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TRANSFORMATIONS ON DENSITY OPERATORS PRESERVING GENERALISED ENTROPY OF A CONVEX COMBINATION

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Abstract

We aim to characterise those transformations on the set of density operators (which are the mathematical representatives of the states in quantum information theory) that preserve a so-called generalised entropy of *one* fixed convex combination of operators. The characterisation strengthens a recent result of Karder and Petek where the preservation of the same quantity was assumed for *all* convex combinations.

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1. Introduction and statement of the main result

We begin with a brief survey of results which have led us to the investigation of the problem treated in this paper. The study of automorphisms, symmetries, or transformations which preserve certain numerical quantities attached to the underlying structure is an important task in several branches of mathematics. One of the most fundamental results in this direction is the celebrated theorem of Wigner on quantum mechanical symmetry transformations. The theorem asserts that every transformation of the set of all rank-one projections on a Hilbert space preserving the transition probability (that is, the trace of the product) is necessarily implemented by either a unitary or an antiunitary similarity transformation. Wigner's theorem has a considerable literature. (See [10, Sections 2.1 and 2.2] and the comprehensive survey article [7]. For recent elementary proofs, the reader can consult [1, 6].)

In the series of papers [2, 4, 11, 17] motivated by Wigner's theorem, there are several Wigner-type results concerning transformations on the set of density matrices (that is, positive semidefinite matrices with unit trace) which preserve different sorts of relative entropies and quantum divergences. These quantities are very important in quantum information theory. In the recent paper [8], Karder and Petek considered a

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problem which is somewhat different in nature. They completely described maps on density operators preserving certain entropy-like quantities of *any* convex combination of operators. More precisely, for a given strictly convex function $f : [0,1] \to \mathbb{R}$ and density operator A on a finite-dimensional Hilbert space H, they introduced the numerical quantity

$$F(A) = \sum_{j=1}^{d} f(\lambda_j),$$

where the numbers λ_j are the eigenvalues of A counted with multiplicity. They described the structure of those transformations of the set formed by all density operators on H which satisfy

$$F(\alpha\phi(A) + (1-\alpha)\phi(B)) = F(\alpha A + (1-\alpha)B) \quad \text{for all } \alpha \in]0,1[.$$

For further results on preserver problems which are related to convex combinations, the interested reader is referred to [5, 13, 14].

In the present paper, we consider the more general problem where the preservation of $F(\alpha A + (1-\alpha)B)$ is assumed only for *one fixed* real number $\alpha \in]0,1[$, not for all. It turns out that the proof of our result is much shorter than that presented in [8]. We obtain the interesting fact that the corresponding symmetries are exactly the unitary–antiunitary similarity transformations. Thus, the main result of the current paper can be viewed as a Wigner type characterisation of such maps. Note that from the quantity $F(A)$, by particular choices of the function f, one can recover some widely used entropies in quantum information theory, as demonstrated in the following examples.

EXAMPLE 1.1. If

$$f(x) = \begin{cases} x\log_2 x, & x > 0, \\ 0, & x = 0, \end{cases}$$

and A is an operator in $\mathcal{S}(H)$, then $F(A)$ is the negative of the usual von Neumann entropy of A.

EXAMPLE 1.2. Let $p > 0$, $p \neq 1$, be fixed and take $f(x) = (x^p - x)/(p-1)$ for $x \in [0,1]$. For an operator $A \in \mathcal{S}(H)$, the quantity $F(A) = (p-1)^{-1}(\operatorname{Tr} A^p - 1)$ is the negative of the Tsallis entropy of A.

EXAMPLE 1.3. Let $p > 0$, $p \neq 1$, be fixed and consider $f(x) = \operatorname{sgn}(p-1)x^p$ for $x \in [0,1]$. For an operator $A \in \mathcal{S}(H)$, the quantity $F(A) = \operatorname{sgn}(p-1)\operatorname{Tr} A^p$ is closely related to the Rényi entropy $D(A) = (1-p)^{-1}\log_2|F(A)|$.

For more on entropy in operator algebras and entropy preserver problems, see, for instance, [3, 9, 12, 15, 16].

In what follows, the symbol H stands for a complex Hilbert space with dimension $2 \leq d := \dim H < \infty$ and $\mathcal{S}(H)$ denotes the set of density operators acting on H. We now state our solution to the problem formulated in the preceding paragraphs.

THEOREM 1.4. *Let α be a real number in $]0,1[$ and let $\phi\colon \mathcal{S}(H)\to\mathcal{S}(H)$ be a transformation satisfying*

$$F(\alpha\phi(A)+(1-\alpha)\phi(B)) = F(\alpha A+(1-\alpha)B), \quad A,B\in\mathcal{S}(H). \tag{1.1}$$

Then there exists either a unitary or an antiunitary operator U on H such that

$$\phi(A) = UAU^*, \quad A\in\mathcal{S}(H).$$

2. Proof of Theorem 1.4

First observe that an affine perturbation of the function f does not have any influence on the preservation of the quantity $F(\alpha A+(1-\alpha)B)$. Indeed, defining $g\colon [0,1]\to\mathbb{R}$, $g(x)=f(x)+ax+b$ $(x\in[0,1])$ with some $a,b\in\mathbb{R}$,

$$\sum_{j=1}^{d} g(\lambda_j) = \sum_{j=1}^{d} f(\lambda_j) + a\cdot\sum_{j=1}^{d}\lambda_j + d\cdot b = F(A)+a+d\cdot b, \quad A\in\mathcal{S}(H),$$

which verifies our claim. So assume, as we may, that $f(0)=f(1)=0$. In what follows, rank-one (orthogonal) projections on H will appear several times; their set is denoted by $\mathcal{P}_1(H)$. For any self-adjoint operators A,B on H, we write $A\le B$ if $B-A$ is positive. We now derive a formula which will be needed later.

LEMMA 2.1. *If $P,Q\in\mathcal{P}_1(H)$ are projections, then*

$$F(\alpha P+(1-\alpha)Q) = f\Big(\frac{1+\sqrt{1+c_\alpha(\operatorname{Tr}PQ-1)}}{2}\Big) + f\Big(\frac{1-\sqrt{1+c_\alpha(\operatorname{Tr}PQ-1)}}{2}\Big)$$

with a real number c_α that does not depend on P,Q. Furthermore, the range of the function $g_\alpha\colon [0,1]\to\mathbb{R}$, $g_\alpha(x)=\sqrt{1+c_\alpha(x-1)}/2$ for $x\in[0,1]$ is contained in $[0,1/2]$.

PROOF. Choose an orthonormal basis $\{e_1,\dots,e_d\}$ in H such that $\operatorname{rng} Q=\operatorname{span}\{e_1\}$ and $\operatorname{rng} P\subseteq\operatorname{span}\{e_1,e_2\}$. With respect to this basis, we can write P in the matrix form

$$P = \begin{pmatrix} \operatorname{Tr}PQ & \varepsilon\sqrt{\operatorname{Tr}PQ(1-\operatorname{Tr}PQ)} \\ \bar{\varepsilon}\sqrt{\operatorname{Tr}PQ(1-\operatorname{Tr}PQ)} & 1-\operatorname{Tr}PQ \end{pmatrix} \bigoplus \mathbf{0}_{d-2}$$

with some complex number ε of modulus one, and henceforth

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \bigoplus \mathbf{0}_{d-2}.$$

Thus,

$$\alpha P+(1-\alpha)Q = \begin{pmatrix} \alpha\operatorname{Tr}PQ+(1-\alpha) & \alpha\varepsilon\sqrt{\operatorname{Tr}PQ(1-\operatorname{Tr}PQ)} \\ \alpha\bar{\varepsilon}\sqrt{\operatorname{Tr}PQ(1-\operatorname{Tr}PQ)} & \alpha(1-\operatorname{Tr}PQ) \end{pmatrix} \bigoplus \mathbf{0}_{d-2}.$$

It is an elementary task to check that the characteristic polynomial of the matrix of $\alpha P+(1-\alpha)Q$ is

$$\varphi(s) = (-1)^d s^{d-2}(s^2-s+\alpha(1-\alpha)(1-\operatorname{Tr}PQ))$$

and thus its nonzero eigenvalues are

$$\frac{1 \pm \sqrt{1 + 4\alpha(1-\alpha)(\operatorname{Tr} PQ - 1)}}{2}.$$

Now defining $c_\alpha := 4\alpha(1-\alpha)$ yields the first statement of Lemma 2.1, and then the second follows very easily. □

Lemma 2.2. *Keeping the notation in Lemma 2.1, the function*

$$x \mapsto f(\tfrac{1}{2} + g_\alpha(x)) + f(\tfrac{1}{2} - g_\alpha(x)), \quad x \in [0,1],$$

is injective.

Proof. The function $h\colon [0,1] \to \mathbb{R}$ defined by

$$h(x) = f(\tfrac{1}{2} + x) + f(\tfrac{1}{2} - x), \quad x \in [0, \tfrac{1}{2}],$$

is strictly increasing since f is strictly convex. Since g_α is injective, the result follows. □

The following lemma characterises the rank-one projections in terms of the quantity $F(A)$.

Lemma 2.3. *For any $A \in \mathcal{S}(H)$, we have $F(A) = 0$ if and only if $A \in \mathcal{P}_1(H)$.*

Proof. Since $f(0) = f(1) = 0$ and f is strictly convex, $f|_{]0,1[} < 0$. For any $A \in \mathcal{S}(H)$, it follows that $F(A) = 0$ exactly when the spectrum of A lies in $\{0, 1\}$. But this property is equivalent to A being a projection, that is, $A \in \mathcal{P}_1(H)$. □

In the remaining part of this section, $\phi\colon \mathcal{S}(H) \to \mathcal{S}(H)$ denotes a transformation admitting the property (1.1). Substituting $B = A$ into (1.1) shows that ϕ satisfies $F(\phi(A)) = F(A)$ for all operators $A \in \mathcal{S}(H)$. Thus, we have the following immediate consequence.

Corollary 2.4. *The transformation ϕ maps $\mathcal{P}_1(H)$ into itself.*

By Lemmas 2.1 and 2.2, we infer that $F(\alpha P + (1-\alpha)Q)$ for $P, Q \in \mathcal{P}_1(H)$ is an injective function of the transition probability $\operatorname{Tr} PQ$ which yields the following corollary.

Corollary 2.5. *The map $\phi|_{\mathcal{P}_1(H)}$ preserves the transition probability, whence it is either a unitary or an antiunitary similarity transformation, by the nonsurjective version of Wigner's theorem (see, for instance, [6]).*

Since the quantity $F(tA + (1-t)B)$ is clearly invariant under such transformations, composing ϕ by an appropriate one, we may suppose that ϕ acts as the identity on $\mathcal{P}_1(H)$. The remainder of the section is concerned with verifying that such a ϕ is the identity on the whole of $\mathcal{S}(H)$.

LEMMA 2.6. *Let A be an arbitrary self-adjoint operator on H with spectrum in* $[0, 1]$. *Denote by $\mathcal{M}_A$ the set of rank-one projections on H with range contained in the eigensubspace of A corresponding to its largest eigenvalue. Then*

$$F(\alpha A + (1-\alpha)P) \leq F(\alpha A + (1-\alpha)P_A), \quad P \in \mathcal{P}_1(H),\ P_A \in \mathcal{M}_A,$$

with equality if and only if $P \in \mathcal{M}_A$.

PROOF. The proof parallels the proof of [8, Proposition 2.4] and is omitted. □

From now on, we shall use the argument in the proof of [17, Claim 8]. Fix an operator $A \in \mathcal{S}(H)$ and set $B := \phi(A)$. Suppose that the spectral decompositions of A and B are

$$A = \sum_{i=1}^{m} \lambda_i P_i, \quad B = \sum_{j=1}^{n} \mu_j Q_j,$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_m$ and $\mu_1 > \mu_2 > \cdots > \mu_n$. Using (1.1) and the assumption that ϕ is the identity on $\mathcal{P}_1(H)$,

$$F(\alpha A + (1-\alpha)P) = F(\alpha B + (1-\alpha)P), \quad P \in \mathcal{P}_1(H).$$

By Lemma 2.6, for any projection $R \in \mathcal{P}_1(H)$,

$$F(\alpha A + (1-\alpha)R) = \max_{X \in \mathcal{P}_1(H)} F(\alpha A + (1-\alpha)X)$$

if and only if $R \leq P_1$, and in this case

$$F(\alpha A + (1-\alpha)R) = f(\alpha\lambda_1 + 1 - \alpha) - f(\alpha\lambda_1) + \sum_{i=1}^{m} \operatorname{rank}(P_i) f(\alpha\lambda_i).$$

Similarly, for each operator $R \in \mathcal{P}_1(H)$,

$$F(\alpha B + (1-\alpha)R) = \max_{X \in \mathcal{P}_1(H)} F(\alpha B + (1-\alpha)X)$$

exactly when $R \leq Q_1$, and then

$$F(\alpha B + (1-\alpha)R) = f(\alpha\mu_1 + 1 - \alpha) - f(\alpha\mu_1) + \sum_{j=1}^{n} \operatorname{rank}(Q_j) f(\alpha\mu_j).$$

Notice that for any projection $R \in \mathcal{P}_1(H)$, the quantity $F(\alpha A + (1-\alpha)R)$ is maximal if and only if $F(\alpha B + (1-\alpha)R)$ is so. Equivalently, if $R \in \mathcal{P}_1(H)$ is an operator, then $R \leq P_1 \Leftrightarrow R \leq Q_1$, hence $P_1 = Q_1$.

Now set $S := P_1 = Q_1$. Then for each projection $R \in \mathcal{P}_1(H)$ with $RS = 0$, the equality

$$F(\alpha A + (1-\alpha)R) = \max_{X \in \mathcal{P}_1(H): XS = 0} F(\alpha A + (1-\alpha)X)$$

holds exactly when $R \leq P_2$, and in this case

$$F(\alpha A + (1-\alpha)R) = f(\alpha\lambda_2 + 1 - \alpha) - f(\alpha\lambda_2) + \sum_{i=1}^{m} \operatorname{rank}(P_i) f(\alpha\lambda_i).$$

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Analogously, for each operator $R \in \mathcal{P}_1(H)$ with $RS = 0$,

$$F(\alpha B + (1-\alpha)R) = \max_{X \in \mathcal{P}_1(H):\, XS=0} F(\alpha B + (1-\alpha)X)$$

if and only if $R \leq Q_2$, and then

$$F(\alpha B + (1-\alpha)R) = f(\alpha\mu_2 + 1 - \alpha) - f(\alpha\mu_2) + \sum_{j=1}^{n} \operatorname{rank}(Q_j) f(\alpha\mu_j).$$

Using the conditions on A and B, we infer that for any projection $R \in \mathcal{P}_1(H)$ with $RS = 0$, the quantity $F(\alpha A + (1-\alpha)R)$ is maximal if and only if $F(\alpha B + (1-\alpha)R)$ is so. This means that for each such operator R, the equivalence $R \leq P_2 \Leftrightarrow R \leq Q_2$ holds. Thus $P_2 = Q_2$. We can continue this argument and arrive at the conclusion that $m = n$ and $P_i = Q_i$ for each integer i with $1 \leq i \leq m$.

Now we define the function g on $[0, 1]$ by

$$g(x) := f(\alpha x + 1 - \alpha) - f(\alpha x), \quad x \in [0, 1].$$

We infer that g is strictly increasing. The variational formulas above yield

$$\max_{R \in \mathcal{P}_1(H): R \leq P_l + \cdots + P_m} F(\alpha A + (1-\alpha)R) - \max_{R \in \mathcal{P}_1(H): R \leq P_{l+1} + \cdots + P_m} F(\alpha A + (1-\alpha)R) = g(\lambda_l) - g(\lambda_{l+1})$$

and

$$\max_{R \in \mathcal{P}_1(H): R \leq P_l + \cdots + P_m} F(\alpha B + (1-\alpha)R) - \max_{R \in \mathcal{P}_1(H): R \leq P_{l+1} + \cdots + P_m} F(\alpha B + (1-\alpha)R) = g(\mu_l) - g(\mu_{l+1})$$

for $l = 1, 2, \ldots, m-1$. Using the conditions on A and B, we deduce that

$$g(\lambda_l) - g(\lambda_{l+1}) = g(\mu_l) - g(\mu_{l+1}) \quad \text{for } l = 1, 2, \ldots, m-1.$$

We will now show that $\lambda_i = \mu_i$ for all integers $1 \leq i \leq m$. Suppose to the contrary that $\lambda_k > \mu_k$ for a certain natural number k with $1 \leq k \leq m$. Since $\sum_{i=1}^{m} \lambda_i = \sum_{i=1}^{m} \mu_i = 1$, it follows that $\lambda_s < \mu_s$ for some integer s with $1 \leq s \leq m$. There is no loss of generality in supposing that $s > k$. In this case, there is a number $t \in \{k, k+1, \ldots, s-1\}$ for which

$$[\lambda_{t+1}, \lambda_t] \supsetneq [\mu_{t+1}, \mu_t]$$

and this gives, for the strictly increasing function g, the relation

$$g(\lambda_t) - g(\lambda_{t+1}) > g(\mu_t) - g(\mu_{t+1}),$$

which is a contradiction. By what we have proved so far, the equality $A = B (:= \phi(A))$ is valid and, since $A \in \mathcal{S}(H)$ was an arbitrary density operator, it follows that ϕ is the identity. Having in mind the reduction we made so that ϕ was the identity on $\mathcal{P}_1(H)$, we conclude that ϕ is of the desired form, and this completes the proof of Theorem 1.4.

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