ON FOURIER-STIELTJES TRANSFORMS

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Let $\mathscr{M}$ be the class of bounded non-decreasing functions defined on the real line which are normalized by the conditions $\phi(-\infty) = 0$, $\phi(t+0) = \phi(t)$. Let $\mathscr{F}$ be the class of Fourier-Stieltjes transforms of elements of $\mathscr{M}$, i.e. the elements of $\mathscr{M}$ and $\mathscr{F}$ are connected by the relation[1]

$$\Phi(x) = \int e^{itx} d\phi(t),$$

where $\phi \in \mathscr{M}$ and $\Phi \in \mathscr{F}$. It is well known, and easy to verify that this mapping from $\mathscr{M}$ to $\mathscr{F}$ is one to one (**1**, p. 67, Satz 18).

It is the purpose of this paper to give various topologies to $\mathscr{F}$ and $\mathscr{M}$ so that the mapping from $\mathscr{F}$ to $\mathscr{M}$ will be continuous or at least continuous at certain points of $\mathscr{F}$ depending on the topologies. The topologies which we shall have occasion to use are enumerated below.

A. The almost weak topology on $\mathscr{F}$. As a neighbourhood basis of an element $\Phi_0 \in \mathscr{F}$ we shall take the sets in $\mathscr{F}$ which satisfy the relations

$$\left| \int f_k(x)[\Phi(x) - \Phi_0(x)]\, dx \right| < \delta, \qquad k = 1, 2, \ldots, n,$$

and

$$\Phi(0) < \Phi_0(0) + \delta$$

where $\{f_k\}_1^n$ is any finite set of elements in the Lebesgue class $L^1(-\infty, \infty)$, and δ is any positive number. We shall designate such neighborhoods by

$$\mathfrak{M}[\{f_k\}; \delta; \Phi_0].$$

B. The mean value topology in $\mathscr{F}$. As a neighborhood basis of an element $\Phi_0 \in \mathscr{F}$ we shall take the sets in $\mathscr{F}$ which satisfy the relation

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\Phi(x) - \Phi_0(x)|\, dx < \delta,$$

and

$$\Phi(0) < \Phi_0(0) + \delta$$

where $\delta > 0$. In case a $\Phi \in U$ satisfies the above two relations we shall write

$$||\Phi - \Phi_0||_m < \delta.$$

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[1]Absence of limits of integration will mean that the integral is taken over the interval $(-\infty, \infty)$.

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C. The mean almost weak topology in $\mathscr{F}$. As a neighborhood basis of a $\Phi_0 \in \mathscr{F}$ we shall take those sets which for any δ satisfy simultaneously the relations in A and B. We shall designate such neighborhoods by

$$\mathfrak{M}_m[\{f_k\};\ \delta;\ \Phi_0].$$

D. The uniform topology in $\mathscr{F}$ and $\mathscr{M}$. Let us write

$$||\Phi - \Phi_0|| = \sup |\Phi(x) - \Phi_0(x)|,$$

where the sup is taken over all x on the real line. Then as a neighborhood basis of Φ_0 we shall take the sets which satisfy

$$||\Phi - \Phi_0|| < \delta.$$

The same type of topology on $\mathscr{M}$ will be called the uniform topology on $\mathscr{M}$.

E. The variational topology on $\mathscr{M}$. We shall write

$$||\phi - \phi_0||_v = \text{total variation } [\phi(t) - \phi_0(t)],$$

and as a neighborhood basis of ϕ_0 take the sets in $\mathscr{M}$ which satisfy

$$||\phi - \phi_0||_v < \delta.$$

Suppose now that $\phi \in \mathscr{M}$ and t a point where $\phi(t) - \phi(t-0) \geqslant \delta > 0$. Let $I(\phi;\delta;t)$ be a generic symbol for an open interval which contains the point t and let $\{I(\phi;\delta;t_k)\}$ represent a class of such intervals where the t_k run over all points for which the jump of $\phi(t)$ is greater than or equal to δ. Each such class of course contains only a finite number of members.

THEOREM 1. *Let $\Phi_0 \in \mathscr{F}$ and $\epsilon > 0$ be given. There exists a $\delta > 0$ such that if we exclude a small interval about each point of the real axis where the jump of $\phi_0(t)$ is greater than or equal to δ, we can find an almost weak neighborhood of Φ_0, $\mathfrak{M}[\{f_k\};\delta;\ \Phi_0]$, so that outside the excluded intervals each element of $\mathscr{M}$ which corresponds to an element of $\mathfrak{M}[\{f_k\};\delta;\ \Phi_0]$ is uniformly within ϵ of $\phi_0(t)$.*

In more technical language the above theorem can be stated as follows: *Given $\Phi_0 \in \mathscr{F}$ and $\epsilon > 0$. There exists a $\delta > 0$ such that for any $\{I(\phi_0;\delta;t_k)\}$ there exists an $\mathfrak{M}[\{f_k\};\delta;\ \Phi_0]$ so that $\Phi \in \mathfrak{M}[\{f_k\};\delta;\ \Phi_0]$ implies $|\phi(t) - \phi_0(t)| < \epsilon$ for all $t \notin \bigcup I(\phi_0;\delta;t_k)$.*

Proof. Let $\delta > 0$ be given and choose R sufficiently large so that

$$\int_{|t|\geqslant R} d\ \phi_0(t) < \delta$$

Further, choose $f_0^*(t)$ to be of class C^2 (continuous second derivatives) such that $0 \leqslant f_0^*(t) \leqslant 1$ and

$$f_0^*(t) = \begin{cases} 1, & |t| \leqslant R \\ 0, & |t| \geqslant R+1. \end{cases}$$

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Let

$$f_0(x) = \frac{1}{2\pi}\int e^{-ixt} f_0^*(t)\, dt.$$

Integrating by parts twice will immediately show that $f_0(x) \in L^1(-\infty, \infty)$. Further, since $f_0^*(t)$ itself belongs to $L^1(-\infty, \infty)$, is continuous and of bounded variation over the whole real axis, we have the inversion formula (**1**, p. 42).

$$f_0^*(t) = \int f_0(x)\, e^{ixt} dx.$$

Therefore,

$$\int f_0^*(t)\, d\phi(t) = \int \left[\int f_0(x)\, e^{ixt}\, dx\right] d\phi(t).$$

Since $f_0(x) \in L^1(-\infty, \infty)$ and $\phi(t)$ is bounded we may apply Fubini's theorem (**4**, p. 77) and we get the Parseval relation

$$\int f_0^*(t)\, d\phi(t) = \int f_0(x)\, \Phi(x)\, dx,$$

Therefore, if we choose any Φ such that

(1) $$\left|\int f_0(x)[\Phi(x) - \Phi_0(x)]\, dx\right| < \delta,$$

we have for the corresponding $\phi(t)$,

$$\left|\int f_0^*(t)\, d[\phi(t) - \phi_0(t)]\right| < \delta.$$

If Φ satisfies the further condition

(2) $$\Phi(0) < \Phi_0(0) + \delta,$$

then we have

(3) $$\Phi_0(0) + \delta > \Phi(0) \geqslant \int f_0^*(t)\, d\phi(t) > \int f_0^*(t)\, d\phi_0(t) - \delta > \Phi_0(0) - 2\delta.$$

Therefore,

$$0 \leqslant \int d\phi(t) - \int_{|t|<R+1} d\phi(t) \leqslant \int d\phi(t) - \int f_0^*(t)\, d\phi(t) < 3\delta,$$

from which we get

(4) $$\phi(-R-1) < 3\delta.$$

Now, choose a set $\{I(\phi_0; \delta; t_k)\}$ and suppose there exists a t_0 in the complement of $\bigcup I(\phi_0; \delta; t_k)$ which lies to the right of $-R-1$. There exists an $h > 0$ such that

(5) $$|\phi_0(t_0 \pm h) - \phi_0(t_0)| < \delta.$$

Choose $f_1^*(t)$ and $f_2^*(t)$ to be in C^2 with range in $[0, 1]$ and defined in the following way:

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$$f_1^*(t) = \begin{cases} 1, & -R-1 \leqslant t \leqslant t_0 - h, \\ 0, & t \leqslant -R-2, \quad t \geqslant t_0, \end{cases}$$

$$f_2^*(t) = \begin{cases} 1, & -R-1 \leqslant t \leqslant t_0, \\ 0, & t \leqslant -R-2, \quad t \geqslant t_0 + h. \end{cases}$$

If $f_1(x)$ and $f_2(x)$ are the Fourier transforms respectively of $f_1^*(t)$ and $f_2^*(t)$, then f_1 and f_2 are in $L^1(-\infty, \infty)$.

Let $\Phi(x)$ be any element of $\mathscr{F}$ which satisfies (1), (2) and the further conditions

$$\left| \int f_k(x)[\Phi(x) - \Phi_0(x)]\, dx \right| < \delta, \qquad k = 1, 2.$$

By the Parseval relation we have for $k = 1, 2$,

$$(6) \qquad \left| \int f_k^*(t)\, d[\phi(t) - \phi_0(t)] \right| < \delta$$

Consequently, by (4), (5) and (6) we get

$$\phi_0(t_0) - 3\delta < \int f_1^*(t)\, d\phi(t) < \phi(t_0),$$

$$\phi(t_0) - 3\delta < \int f_2^*(t)\, d\phi(t) < \phi_0(t_0) + 2\delta.$$

From this it follows that

$$-5\delta < \phi_0(t_0) - \phi(t_0) < 3\delta.$$

The complement of $\bigcup I(\phi_0; \delta; t_k)$ (which we may as well suppose is not the null set) which lies in the interval $(-R-1, \infty)$ consists of a finite number of mutually disjoint intervals. In each such interval it is possible to find a finite set of numbers $\tau_1 < \tau_2 < \ldots < \tau_n$ such that τ_1 and τ_n are the endpoints of the interval and

$$\phi_0(\tau_{k+1}) - \phi_0(\tau_k) < \delta.$$

Therefore, there exist functions $\{f_k(x)\}$ each of which belongs to $L^1(-\infty, \infty)$ such that if $\Phi(x) \in \mathfrak{M}[\{f_k\}; \delta; \Phi_0]$ we have

$$|\phi(\tau_k) - \phi_0(\tau_k)| < 5\delta.$$

((2) and (3) also give us this relation for $\tau_k = \infty$.)

Suppose $\tau_k \leqslant t \leqslant \tau_{k+1}$. Then

$$\phi_0(\tau_k) \leqslant \phi_0(t) \leqslant \phi_0(\tau_{k+1}), \quad \phi(\tau_k) \leqslant \phi(t) \leqslant \phi(\tau_{k+1}).$$

Therefore

$$(7) \qquad -6\delta < \phi(\tau_k) - \phi_0(\tau_{k+1}) \leqslant \phi(t) - \phi_0(t) \leqslant \phi(\tau_{k+1}) - \phi_0(\tau_k) < 6\delta.$$

Since we are dealing with only a finite number of intervals in the complement of $\bigcup I(\phi_0; \delta; t_k)$ which lies in $(-R-1, \infty)$ we can find an almost weak

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neighborhood of Φ_0 such that if Φ belongs to this neighborhood, then the corresponding functions satisfy (7). If we now choose $\delta = \frac{1}{6}\epsilon$ we have our theorem.

COROLLARY. *If $\phi_0(t)$ is continuous then the mapping from $\mathscr{F}$, with the almost weak topology, to $\mathscr{M}$, with the uniform topology, is continuous at Φ_0.*

THEOREM 2. *Let $\phi_0(t) \in \mathscr{M}$ be a step function. Then given $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$||\Phi - \Phi_0||_m < \delta$$

implies

$$||\phi - \phi_0||_v < \epsilon.$$

Proof. Given $\phi(t)$ and $\phi_0(t)$, let t_n be the set of points where either $\phi(t)$ or $\phi_0(t)$ has a jump. Let a_n and b_n be respectively the jump of $\phi_0(t)$ and $\phi(t)$ at t_n. Let us write

$$\phi(t) = S(t) + D(t),$$

where $S(t)$ is a step function and $D(t)$ is a continuous function. We then have

$$\phi(t) - \phi_0(t) = \{S(t) - \phi_0(t)\} + D(t).$$

Since $\phi_0(t)$ is a step function, $S(t) - \phi_0(t)$ is either a step function or identically zero since $S(-\infty) = \phi_0(-\infty) = 0$. This gives us the decomposition of $\phi(t) - \phi_0(t)$ into a step function and a continuous function. Therefore (**2**, pp. 189–190)

$$||\phi - \phi_0||_v = ||S - \phi_0||_v + ||D||_v$$

Now, let $\psi(t) = S(t) - \phi_0(t)$. Then (**2**, pp. 188–190),

$$||S - \phi_0||_v = ||\psi||_v = \sum_{n=1}^{\infty} \{|\psi(t_n + 0) - \psi(t_n)| + |\psi(t_n) - \psi(t_n - 0)|\}$$

By normalization of the functions in $\mathscr{M}$ we have

$$|\psi(t_n + 0) - \psi(t_n)| = 0$$

Therefore

$$||S - \phi_0||_v = \sum_{n=1}^{\infty} |b_n - a_n|.$$

Consequently

$$\begin{aligned} ||\phi - \phi_0||_v &= \sum_{n=1}^{\infty} |b_n - a_n| + ||D||_v \\ &\leqslant \sum_{n=1}^{N} |b_n - a_n| + ||D||_v + \sum_{n=N+1}^{\infty} b_n + \sum_{n=N+1}^{\infty} a_n. \end{aligned}$$

Since

$$\Phi(0) = ||\phi||_v = \sum_{n=1}^{\infty} b_n + ||D||_v,$$

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we have

$$||\phi - \phi_0||_v = \sum_{n=1}^{N} |b_n - a_n| + \Phi(0) - \sum_{n=1}^{N} b_n + \sum_{n=N+1}^{\infty} a_n.$$

Let us here make the parenthetical remark that if either $\phi(t)$ or $\phi_0(t)$ has a finite number of jumps, then b_n or a_n from some point on will be zero.

Now,

$$\Phi(0) - \sum_{n=1}^{N} b_n = \Phi(0) - \Phi_0(0) + \sum_{n=N+1}^{\infty} a_n - \sum_{n=1}^{N} (b_n - a_n).$$

Therefore

$$||\phi - \phi_0||_v \leqslant 2\sum_{n=1}^{N} |b_n - a_n| + 2\sum_{n=N+1}^{\infty} a_n + \Phi(0) - \Phi_0(0).$$

Choose N so that

$$\sum_{n=N+1}^{\infty} a_n < \epsilon/5$$

and then choose $\delta \leqslant \epsilon/5N$. It is well known (**1**, p. 79, Satz 24) that

$$a_n = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-it_n x}\phi_0(x)\,dx,$$

$$b_n = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-it_n x}\phi(x)\,dx.$$

Therefore

$$|b_n - a_n| \leqslant \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |\phi(x) - \phi_0(x)|\,dx.$$

From this inequality we get the desired result.

From the two preceding results we might expect that if $\mathscr{F}$ is given the mean almost weak topology and $\mathscr{M}$ the uniform topology, then the mapping from $\mathscr{F}$ to $\mathscr{M}$ is continuous. This is shown by the next theorem.

THEOREM 3. *Given* $\Phi_0 \in \mathscr{F}$ *and* $\epsilon > 0$, *there exists a neighborhood* $\mathfrak{M}_m(\{f_k\}; \delta; \Phi_0)$ *such that* $\Phi \in \mathfrak{M}_m$ implies

$$||\phi - \phi_0|| < \epsilon.$$

Proof. As in the proof of Theorem 1, let $\delta > 0$ be given and choose R sufficiently large so that

$$\int_{|t|\geqslant R} d\phi_0(t) < \delta.$$

Also, choose $f_0^*(t)$ as in Theorem 1 and let $f_0(x)$ be its Fourier transform. Then if $\Phi \in \mathscr{F}$ is such that

(2) $$\Phi(0) \leqslant \Phi_0(0) + \delta$$

and

$$\left|\int f_0(x)[\Phi(x) - \Phi_0(x)]\,dx\right| < \delta;$$

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as in Theorem 1 we get, for $t \leqslant -R-1$,

$$0 \leqslant \phi(t) \leqslant \phi(-R-1) < 3\delta,$$

and

$$\Phi(0) > \Phi_0(0) - 2\delta.$$

Suppose now that $\{\tau_k\}$ is the finite set of points to the right of $-R-1$ for which $\phi_0(\tau_k) - \phi_0(\tau_k - 0) \geqslant \delta$. The interval $[\tau_k, \tau_{k+1}]$ may be subdivided by a finite number of points

$$\tau_k = \tau_{0,k} < \tau_{1,k} < \ldots < \tau_{m,k} = \tau_{k+1}$$

such that

$$\phi_0(\tau_{j+1,k}) - \phi_0(\tau_{j,k}) < \delta, \qquad j = 0, 1, \ldots, m-1,$$

and

$$\phi_0(\tau_{k+1} - 0) - \phi_0(\tau_{m-1,k}) < \delta.$$

Therefore, there exists a finite set of points, $-R-1 = t_0 < t_1 < \ldots < t_n = \infty$, which includes the set $\{\tau_k\}$ and such that

$$\phi_0(t_{k+1}) - \phi_0(t_k) < \delta, \qquad t_{k+1} \notin \{\tau_k\},$$

and

$$\phi_0(t_{k+1} - 0) - \phi_0(t_k) < \delta, \qquad t_{k+1} \in \{\tau_k\}.$$

For $k = 1, \ldots, n-2$, choose, as in Theorem 1, $f_k^*(t) \in C^2$ and with range in $[0, 1]$ in the following manner:

$$f_k^*(t) = \begin{cases} 1, & t_0 \leqslant t \leqslant t_k \\ 0, & t \geqslant t_{k+1}, \quad t \leqslant t_0 - 1. \end{cases}$$

Further, choose $f_{n-1}^*(t) \in C^2$ such that $0 \leqslant f_{n-1}^*(t) \leqslant 1$ and

$$f_{n-1}^*(t) = \begin{cases} 1, & t_0 \leqslant t \leqslant t_{n-1} \\ 0, & t \geqslant t_{n-1} + 1, \quad t \leqslant t_0 - 1. \end{cases}$$

Let $f_k(x)$ be the Fourier transform of $f_k^*(t)$. Then if we choose Φ to satisfy (2) and

$$\left| \int f_k(x)[\Phi(x) - \Phi_0(x)]\, dx \right| < \delta, \qquad k = 0, 1, \ldots, n-1, \tag{8}$$

then for $t_k \notin \{\tau_k\}$, by the same method of proof as in Theorem 1 we have

$$|\phi(t_k) - \phi_0(t_k)| < 5\delta.$$

If $t_k \in \{\tau_k\}$ then we have

$$\phi_0(t_k - 0) - 3\delta < \int f_{k-1}^*(t)\, d\phi(t) \leqslant \phi(t_k - 0),$$

from which

$$\phi_0(t_k - 0) - \phi(t_k - 0) < 3\delta.$$

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Further, for the same t_k

$$\phi(t_k) - \phi(-R-1) \leqslant \int f_k^*(t)\, d\phi(t) < \phi_0(t_k) + 2\delta,$$

from which

$$\phi_0(t_k) - \phi(t_k) > -5\delta.$$

In addition to (2) and (8) let us now pick $\Phi \in \mathrm{U}$ to also satisfy

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\Phi(x) - \Phi_0(x)|\, dx < \delta. \tag{9}$$

Suppose

$$\phi_0(t_k - 0) - \phi(t_k - 0) \leqslant -6\delta \text{ or } \phi_0(t_k) - \phi(t_k) \geqslant 5\delta.$$

Then, if a_k and b_k are respectively the jump of $\phi_0(t)$ and $\phi(t)$ at t_k we have

$$a_k - b_k \geqslant \delta.$$

But since

$$|a_k - b_k| \leqslant \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\Phi(x) - \Phi_0(x)|\, dx < \delta,$$

we get a contradiction. Therefore,

$$|\phi_0(t_k - 0) - \phi(t_k - 0)| < 6\delta,$$

and

$$|\phi_0(t_k) - \phi(t_k)| < 5\delta.$$

If we now proceed as in Theorem 1, the proof of our theorem is complete.

From this theorem we get the following corollary, which was originally proved by Dyson **(3)**.

COROLLARY. *Given* $\Phi_0 \in \mathscr{F}$ *and* $\epsilon > 0$, *there exists a* $\delta_1 > 0$ *such that*

$$||\Phi - \Phi_0|| < \delta_1 \textit{ implies } ||\phi - \phi_0|| < \epsilon.$$

Proof. Let

$$M = \max_k \int |f_k(x)|\, dx,$$

where $\{f_k\}$ is the set in Theorem 3. Then choose $\delta_1 = \delta/M$, where δ is that of Theorem 3.

In closing this paper we wish to remark that if we replace the space $\mathscr{M}$ by the space $\mathscr{B}$ of all functions of total bounded variation defined on the line and normalized in the same way as in $\mathscr{M}$, then our previous theorems can be given a meaning. We shall write down these corresponding theorems without proof and only remark that the proofs follow the pattern we have established before with only some slight modification.

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THEOREM 1′. *Let a continuous $\phi_0 \in \mathscr{B}$ and $\epsilon > 0$ be given. Then there exists a $\delta > 0$ and functions $\{f_k\}_1^n \subset C^2$ such that*

$$\left| \int f_k \, d[\phi - \phi_0] \right| < \delta,$$

and

$$||\phi||_v \leqslant ||\phi_0||_v + \delta$$

implies

$$||\phi - \phi_0|| < \epsilon.$$

THEOREM 2′. *Let $\phi_0 \in \mathscr{B}$ be a step function. Then given $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$\max |\text{saltus } [\phi(t) - \phi_0(t)]| < \delta$$

and

$$||\phi||_v \leqslant ||\phi_0||_v + \delta$$

implies

$$||\phi - \phi_0||_v < \epsilon.$$

THEOREM 3′. *Let $\phi_0 \in \mathscr{B}$ and $\epsilon > 0$ be given. Then there exist a $\delta > 0$ and $\{f_k\}_1^n \subset C^2$ such that*

$$\left| \int f_k \, d[\phi - \phi_0] \right| < \delta, \quad \max|\text{saltus}[\phi(t) - \phi_0(t)]| < \delta,$$

and

$$||\phi||_v \leqslant ||\phi_0||_v + \delta$$

implies

$$||\phi - \phi_0|| < \epsilon.$$

In the above theorems it is of course understood that ϕ belongs to $\mathscr{B}$.

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