An Application of Abel's Lemma to Double Series.[1]

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Let b_{mn} be a positive function of m and n which decreases steadily with n, so that $b_{mn} \geqslant b_{m,\,n+1}$ for all values of m and n. Assume also that $|a_{m1} + a_{m2} + \ldots + a_{mn}| < K$ for all values of m and n, K being finite. Denote by S_{mn} the sum of the first n terms in the first m rows of the double series

$$\begin{array}{l} a_{11}b_{11},\ a_{12}b_{12},\ \ldots,\ a_{1n}b_{1n},\ldots \\ a_{21}b_{21}\ \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}b_{m1},\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

Then it can be shewn that if

(1) $\sum_{m=1}^{\infty} b_{m1}$ converges, and b_{mn} tends to zero as $n \to \infty$ (m being fixed) the double series S_{mn} converges.

(2) $\sum_{1}^{\infty} b_{m1}$ converges, and the sum of S_{mn} by rows exists, being S, the double series S_{mn} will also converge to S.

(3) The sum of S_{mn} by columns exists, being S, and $\sum_{m=1}^{\infty} b_{mn}$ converges for all values of n greater than a fixed value N_1, tending to zero as $n \to \infty$, then the double series converges to S.

Of course in cases (2) and (3) by Pringsheim's Theorem it is merely necessary to prove that S_{mn} converges, but to prove that it

[1] For various similar results, and for other applications of this Lemma to multiple series, together with extensions of it, see *e.g.* Hardy :—*Proc. London. Math. Soc.* 2, **1** (1903), 124-128 ; **2** (1904), 190 ; *Proc. Cambridge Phil. Soc.* **19** (1919), 86, etc. Also, Bromwich, *Proc. London Math. Soc.* 2, **6** (1907), 58-76 ; and papers and theorems by Hadamard, Ferrar, *Proc. London Math. Soc.*, **29** (1929), and others.

converges to S is just as short. For all values of m and n, and for all positive values of p we have

$$\left|\sum_{r=n+1}^{n+p} a_{mr}\right| = \left|\sum_{r=1}^{n+p} a_{mr} - \sum_{r=1}^{n} a_{mr}\right| < 2K.$$

Hence since b_{mn} decreases steadily with n, by Abel's Lemma $\left|\sum_{r=n+1}^{n+p} b_{mr}\,a_{mr}\right| < b_{m,\,n+1}.\ V$, where V is the greatest of the moduli

$$|a_{m,\,n+1}|,\ \left|\sum_{r=n+1}^{n+2} a_{mr}\right|, \ldots, \left|\sum_{r=n+1}^{n+p} a_{mr}\right|,\ i.e.\ V < 2K.\quad \text{Thus}$$

$$\left|\sum_{n+1}^{n+p} b^{mr}\,a_{mr}\right| < 2Kb_{m,\,n+1}.$$

Putting $m = 1, 2, \ldots . M$, we have for all values of M, n and p,

$$|S_{M,\,n+p} - S_{Mn}| = \left|\sum_{m=1}^{M}\ \sum_{r=n+1}^{n+p} b_{mr}\,a_{mr}\right| \leqslant \sum_{m=1}^{M}\left|\sum_{r=n+1}^{n+p} b_{mr}\,a_{mr}\right| <$$

$$2K\,(b_{1\,n+1} + b_{2,\,n+1} + \ldots . + b_{M,\,n+1}). \tag{A}$$

Also putting $n = 0$ in the inequality $\left|\sum_{n+1}^{n+p} b_{mr}a_{mr}\right| < 2Kb_{m,\,n+1}$, and letting $m = M+1, M+2, \ldots, M+s$, in succession, it follows similarly that

$$|S_{M+s,\,p} - S_{Mp}| \leqslant \sum_{m=M+1}^{M+s}\left|\sum_{r=1}^{p} b_{mr}\,a_{mr}\right|$$

$$< 2K\,(b_{M+1,\,1} + b_{M+2,\,1} + \ldots\ldots + b_{M+s,\,1}), \tag{B}$$

M, s, and p being arbitrary.

In case (1), given ϵ, M can be found so that if $m > M$, $\sum_{r=m}^{\infty} b_{r\,1} < \epsilon$, hence from (B) if $m > M$,

$|S_{mn} - S_{Mn}| < 2K\epsilon$, for all values of n.

Now N can be found so that if $n > N$, $b_{1n}, b_{2n}, \ldots, b_{Mn}$ are each less than $\frac{\epsilon}{M}$. Accordingly from (A) if $n > N$, $q > N$,

$$|S_{Mn} - S_{Mq}| < 2K\,.\,M\,.\,\frac{\epsilon}{M},\ i.e. < 2K\epsilon.$$

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and so when $p, m, > M, q, n > N$,

$$|S_{mn} - S_{pq}| \leqslant |S_{mn} - S_{Mn}| + |S_{Mn} - S_{Mq}| + |S_{Mq} - S_{pq}| < 6K\epsilon$$

which is arbitrarily small, thus S_{mn} converges.

For case (2), if r_m denote the sum of the first m rows of S_{mn} M can be found such that $|r_m - S| < \epsilon$, and $\sum_m^\infty b_{r1} < \epsilon$, if $m \geqslant M$. Then we can find N so that $|S_{Mn} - r_M| < \epsilon$ if $n > N$. Thus

$|S_{Mn} - S| < 2\epsilon$ if $n > N$, and as in case (1), $|S_{mn} - S_{Mn}| < 2K\epsilon$ for all values of n, if $m > M$. Hence if $m > M, n > N$,

$$|S_{mn} - S| \leqslant |S_{mn} - S_{Mn}| + |S_{Mn} - S| < 2(K+1)\epsilon$$

which is arbitrarily small, and the result follows.

In case (3) if the sum of the first n columns be denoted by l_n, $N(> N_1)$ can be found so that if $n \geqslant N$, $|l_n - S| < \epsilon$, and $\sum_{m=1}^\infty b_{mn} < \epsilon$. Then we can find M so that if $m > M$, $|S_{mN} - l_N| < \epsilon$, and consequently $|S_{mN} - S| < 2\epsilon$. But from (A), we have for all values of m, if p is positive

$$|S_{m, N+p} - S_{mN}| < 2K(b_{1\, N+1} + b_{2, N+1} + \ldots + b_{m, N+1}) < 2K\epsilon.$$

Therefore if $m > M$, $n > N$

$$|S_{mn} - S| \leqslant |S_{mn} - S_{mN}| + |S_{mN} - S| < 2(K+1)\epsilon$$

so S_{mn} converges to S.

[*Note added* 30*th March* 1929.—It was noticed, too late to alter proof sheets, that (1) and (2) above are merely particular cases of

(a) The double series S_{mn} converges if $\sum_{m=1}^\infty b_{m1}$, and each row of S_{mn}, converges. For M can then be found so that $\sum_{m=M}^\infty b_{m1} < \epsilon$, hence from ($B$), if $m > M$, then $|S_{mn} - S_{Mn}| < 2K\epsilon$ for all values of n. As $\lim_{n \to \infty} S_{Mn}$ exists we can find N so that if $n, q > N$,

$$|S_{Mn} - S_{Mq}| < \epsilon.$$

Thus when $m, p > M$, $n, q > N$,

$$|S_{mn}-S_{pq}| \leqslant |S_{mn}-S_{Mn}| + |S_{Mn}-S_{Mq}| + |S_{Mq}-S_{pq}| < (4K+1)\epsilon,$$

so that the double series converges.

Also (3) above is only a particular case of

(*b*) If the columns of S_{mn} converge, and $\sum_{m=1}^{\infty} b_{mn}$ converges when $n > N_1$, tending to zero as $n \to \infty$, S_{mn} will converge.

To shew this; $N\ (> N_1)$ can be found so that if

$$n \geqslant N,\ \sum_{m=1}^{\infty} b_{mn} < \epsilon;$$

hence as in (3), $|S_{m,\,N+r} - S_{mN}| < 2K\epsilon$ for all values of m and r. As $\lim_{m \to \infty} S_{mN}$ exists, M can be chosen such that

$$|S_{mN} - S_{pN}| < \epsilon, \text{ if } m, p > M.$$

And so when $m, p > M$, $n, q > N$

$$|S_{mn}-S_{pq}| \leqslant |S_{mn}-S_{mN}| + |S_{mN}-S_{pN}| + |S_{pN}-S_{pq}| < (4K+1)\epsilon$$

i.e. S_{mn} converges.]

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