TOPOLOGICAL HOMOTHETIES ON COMPACT METRIZABLE SPACES

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Notation and definitions.

Definition 1. Let (X, ρ) be a metric space and $\phi: X \to X$ a continuous self-mapping of X. We shall call ϕ and *α-contraction on* (X, ρ) if and only if $\alpha \in [0, 1)$ and $\forall x, y \in X$: $\rho(\phi(x), \phi(y)) \leqq \alpha\rho(x, y)$. We shall call ϕ an *α-homothety on* (X, ρ) if and only if $\alpha > 0$ and $\forall x, y \in X$: $\rho(\phi(x), \phi(y)) = \alpha\rho(x, y)$.

Definition 2. Let X be a metrizable topological space and $\phi: X \to X$ a continuous self-mapping of X. We shall call ϕ a *topological α-contraction on* X if and only if there exists a metric ρ on X inducing the given topology and such that ϕ is an α-contraction on (X, ρ). Similarly, we introduce a *topological α-homothety.*

Remark. If $\phi: X \to X$ is a homeomorphism and at the same time a topological α-contraction on X, we say that ϕ is a topologically α-contractive homeomorphism on X. If $\phi: X \to X$ is defined on the metric space (X, ρ), then the statement: ϕ is a topological α-contraction on X is to be understood *without regarding* the particular metric, taking into account only the topology on X defined by ρ.

Our main objective in this paper is to characterize topological α-homotheties of compact metrizable spaces by a very simple condition, namely:

If $\phi: X \to X$ is a homeomorphism of a compact metrizable space X into itself and $\alpha \in (0, 1)$, then ϕ is a topological α-homothety on X if and only if the intersection $\bigcap_{n=1}^{\infty}\phi^n(X)$ of all iterated images of X is a singleton.

LEMMA 1. *Let (A, ρ) be a bounded metric space and $\psi: A \to A$ a continuous mapping of A into itself. Then, for any $\alpha \in (0, 1)$ the expression $\rho^*(x, y)$, defined by*

$$\rho^*(x, y) = \sup_n\{\alpha^n\rho(\psi^n(x), \psi^n(y))\},$$

is a metric on A, and is topologically equivalent to ρ, where the supremum is taken over the set of all non-negative integers $n = 0, 1, 2, \ldots$ and $\psi^0(x)$ stands for x.

Proof. To show that ρ^* is a metric it is only necessary to check the triangle inequality for ρ^*. Let $x, y \in A$; then, following the definition of ρ^*, the number

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$\rho^*(x, y)$ is the supremum of the set $\{\rho(x,y), \alpha\rho(\psi(x), \psi(y)), \alpha^2\rho(\psi^2(x), \psi^2(y)), \ldots\}$. Since ρ is bounded and $\alpha \in (0, 1)$, the supremum is attained on this set, and, therefore, for each pair $x, y \in A$ there exists an integer n such that $\rho^*(x, y) = \alpha^n\rho(\psi^n(x), \psi^n(y))$. We now let $x, y, z \in A$ be given points in A, then $\rho^*(x, z) = \alpha^k\rho(\psi^k(x), \psi^k(z))$ for some k and applying the triangle inequality for ρ on the points $\psi^k(x)$, $\psi^k(y)$, and $\psi^k(z)$, we have that

$$\alpha^k\rho(\psi^k(x), \psi^k(z)) \leqq \alpha^k\rho(\psi^k(x), \psi^k(y)) + \alpha^k\rho(\psi^k(y), \psi^k(z)),$$

and since $\alpha^k\rho(\psi^k(x), \psi^k(y)) \leqq \rho^*(x, y)$ and $\alpha^k\rho(\psi^k(y), \psi^k(z)) \leqq \rho^*(y, z)$, the triangle inequality follows. To prove the equivalence of ρ^* with ρ, we observe that $\rho(x, y) \leqq \rho^*(x, y)$; thus, there is only to show that

$$\rho(x_n, x) \to 0 \Rightarrow \rho^*(x_n, x) \to 0.$$

Let us suppose that this is not the case. Then, since ρ (hence, also, ρ^*) is bounded, there exists a sequence $\{x_n\}$ and a point $x \in A$ such that

$$\rho(x_n, x) \to 0 \quad \text{and} \quad \rho^*(x_n, x) \to a > 0$$

for some positive a. Since for each $n = 1, 2, \ldots$ there exists a non-negative integer k_n such that $\rho^*(x_n, x) = \alpha^{k_n}\rho(\psi^{k_n}(x_n), \psi^{k_n}(x))$, we have that

$$\alpha^{k_n}\rho(\psi^{k_n}(x_n), \psi^{k_n}(x)) \to a > 0.$$

If the sequence $\{k_n\}$ were not bounded, this is not possible, since then,

$$\liminf \alpha^{k_n}\rho(\psi^{k_n}(x_n), \psi^{k_n}(x)) = 0.$$

If the sequence $\{k_n\}$ is bounded, then at least one of the integers k_n, say k, is infinitely repeated, and there exists a subsequence $\{x_{ln}\}$ of $\{x_n\}$ such that

$$\alpha^k\rho(\psi^k(x_{ln}), \psi^k(x)) \to a > 0.$$

This, however, contradicts the supposition that $\rho(x_n, x) \to 0$ since ψ is assumed to be continuous, and our theorem follows.

LEMMA 2. *Let (X, ρ) be a bounded metric space and ϕ: $X \to X$ an α-contractive homeomorphism of (X, ρ) into itself, and suppose that there exists a bounded metric space (X^*, ρ^*) such that*

(i) *$X \subseteq X^*$ and $\rho(x, y) = \rho^*(x, y)$ on X and*

(ii) *there exists a continuous mapping ψ, ψ: $X^* \to X^*$ such that $\psi(x) = \phi^{-1}(x)$ for $x \in \phi(X)$.*

Then ϕ is a topological α-homothety.

Proof. Since ϕ is an α-contraction with respect to ρ, we have that $\rho^*(\phi(x), \phi(y)) \leqq \alpha\rho^*(x, y)$ for all $x, y \in X$ since ρ^* coincides with ρ on X. Following Lemma 1, the metric ρ^{**}, defined by

$$\rho^{**}(x, y) = \sup_n\{\alpha^n\rho^*(\psi^n(x), \psi^n(y))\},$$

defines a metric on X^*, equivalent to ρ^*. Let now $x, y \in X$; then, the number

$\rho^{**}(x, y)$ is the maximum of the set

$$\{\rho^*(x, y), \alpha\rho^*(\psi(x), \psi(y)), \alpha^2\rho^*(\psi^2(x), \psi^2(y)), \ldots\}$$

and similarly, the number $\rho^{**}(\phi(x), \phi(y))$ is (taking into account that $\psi(\phi(x)) = x$ on X) the maximum of the set

$$\{\rho^*(\phi(x), \phi(y)), \alpha\rho^*(x, y), \alpha^2\rho^*(\psi(x), \psi(y)), \ldots\}.$$

But since $\rho^*(x, y) = \rho(x, y)$ and $\rho^*(\phi(x), \phi(y)) = \rho(\phi(x), \phi(y))$ $(x, y \in X)$, we have that $\rho^*(\phi(x), \phi(y)) \leqq \alpha\rho^*(x, y)$, therefore the maximum is equal to the maximum of the set $\{\alpha\rho^*(x, y), \alpha^2\rho^*(\psi(x), \psi(y)), \ldots\}$, and we have the equality $\rho^{**}(\phi(x), \phi(y)) = \alpha\rho^{**}(x, y)$ for $x, y \in X$.

Now we have prepared our way to prove the crucial lemma.

LEMMA 3. *Let X be a compact metrizable space and ϕ: $X \to X$ a topologically α-contractive homeomorphism on X. Then ϕ is a topological α-homothety on X.*

Proof. There exists a topological embedding μ: $X \to H$ of X into the Hilbert cube H, and identifying X with $\mu(X)$ we can consider X to be a closed subset of H. Since $\phi(X)$ is compact in X and ϕ^{-1} is continuous on $\phi(X)$, the theorem of Tietze ensures that the function ϕ^{-1} can be extended over H, i.e., there exists ψ: $H \to H$ such that $\psi(x) = \phi^{-1}(x)$ for $x \in \phi(X) \subseteq X \subseteq H$.

Since ϕ is a topological α-contraction on X, there exists a metric ρ on X such that ϕ is an α-contraction on (X, ρ). Since X is closed in H, the metric ρ defined on X can be extended over H (see **1**). Denoting this extension of ρ by ρ^*, we have a metric space (H, ρ^*), homeomorphic to the Hilbert cube, and therefore bounded, and we see that the metric space (H, ρ^*), together with the mapping ψ: $H \to H$, satisfies the conditions imposed on (X^*, ρ^*) in Lemma 2, which proves our assertion. The consequence of this lemma is our main theorem.

THEOREM. *Let X be a compact metrizable space, ϕ a homeomorphism of X into itself, and $\alpha \in (0, 1)$. Then ϕ is a topological α-homothety on X if and only if the intersection of all iterated images $\phi^n(X)$ of X is a one-point set, i.e., if and only if there exists an $a \in X$ such that*

$$\bigcap_{n=1}^{\infty} \phi^n(X) = \{a\}.$$

Proof. In view of (**2**), the last condition implies that ϕ *is a topological α-contraction and, therefore, a topological α-homothety because of our Lemma* 3. If, on the other hand, ϕ is a topological α-homothety, then $\phi^n(X)$ shrinks, evidently, to the fixed point a.

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