10.4153/CJM-1994-041-3 Published online by Cambridge University Press

Note that $Z = \{\xi \in \beta X : \hat{g}(\xi) = \infty\}$. Now let A be the unital subalgebra of $C(X)$ generated by g and A_η, that is:

$$A = \{f_0 + f_1 g + \cdots + f_n g^n : f_0, f_1, \ldots, f_n \in A_\eta; n \in \mathbf{N}\}.$$

The algebra A has the following properties:

(1) A is closed under bounded inversion.

This is clear since $C^*(X) \subset A_\eta \subset A$.

(2) For each $\xi \in \beta X \setminus X$ there exists $f \in A$ such that $\hat{f}(\xi) = \infty$.

First note that $\hat{g}(\eta) = \infty$. Now if $\xi \in \beta X \setminus X$, $\xi \neq \eta$, there exists $h \in C(\beta X)$ such that $h(\xi) = 0$ and $h(y) > 0$ for every $y \in X \cup \{\eta\}$. Then $f = \frac{1}{h}|_X \in A_\eta \subset A$ and $\hat{f}(\xi) = \infty$.

(3) $\operatorname{Hom} A = X$.

It is a consequence of Proposition 1.4.

(4) If $h \in C(X)$ satisfies $\hat{h}(\xi) = \infty$, for every $\xi \in \beta X \setminus X$, then $h \notin A$.

Indeed, if $f = f_0 + f_1 g + \cdots + f_n g^n \in A$ (where $f_0, f_1, \ldots, f_n \in A_\eta$), then $f_0, f_1, \ldots, f_n$ extend finite and continuously to a neighbourhood of η in βX, and therefore f extends finite and continuously to some point of $\beta X \setminus X$.

(5) A is not inverse-closed.

Otherwise we would have $A = C(X)$ (since $C^*(X) \subset A$ and every function in $C(X)$ is the quotient of two functions in $C^*(X)$). But (4) shows that this is not the case.

Now suppose that in Proposition 1.4 the condition on A is not fulfilled, *i.e.* there exists $\xi \in \beta X \setminus X$ such that $\hat{f}(\xi) \neq \infty$ for every $f \in A$. We can then consider the algebra homomorphism δ_ξ on A defined by $\delta_\xi(f) = \hat{f}(\xi)$ for every $f \in A$. Suppose that, in addition, A separates points and closed sets of X (that is, if $C \subset X$ is closed and $a \in X \setminus C$, there exists $f \in A$ such that $f(a) \notin \overline{f(C)}$. Then for each $a \in X$ there exists $f \in A$ so that $f(a) \neq \hat{f}(\xi)$, and therefore δ_ξ is not given by evaluation at any point of X. Summarizing, we have the following.

THEOREM 1.6. *Let X be a completely regular space and let $A \subset C(X)$ be a subalgebra with unit, closed under bounded inversion, which separates points and closed sets of X. Then the following are equivalent:*

(i) $\operatorname{Hom} A = X$

(ii) For each $\xi \in \beta X \setminus X$ there exists $f \in A$ such that $\hat{f}(\xi) = \infty$.

Motivated by the results of [21], we give a simple application for algebras of continuous functions over an arbitrary product of real lines.

COROLLARY 1.7. *Let $X \subset \mathbf{R}^I$ be a closed set and let $A \subset C(X)$ be a subalgebra with unit, closed under bounded inversion. Suppose that $\pi_i|_X \in A$ for each projection $\pi_i \colon \mathbf{R}^I \longrightarrow \mathbf{R}$ $(i \in I)$. Then* $\operatorname{Hom} A = X$.

PROOF. Let $\mathcal{R}$ be the smallest subalgebra with unit of $C(X)$ which is closed under bounded inversion and contains $\pi_i|_X$ for every $i \in I$. That is, each $g \in \mathcal{R}$ is a function of the form $g = P/Q$, where P and Q are polynomials in a finite number of projections and $\inf\{|Q(x)| : x \in X\} > 0$. Note that $\mathcal{R} \subset A$.

Consider $\varphi \in \operatorname{Hom} \mathcal{R}$. For each $i \in I$ define $a_i = \varphi(\pi_i|_X)$, and set $a = (a_i)_{i \in I}$. We shall show that $a \in X$. Indeed, suppose that $U \cap X = \emptyset$ for some open set U of the form $U = \prod_{i \in I} U_i$, where $U_i = (a_i - \varepsilon, a_i + \varepsilon)$ if $i \in \{i_1, \ldots, i_m\}$, being $\varepsilon > 0$, and $U_i = \mathbf{R}$ if $i \in I \setminus \{i_1, \ldots, i_m\}$. Then $P = (\pi_{i_1} - a_{i_1})^2 + \cdots + (\pi_{i_m} - a_{i_m})^2 \in \mathcal{R}$ satisfies $\varphi(P) = 0$ and $P(x) \geq \varepsilon^2 > 0$ for every $x \in X$, and this contradicts Proposition 1.2.

Now since $\varphi(\pi_{i|X}) = \pi_i(a)$ for every $i \in I$, it follows that $\varphi(g) = g(a)$ for every $g \in \mathcal{R}$. This shows that Hom $\mathcal{R}$=X. On the other hand, since $\mathcal{R}$ separates points and closed sets of X, Theorem 1.6 implies that $\operatorname{Hom} A = X$.

THEOREM 1.8. *Let X be a realcompact space and let $A \subset C(X)$ be a subalgebra with unit, closed under bounded inversion. If A is uniformly dense in $C(X)$, then* $\operatorname{Hom} A = X$.

PROOF. Let $\xi \in \beta X \setminus X$. Since X is realcompact, there exists $g \in C(X)$ such that $\hat{g}(\xi) = \infty$ (see *e.g.* [11]). Choosing $f \in A$ with $|f(x) - g(x)| < 1$ for every $x \in X$, we have that $\hat{f}(\xi) = \infty$. Then the result follows from Proposition 1.4.

Now Corollary 1.9 below can be obtained as an easy consequence of Theorem 1.8 and the results of [12] on uniform density (see also [1]). This corollary extends Theorem 3.2 of [21] and Theorem 2 of [17].

First recall that a zero-set in X is a set of the form $Z(f) = f^{-1}(0)$, for some $f \in C(X)$. Also, for $f \in C(X)$ we denote $\operatorname{coz}(f) = X \setminus Z(f)$.

COROLLARY 1.9. *Let X be a realcompact space and let $A \subset C(X)$ be a subalgebra with unit satisfying:*

(i) A is closed under bounded inversion.

(ii) If $Z_0, Z_1 \subset X$ are (nonempty) disjoint zero-sets, then there exists $f \in A$ such that $f(Z_0) = 0$ and $f(Z_1) = 1$.

(iii) If (f_n) is a sequence of functions in A such that $\operatorname{coz}(f_n) \cap \operatorname{coz}(f_m) = \emptyset$ *for $|n - m| > 1$, then $\sum_{n=1}^{\infty} f_n \in A$.*

Then A is uniformly dense in $C(X)$, and therefore $\operatorname{Hom} A = X$.

Our next result, which follows the lines of Theorem 1 of [4], will be useful in the sequel.

PROPOSITION 1.10. *Let X be a completely regular space and let $A \subset C(X)$ be an inverse-closed subalgebra with unit.*

(1) Suppose that $(f_n) \subset A$ is a sequence such that, for every summable sequence (α_n) of positive numbers, $\sum_n \alpha_n f_n$ and $\sum_n \alpha_n f_n^2$ belong to A. Then for each $\varphi \in \operatorname{Hom} A$ there exists $a \in X$ such that $\varphi(f_n) = f_n(a)$ for all n.

(2) Suppose that, in addition, (f_n) separates the points of X. Then $\operatorname{Hom} A = X$.

PROOF. (1) Let $\varphi \in \operatorname{Hom} A$ be given. By Proposition 1.2 there exists $\xi \in \beta X$ such that $A \subset A_\xi$ and $\varphi(f) = \hat{f}(\xi)$ for every $f \in A$. Suppose that there exists no $a \in X$ such that $\varphi(f_n) = f_n(a)$ for all n. Now choose a summable sequence (α_n) of positive numbers

https://doi.org/10.4153/CJM-1994-041-3 Published online by Cambridge University Press

such that $g_0 = \sum_n \alpha_n \big(f_n - \varphi(f_n)\big)^2$ and $g_1 = \sum_n 2^{-n}\alpha_n \big(f_n - \varphi(f_n)\big)^2$ belong to A. Since g_0 is never zero on X, we have that $1/g_0 \in A$ and $\hat{g}(\xi) \neq 0$. Then consider

$$h = \frac{g_1}{g_0} = \sum_n 2^{-n} \frac{\alpha_n \big(f_n - \varphi(f)\big)^2}{g_0} \in A.$$

The series defining h is uniformly convergent on X, and therefore $0 = \hat{h}(\xi) = \varphi(h)$. But this is a contradiction, since h is a unit in A.

(2) is a consequence of (1).

2. **Functions on Banach spaces.** We now turn our attention to the case of functions over a real Banach space E. Let $\mathcal{P}(E)$ denote the algebra of all continuous polynomials on E and, for $j = 0, 1, 2, \ldots$, let $\mathcal{P}(^jE)$ denote the space of all continuous j-homogeneous polynomials on E. That is, each $P_j \in \mathcal{P}(^jE)$ is a function of the form $P_j(x) = T_j(x, \ldots, x)$, where T_j is a continuous j-linear functional on $E \times \cdots \times E$ (thus for $j = 0$, P_0 is constant), and each $P \in \mathcal{P}(E)$ is a finite sum $P = P_0 + P_1 + \cdots + P_m$, where $P_j \in \mathcal{P}(^jE)$ for $j = 0, 1, 2, \ldots, m$. Recall that a function f defined on an open subset U of E is said to be *real-analytic* on U if, for every $x \in U$ there exist a neighbourhood W of 0 and a sequence (P_j) with each $P_j \in \mathcal{P}(^jE)$, such that $f(x+h) = \sum_{j=0}^{\infty} P_j(h)$, for every $h \in W$. Now let Ω be any subset of E. We denote by $\mathcal{R}(\Omega)$ the algebra of all rational functions on Ω, *i.e.* the functions of the form P/Q, where $P, Q \in \mathcal{P}(E)$ and $Q(x) \neq 0$ for every $x \in \Omega$. Also, we denote by $\mathcal{A}(\Omega)$ (respectively, $C^m(\Omega)$, $1 \leq m \leq \infty$) the algebra of all real functions on Ω which can be extended to a real-analytic function (respectively, an m-times continuously Fréchet differentiable function) on an open subset of E containing Ω. Note that $\mathcal{R}(\Omega) \subset \mathcal{A}(\Omega) \subset C^m(\Omega)$, and they are inverse-closed subalgebras of $C(\Omega)$.

We start with the special case of the separable Hilbert space $E = \ell_2$.

PROPOSITION 2.1. *Let $A \subset C(\ell_2)$ be an inverse-closed subalgebra with unit. Suppose that A contains the dual space ℓ_2^* and the polynomials $P(x) = \sum_{n=1}^{\infty} x_n^2$ and $Q(x) = \sum_{n=1}^{\infty} s_n x_n^2$, where (s_n) is a given summable sequence of positive numbers. Then* $\operatorname{Hom} A = \ell_2$.

PROOF. We shall denote $\ell_2 = X$. Let $\varphi \in \operatorname{Hom} A$ be given. By Proposition 1.2, there exists $\xi \in \beta X$ such that $A \subset A_\xi$ and $\varphi(f) = \hat{f}(\xi)$ for every $f \in A$. Let (x_α) be a net in X such that $x_\alpha \longrightarrow \xi$ in βX. Then $\|x_\alpha\|^2 = P(x_\alpha) \longrightarrow \hat{P}(\xi) = \varphi(P) \neq \infty$ and therefore we can suppose that (x_α) is a bounded net in X. Let $B \subset X$ be a closed ball containing (x_α). By the weak compactness of B, we can also suppose that the net (x_α) is weakly convergent to some point $b = (b_n) \in B$. We now consider the functions on X:

$$g(x) = \big(P(x) - \varphi(P)\big)^2 \quad (\text{for } x \in X)$$
$$h(x) = \sum_n s_n (x_n - b_n)^2 \quad (\text{for } x = (x_n) \in X).$$

It is clear that $g, h \in A$, and $\varphi(g) = 0$. On the other hand, it is not difficult to check that h is weakly continuous on B, and therefore $\varphi(h) = \lim_\alpha h(x_\alpha) = h(b) = 0$. Then

https://doi.org/10.4153/CJM-1994-041-3 Published online by Cambridge University Press

Lemma 1.1 implies that there exists $a \in X$ so that $g(a) = h(a) = 0$. Thus $a = b$ and $\lim_\alpha \|x_\alpha\|^2 = \varphi(P) = P(a) = \|a\|^2$.

Since the net (x_α) is weakly convergent to b and $\|x_\alpha\| \longrightarrow \|b\|$, it follows that (x_α) is norm-convergent to b; hence $\xi = b \in X$.

Our next lemma is taken from [17].

LEMMA 2.2. *Let X and Y be topological spaces and let $A \subset C(X)$ and $B \subset C(Y)$ be inverse-closed subalgebras with unit. Suppose that:*

(i) $H(B)=Y$.

(ii) For each $b \in Y$, there exists $f_b \in B$ such that $f_b^{-1}(0) = \{b\}$.

(iii) There exists a one-to-one map $h\colon X \longrightarrow Y$ such that $f \circ h \in A$ for every $f \in B$.

Then $\operatorname{Hom} A = X$.

REMARK 2.3. Let E be a real Banach space such that there exists a sequence $(\psi_n) \subset E^*$ of norm-one functionals separating the points of E (for example, if E is separable or E is the dual of a separable space). Consider any set $\Omega \subset E$ and let $A \subset C(\Omega)$ be an inverse-closed subalgebra with unit. Suppose that A contains the dual E^* and the polynomials $P = \sum_n r_n^2\psi_n^2$ and $Q = \sum_n s_n r_n^2 \psi_n^2$, where (r_n) and (s_n) are two summable sequences of positive numbers. Then $\operatorname{Hom} A = \Omega$. This is a direct application of Proposition 2.1 and Lemma 2.2, using the map $h\colon \Omega \longrightarrow \ell_2$ defined by $h(x) = \big(r_n\psi_n(x)\big)_{n\in\mathbf{N}}$.

Next we give the main result of the paper. First recall that a set Γ is said to have nonmeasurable cardinal if there exists no nontrivial two-valued measure defined on the power set of Γ (see *e.g.* [13] or [19]).

THEOREM 2.4. *Let Ω be any subset of a real Banach space E such that there exists a continuous, linear, one-to-one operator from E into $\ell_p(\Gamma)$, for some p $(1 < p < \infty)$ and some index set Γ of nonmeasurable cardinal. Suppose that $A \subset C(\Omega)$ is an inverse-closed subalgebra, such that $P|_\Omega \in A$ for every $P \in \mathcal{P}(E)$. Then* $\operatorname{Hom} A = \Omega$. *In particular* $\operatorname{Hom} \mathcal{R}(\Omega) = \operatorname{Hom} \mathcal{A}(\Omega) = \operatorname{Hom} C^m(\Omega) = \Omega$, $(1 \leq m \leq \infty)$.

PROOF. We shall consider two cases.

CASE (i). Suppose first that $\Omega = E = \ell_2(\Gamma)$, where Γ has nonmeasurable cardinal. Let $(\pi_\gamma)_{\gamma\in\Gamma}$ denote the biorthogonal functionals in $\ell_2(\Gamma)^*$ associated to the unit vectors $(e_\gamma)_{\gamma\in\Gamma}$ of $\ell_2(\Gamma)$.

Now let $\varphi \in \operatorname{Hom} A$ be given. For each $\gamma \in \Gamma$ define $a_\gamma = \varphi(\pi_\gamma)$. It follows from Lemma 1.1 that φ is a positive functional on A, and therefore, for each finite subset $\Lambda \subset \Gamma$:

$$\sum_{\gamma\in\Lambda} a_\gamma^2 = \varphi\Big(\sum_{\gamma\in\Lambda} \pi_\gamma^2\Big) \leq \varphi\Big(\sum_{\gamma\in\Gamma} \pi_\gamma^2\Big) < +\infty.$$

Hence $a = (a_\gamma) \in \ell_2(\Gamma)$.

For each $u \in \ell_\infty(\Gamma)$ we define the polynomial $P_u = \sum_{\gamma\in\Gamma} \pi_\gamma(u)(\pi_\gamma - a_\gamma)^2$ on $\ell_2(\Gamma)$.

CLAIM 1. Given a sequence $(u_n)_{n\in\mathbf{N}} \subset \ell_\infty(\Gamma)$, there exists some $b \in \ell_2(\Gamma)$ such that $\varphi(P_{u_n}) = P_{u_n}(b)$, for all $n \in \mathbf{N}$.

https://doi.org/10.4153/CJM-1994-041-3 Published online by Cambridge University Press

Let $\beta_n = (1 + \|u_n\|_\infty)^{-1}$ and consider $f_n = \beta_n P_{u_n}$. It is clear that for each summable sequence (α_n) of positive numbers, $\sum_n \alpha_n f_n, \sum_n \alpha_n f_n^2 \in \mathcal{P}(E) \subset A$. Thus Claim 1 follows from Proposition 1.10.

Next we define the linear functional $\Psi: \ell_\infty(\Gamma) \longrightarrow \mathbf{R}$ by

$$\Psi(u) = \varphi(P_u).$$

CLAIM 2. The restriction $\Psi_{|F}$ is $\sigma\big(\ell_\infty(\Gamma), \ell_1(\Gamma)\big)$-continuous on each $\sigma\big(\ell_\infty(\Gamma), \ell_1(\Gamma)\big)$-separable subspace F of $\ell_\infty(\Gamma)$.

To prove Claim 2, let $(v_n) \subset F$ be a $\sigma(\ell_\infty\big(\Gamma), \ell_1(\Gamma)\big)$-dense sequence in F. Claim 1 provides us with some $b \in \ell_2(\Gamma)$ such that $\varphi(P_{v_n}) = P_{v_n}(b)$ for all $n \in \mathbf{N}$. It is sufficient to show that, in fact, $\Psi(u) = P_u(b)$ for every $u \in F$. Then fix $u \in F$. Again by Claim 1 there exists $c \in \ell_2(\Gamma)$ so that $\varphi(P_u) = P(c)$ and $\varphi(P_{v_n}) = P_{v_n}(c)$ for all $n \in \mathbf{N}$. By the $\sigma\big(\ell_\infty(\Gamma), \ell_1(\Gamma)\big)$-density of (v_n), for each $k \in \mathbf{N}$, we can find some $n_k \in \mathbf{N}$ such that

$$|P_u(b) - P_{v_{n_k}}(b)| < \frac{1}{k} \quad \text{and} \quad |P_u(c) - P_{v_{n_k}}(c)| < \frac{1}{k}.$$

Therefore $\Psi(u) = P(c) = \lim_k \varphi(P_{v_{n_k}}) = P_u(b)$.

Now we use the fact that $\ell_1(\Gamma)$ is weakly realcompact when Γ has nonmeasurable cardinal (see [10]). Then we obtain from Claim 2 and the characterization of weak realcompactness given by Corson in [8] that Ψ is $\sigma\big(\ell_\infty(\Gamma), \ell_1(\Gamma)\big)$-continuous on $\ell_\infty(\Gamma)$, and therefore $\Psi(u) = \Sigma_\gamma u_\gamma \Psi(e_\gamma)$ for every $u = (u_\gamma) \in \ell_\infty(\Gamma)$. Since $\Psi(e_\gamma) = \varphi(P_{e_\gamma}) = \varphi\big((\pi_\gamma - a_\gamma)^2\big) = 0$ for each γ, it follows that $\Psi = 0$. In particular, for the polynomial $P_1 = \Sigma_\gamma(\pi_\gamma - a_\gamma)^2$ we have that $\varphi(P_1) = 0$.

Finally, consider any $f \in A$. By Lemma 1.1, there exists $b = (b_\gamma) \in \ell_2(\Gamma)$ such that $\varphi(f) = f(b)$ and $0 = \varphi(P_1) = P_1(b)$. Hence $b = a$, and therefore $\varphi = \delta_a$.

CASE (ii). The general case. Let $T: E \longrightarrow \ell_p(\Gamma)$ be a continuous, linear, one-to-one operator, where $1 < p < \infty$ and Γ has nonmeasurable cardinal. Choose and odd integer N with $2N \geq p$, and consider the map $h: \Omega \longrightarrow \ell_2(\Gamma)$ defined by

$$h(x) = \Big(\big(\pi_\gamma(Tx)\big)^N\Big)_{\gamma\in\Gamma} \quad (x \in \Omega).$$

Now $\mathrm{Hom}\Big(\mathcal{R}\big(\ell_2(\Gamma)\big)\Big) = \ell_2(\Gamma)$ by Case (i). Also, for each $a = (a_\gamma) \in \ell_2(\Gamma)$, the function $f_a = \Sigma_\gamma(\pi_\gamma - a_\gamma)^2 \in \mathcal{R}\big(\ell_2(\Gamma)\big)$ satisfies $f_a^{-1}(0) = \{a\}$. Finally, it is clear that h is one-to-one and $f \circ h \in A$ for every $f \in \mathcal{R}\big(\ell_2(\Gamma)\big)$. Then the conclusion follows from Lemma 2.2.

REMARKS. (1) The hypothesis on E in Theorem 2.4 is satisfied if E is a separable space, or E is the dual of a separable space or, more generally, if E is a closed subspace of $C(K)$, where K is a compact, separable space. In this case, we can indeed consider the operator $T: E \longrightarrow \ell_2$ defined by $Tx = \big(2^{-n}x(t_n)\big)_{n\in\mathbf{N}}$, where (t_n) is a dense sequence in K.

(2) Recall that super-reflexive Banach spaces can be defined as those spaces admitting an equivalent uniformly convex norm (see for instance [9]). It follows from ([20],

https://doi.org/10.4153/CJM-1994-041-3 Published online by Cambridge University Press

Lemma 9) that the hypothesis on E in Theorem 2.4 is also satisfied whenever E is a superreflexive space with nonmeasurable cardinal.

(3) The requirement on the cardinality of Γ in Theorem 2.4 is very mild, since in fact it is not known whether measurable cardinals exist. On the other hand, if we suppose that Γ has measurable cardinal, it follows that $E = \ell_2(\Gamma)$ is not realcompact (see [13]). In this case let υE denote the Hewitt-Nachbin realcompactification of E. Now if $A \subset C(E)$ is a subalgebra as in Theorem 2.4, each point $\xi \in \upsilon E \setminus E$ gives a homomorphism $\varphi(f) = \hat{f}(\xi)$ on A and, since A separates points and closed sets of E, φ is not given by evaluation at any point of E.

(4) In Theorem 2.4 we cannot change the condition "A is inverse-closed" by "A is closed under bounded inversion". Consider as an example $E = \ell_2$, let Ω be the open unit ball of E and define

$$A = \left\{P/Q : P, Q \in \mathcal{P}(\ell_2); \inf_{x\in\Omega} |Q(x)| > 0\right\}.$$

Then $A \subset C(\Omega)$ is a subalgebra with unit, closed under bounded inversion, which contains every polynomial function on Ω. Now let $\xi \in \beta\Omega \setminus \Omega$. Since each function in A is bounded, ξ defines the algebra homomorphism $\varphi(f) = \hat{f}(\xi)$ on A. But since A separates points and closed sets of Ω, φ is not given by evaluation at any point of Ω.

The result of Theorem 2.4 does not hold for arbitrary Banach spaces, as the following example shows. An analogous example can be seen in [15].

EXAMPLE 2.6. Let $E = c_0(\Gamma)$, where Γ is uncountable, and let $\Omega = c_0(\Gamma) \setminus \{0\}$. Then:

(1) For every real-analytic function $f : \Omega \longrightarrow \mathbf{R}$, there exists $\lim_{x\to 0} f(x)$.

(2) The algebra homomorphism $\varphi : \mathcal{A}(\Omega) \longrightarrow \mathbf{R}$ defined by $\varphi(f) = \lim_{x\to 0} f(x)$ is not given by evaluation at any point of Ω.

PROOF. (1) Let $f : \Omega \longrightarrow \mathbf{R}$ be a real-analytic function, and consider any sequence $(u^m) \subset \Omega$ with $u^m \longrightarrow 0$. For each $m \in \mathbf{N}$ there exist $\varepsilon_m > 0$ and a sequence $(P_n^m)_{n\in\mathbf{N}}$ where each $P_n^m \in \mathcal{P}\left({}^n c_0(\Gamma)\right)$, such that

$$(*) \qquad f(u^m + h) = f(u^m) + \sum_{n=1}^{\infty} P_n^m(h), \quad \text{for } \|h\| < \varepsilon_m.$$

It is well-known that every continuous polynomial on $c_0(\Gamma)$ is the uniform limit, on each bounded set, of a sequence of polynomials in a finite number of continuous linear functionals on $c_0(\Gamma)$ (see [3] and [23]). Therefore the set

$$\bigcup_{m,n=1}^{\infty} \{\gamma \in \Gamma : P_n^m(e_\gamma) \neq 0\}$$

is countable, where $(e_\gamma)_{\gamma\in\Gamma}$ denotes the unit vectors of $c_0(\Gamma)$. Thus we can select $\gamma \in \Gamma$ such that $P_n^m(e_\gamma) = 0$ for every $m, n \in \mathbf{N}$ and also $u^m + te_\gamma \neq 0$ for every $m \in \mathbf{N}$ and every $t \in \mathbf{R}$. Now for each $m \in \mathbf{N}$, we obtain from $(*)$ that $f(u^m + te_\gamma) = f(u^m)$ if $|t| < \varepsilon_m$; and

https://doi.org/10.4153/CJM-1994-041-3 Published online by Cambridge University Press

since the function $t \longrightarrow f(u^m + te_\gamma)$ is real-analytic on $\mathbf{R}$, we have that $f(u^m + te_\gamma) = f(u^m)$ for every $t \in \mathbf{R}$. In particular, $f(u^m + e_\gamma) = f(u^m)$ and, since $\lim_m u^m + e_\gamma = e_\gamma$, there exists

$$\lim_m f(u^m) = \lim_m f(u^m + e_\gamma) = f(e_\gamma).$$

(2) For each $a \in \Omega$, we can consider a continuous linear functional f on $c_0(\Gamma)$ such that $f(a) \neq 0$, and then $0 = \varphi(f) \neq f(a)$.

Let Ω be an open subset of $c_0(\Gamma)$, where Γ is uncountable. Since $c_0(\Gamma)$ admits C^∞-partitions of unity (see [24]), it follows from Corollary 1.9 that $\operatorname{Hom} C^m(\Omega) = \Omega$ (see also [17]). However, in the case of real-analytic functions the situation is different. In fact, combining Example 2.6 with Theorem 2.7, we can see that the shape of Ω plays a role.

THEOREM 2.7. *Let Ω be an open ball of $c_0(\Gamma)$, or let $\Omega = c_0(\Gamma)$. Suppose that $A \subset \mathcal{A}(\Omega)$ is an inverse-closed subalgebra, such that $P_{|\Omega} \in A$ for every $P \in \mathcal{P}\big(c_0(\Gamma)\big)$. Then* $\operatorname{Hom} A = \Omega$.

PROOF. First note that if Γ is countable, then the result follows from Theorem 2.4 (see also Remark 2.5) for arbitrary Ω. Therefore we shall consider that Γ is uncountable. We prove the result for $\Omega = c_0(\Gamma)$, only trivial modifications being necessary in the other case.

Let $(\pi_\gamma)_{\gamma\in\Gamma}$ denote the unit vector functionals in $c_0(\Gamma)^*$. Now let $\varphi \in \operatorname{Hom} A$ be given. For each $\gamma \in \Gamma$ define $a_\gamma = \varphi(\pi_\gamma)$, and set $a = (a_\gamma)_{\gamma\in\Gamma}$.

For each countable subset Λ of Γ, consider the natural projection $\pi_\Lambda : c_0(\Gamma) \longrightarrow c_0(\Lambda)$, consider also the algebra

$$A_\Lambda = \big\{g \in C\big(c_0(\Lambda)\big) : g \circ \pi_\Lambda \in A\big\}$$

and define $\varphi_\Lambda \in \operatorname{Hom} A_\Lambda$ by $\varphi_\Lambda(g) = \varphi(g \circ \pi_\Lambda)$. Thus by Theorem 2.4 there exists $b_\Lambda \in c_0(\Lambda)$ such that

$$\varphi_\Lambda(g) = g(b_\Lambda), \quad \text{for all } g \in A_\Lambda.$$

In particular, for every $\gamma \in \Lambda$ we have that

$$a_\gamma = \varphi(\pi_\gamma) = \varphi_\Lambda(\pi_\gamma) = \pi_\gamma(b_\Lambda).$$

That is, $b_\Lambda = (a_\gamma)_{\gamma\in\Lambda}$. It follows that, for each countable set Λ of Γ, $(a_\gamma)_{\gamma\in\Lambda} \in c_0(\Lambda)$; and therefore $a \in c_0(\Gamma)$.

We shall see that $\varphi = \delta_a$. Consider $f \in A$. Since f is real-analytic, there exist an r-ball $B_r(a)$ and a sequence (P_n) where each $P_n \in \mathcal{P}\big({}^n c_0(\Gamma)\big)$ such that

$$f(x) = f(a) + \sum_{n=1}^{\infty} P_n(x-a), \quad \text{for } x \in B_r(a).$$

Using the fact that every continuous polynomial on $c_0(\Gamma)$ is the uniform limit, on each bounded set, of a sequence of polynomials in a finite number of continuous linear functionals ([3], [23]), we obtain that a countable subset Λ of Γ exists, such that $P_n(u) = P_n(v)$

https://doi.org/10.4153/CJM-1994-041-3 Published online by Cambridge University Press

(for all $n \geq 1$) whenever $\pi_\Lambda(u) = \pi_\Lambda(v)$. Therefore if $x, y \in B_r(a)$ and $\pi_\Lambda(x) = \pi_\Lambda(y)$ we have that $f(x) = f(y)$. Now let $i_\Lambda: c_0(\Lambda) \longrightarrow c_0(\Gamma)$ be the natural inclusion given by

$$\pi_\gamma\big(i_\Lambda(x)\big) = \begin{cases} x_\gamma, & \text{if } \gamma \in \Lambda \\ a_\gamma, & \text{if } \gamma \notin \Lambda \end{cases}$$

and consider the function $h = f \circ i_\Lambda \circ \pi_\Lambda$. It is clear that $h_{|B_r(a)} = f_{|B_r(a)}$ and, since f and h are real-analytic functions on $c_0(\Gamma)$, it follows that $f = h$. Then $f \circ i_\Lambda \in A_\Lambda$ and, as we obtained before, $\varphi(f) = \varphi_\Lambda(f \circ i_\Lambda) = (f \circ i_\Lambda)\big(\pi_\Lambda(a)\big) = f(a)$.

In closing, we recall the following open problem:

PROBLEM 2.8. Characterize the subsets Ω of $c_0(\Gamma)$, for uncountable Γ, such that $\mathrm{Hom}\,\mathcal{R}(\Omega) = \mathrm{Hom}\,\mathcal{A}(\Omega) = \Omega$.

REFERENCES

1. F. W. Anderson, *Approximation in systems of real-valued continuous functions*, Trans. Amer. Math. Soc. **103**(1962), 249–271.

2. J. Arias-de-Reyna, *A real-valued homomorphism on algebras of differentiable functions*, Proc. Amer. Math. Soc **104**(1988), 1054–1058.

3. R. M. Aron, *Compact polynomials and compact differentiable mappings between Banach spaces*, Sém. P. Lelong 1974/75, L.N.M. **524**, Springer Verlag, 1976, 231–222.

4. P. Biström, S. Bjon and M. Lindström, *Remarks on homomorphisms on certain subalgebras of $C(X)$*, Math. Japon. **36**(1991).

5. ———, *Homomorphisms on some function algebras*, Monatsh. Math. **111**(1991), 93–97.

6. ———, *Function algebras on which homomorphisms are point evaluations on sequences*, Manuscripta Math. **73**(1991), 179–185.

7. P. Biström and M. Lindström, *Homomorphisms on $C^\infty(E)$ and C^∞-bounding sets*, Monatsh. Math. **115** (1993), 257–266.

8. H. H. Corson, *The weak topology of a Banach space*, Trans. Amer. Math. Soc. **101**(1961), 1–15.

9. J. Diestel, *Geometry of Banach spaces. Selected topics*, L.N.M. **485**, Springer Verlag.

10. G. A. Edgar, *Measurability in a Banach space, II*, Indiana Univ. Math. J. **28**(1979), 559–579.

11. R. Engelking, *General Topology*, Monograf. Math. Warsaw, (1977).

12. M. I. Garrido and F. Montalvo, *Uniform approximation theorems for real-valued continuous functions*, Topology Appl. **45**(1992), 145–155.

13. L. Gillman and M. Jerison, *Rings of continuous functions*, Princeton, New Jersey, 1960.

14. J. Gómez and J. G. Llavona, *Multiplicative functionals on function algebras*, Rev. Mat. Univ. Complutense Madrid **1**(1988), 19–22.

15. A. Hirschowitz, *Sur le non-plongement des varietés analytiques banachiques réeles*, C. R. Acad. Sci. Paris **269**(1969), 844–846.

16. J. A. Jaramillo, *Algebras de funciones continuas y diferenciables. Homomorfismos e interpolación*, Thesis. Univ. Complutense, Madrid, 1987.

17. ———, *Multiplicative functionals on algebras of differentiable functions*, Arch. Math. **58**(1992), 384–387.

18. J. A. Jaramillo and J. G. Llavona, *On the spectrum of $C_b^1(E)$*, Math. Ann. **287**(1990), 531–538.

19. T. Jech, *Set Theory*, Academic Press, 1978.

20. K. John, H. Torunczyk and V. Zizler, *Uniformly smooth partitions of unity on superreflexive Banach spaces*, Studia Math. **70**(1981), 129–137.

21. A. Kriegl, P. Michor and W. Schachermayer, *Characters on algebras of smooth functions*, Ann. Global Anal. Geom. **7**(1989), 85–92.

22. E. A. Michael, *Locally multiplicatively-convex topological algebras*, Mem. Amer. Math. Soc. **11**(1952).

23. A. Pelczynski, *A theorem of Dunford-Pettis type for polynomial operators*, Bull. Acad. Polon. Sci. **11**(1963), 379–386.

24. K. Sundaresan and S. Swaminathan, *Geometry and nonlinear analysis in Banach spaces*, L.N.M. **1131**, Springer Verlag, 1985.

Departamento de Matemáticas
Universidad de Extremadura
Avda. Elvas, s/n
06071-Badajoz
Spain

Departamento de Análisis Matemático
Universidad Complutense
28040-Madrid
Spain

Departamento de Análisis Matemático
Universidad Complutense
28040-Madrid
Spain