HEREDITARY RADICALS IN ASSOCIATIVE AND ALTERNATIVE RINGS

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1. In the first part of this paper we shall consider associative rings, pointing out where associativity is required. In the second part we shall consider not necessarily associative rings and in particular alternative rings.

A property S of rings is said to be a *radical property*, in the sense of Kurosh **(4)**, if it satisfies the following three conditions:

(a) Every homomorphic image of an S-*ring* (i.e. a ring with property S) is again an S-ring.

(b) Every ring R contains an S-*ideal* (i.e. an ideal which is an S ring) $S(R)$, which contains every other S-ideal of R. This maximal S-ideal, $S(R)$, is called the S-*radical of* R.

(c) The factor ring $R/S(R)$ is S-*semi-simple* (i.e. $R/S(R)$ has no non-zero S-ideals).

Many well-known radical properties (Jacobson, Brown–McCoy, Levitzki, etc.) satisfy a further condition:

(d) If I is an ideal of a ring R, if $S(I)$ is the S-radical of the ring I, and if $S(R)$ is the S-radical of R, then

$$S(I) = I \cap S(R).$$

If a radical property satisfies condition (d), we say it is hereditary. However, not every radical property is hereditary. For example, if S is the upper radical property determined by the class of all non-zero nilpotent rings, then S is not hereditary. To explain what is meant by an upper radical property, we remind the reader that if a class M of rings has the property that any non-zero ideal of a ring of M can be mapped homomorphically onto a ring of M, then such a class M can be expanded to be the class of all semi-simple rings with respect to a radical property S_M. A ring is S_M-radical if it cannot be mapped homomorphically onto any ring in the class M. This radical property is the largest radical property for which all rings in M are semi-simple. See **(4)**.

The class of all non-zero nilpotent rings is a class which does determine an upper radical property, because every ideal of a nilpotent ring is itself a nilpotent ring.

This upper radical property S determined by the class of all non-zero nilpotent rings is inverted in some sense, for we usually think of nilpotent rings

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as radical rings whereas for this property they are all semi-simple. Nevertheless this is a legitimate radical property.

To see that this radical property is not hereditary, let I be any non-zero nilpotent ring and let R be the ring obtained from I by adjoining a unity element to I. Then I is semi-simple, i.e. $S(I) = 0$. However, R itself is an S-ring, for it cannot be mapped homomorphically onto a non-zero nilpotent ring. Thus $S(R) = R$. Consequently $S(I) = 0$ but

$$I \cap S(R) = I \cap R = I \neq 0.$$

Therefore $S(I) \neq I \cap S(R)$.

The first two results are interesting and not difficult to prove.

LEMMA 1. *If S is a radical property, then the following two conditions are equivalent:*

(α_1) *Every ideal of an S-ring is itself an S-ring.*

(α_2) *For every ring R and every ideal I of R, we have $S(I) \supseteq I \cap S(R)$.*

LEMMA 2. *If S is a radical property then the following two conditions are equivalent:*

(β_1) *Every ideal of an S-semi-simple ring is itself S-semi-simple.*

(β_2) *For every ring R and every ideal I of R, we have $S(I) \subseteq I \cap S(R)$.*

Proofs. If (α_2) holds and R is an S-ring, then $S(R) = R$ and

$$S(I) \supseteq I \cap S(R) = I \cap R = I.$$

Since $S(I) \subseteq I$, we have $S(I) = I$ and thus (α_1) holds.

If (α_1) holds, we consider $I \cap S(R)$. This is an ideal in the S-ring $S(R)$ and by (α_1) it is itself an S-ring. Since $I \cap S(R)$ is an ideal of I, it must therefore be contained in $S(I)$. Thus we have (α_2).

If (β_2) holds and if R is S-semi-simple, then $S(R) = 0$ and

$$S(I) \subseteq I \cap S(R) = I \cap 0 = 0.$$

Thus $S(I) = 0$ and we have (β_1).

Finally if (β_1) holds, we consider the S-semi-simple ring $R/S(R)$. The ideal

$$[I + S(R)]/S(R) \simeq I/[I \cap S(R)]$$

is an ideal of an S-semi-simple ring and by (β_1) it must be S-semi-simple. Consequently $I/[I \cap S(R)]$ is S-semi-simple. Therefore $S(I) \subseteq I \cap S(R)$. For, if not,

$$\frac{S(I) + [I \cap S(R)]}{I \cap S(R)} \simeq \frac{S(I)}{S(I) \cap I \cap S(R)}$$

would be a non-zero S-ideal (since it is a homomorphic image of $S(I)$) in the S-semi-simple ring $I/[I \cap S(R)]$, and this is impossible. Thus we have (β_2).

Thus heredity is equivalent to conditions (α_2) and (β_2) and by Lemmas 1 and 2 it is equivalent to conditions (α_1) and (β_1). Amitsur **(1)** studied the relationship between conditions (α_1) and (β_1) and he proved that (α_1) implies (β_1). Thus heredity is equivalent to condition (α_1) by itself, and by Lemma 1, to condition (α_2) by itself.

The following theorem shows that the assumption of (α_1) is not necessary in establishing (β_1), that (β_1) holds for all radical properties!

THEOREM 1. *If S is any radical property, then for any (associative) ring R and any ideal I of R, $S(I)$ is an ideal of R.*

Proof. If $S(I)$ is not an ideal of R, then there exists an element x of R such that either $x\cdot S(I)$ or $S(I)\cdot x$ is not contained in $S(I)$. Assume first that $x\cdot S(I)$ is not contained in $S(I)$. Then $x\cdot S(I) + S(I)$ properly contains $S(I)$ and of course it is contained in I. Furthermore $x\cdot S(I) + S(I)$ is an ideal of I. This is clear on the right because $S(I)$ is an ideal of I (we need associativity here). If we multiply $x\cdot S(I) + S(I)$ on the left by I we get

$$I\cdot xS(I) + I\cdot S(I) = Ix\cdot S(I) + I\cdot S(I) \subseteq I\cdot S(I) \subseteq S(I).$$

Again we use associativity.

Since $I/S(I)$ is, by (c), S-semi-simple, the non-zero ideal

$$[x\cdot S(I) + S(I)]/S(I)$$

cannot be an S-ring. We shall, however, show that $[x\cdot S(I) + S(I)]/S(I)$ is a homomorphic image of the S-ring $S(I)$ and is therefore an S-ring, by (a). This contradiction will prove that $x\cdot S(I) \subseteq S(I)$ for every x in R. Similarly, $S(I)\cdot x \subseteq S(I)$ and $S(I)$ is an ideal of R.

To set up the homomorphism, let y be an arbitrary element in $S(I)$ and define:

$$\theta(y) \equiv xy + S(I).$$

Then θ is a mapping from $S(I)$ to $[xS(I) + S(I)]/S(I)$. It is clearly an onto mapping and it preserves addition. To see that $\theta(y_1 y_2) = \theta(y_1)\cdot\theta(y_2)$ we shall show that both of these are the zero coset.

First,

$$\theta(y_1 y_2) = x\cdot y_1 y_2 + S(I) = xy_1\cdot y_2 + S(I),$$

by associativity. Since y_1 is in $S(I) \subseteq I, xy_1 \in I$ and thus $xy_1\cdot y_2$ is in $I\cdot S(I) \subseteq S(I)$. Thus $\theta(y_1 y_2) = 0 + S(I)$.

On the other hand,

$$\theta(y_1)\cdot\theta(y_2) = xy_1\cdot xy_2 + S(I) = xy_1 x\cdot y_2 + S(I),$$

by associativity. Since $xy_1 x$ is in I, $xy_1 x\cdot y_2$ is in $S(I)$. Thus

$$\theta(y_1)\cdot\theta(y_2) = 0 + S(I).$$

This proves that θ is a homomorphism onto, that $[x\cdot S(I) + S(I)]/S(I)$ is an S ring, and that $x\cdot S(I) \subseteq S(I)$. The theorem is thus established.

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COROLLARY 1. *If S is any radical property, then for any (associative) ring R and any ideal I of R we have $S(I) \subseteq I \cap S(R)$.*

Proof. Since $S(I)$ is an ideal of R, and it is an S ring, it must, by (b), be contained in $S(R)$. It is certainly contained in I and thus $S(I) \subseteq I \cap S(R)$.

COROLLARY 2. *If S is any radical property, then every ideal of an S-semi-simple ring (associative) is itself S-semi-simple.*

This follows from Corollary 1 and Lemma 2.

2. Not necessarily associative rings (narings). The Kurosh theory of radicals holds for any class $\mathfrak{R}$ of narings satisfying the following two conditions **(4)**:

(1) Every ideal of a naring in $\mathfrak{R}$ is itself a naring in $\mathfrak{R}$.

(2) Every homomorphic image of a naring in $\mathfrak{R}$ is also a naring in $\mathfrak{R}$.

Thus $\mathfrak{R}$ may be many different classes of narings. In particular it might be the class of all narings, or all alternative narings, or all associative narings (i.e. rings!).

Let us now take $\mathfrak{R}$ to be the class of all narings. The notion of a hereditary radical property is well defined and even the example of a non-hereditary radical property holds, for the notion of an upper radical property and its construction goes through for narings. The notion of adjoining a unity element to a naring in the natural way does not require any associativity. We observe, however, that if $\mathfrak{R}$ is some subclass of the class of all narings, then it is important to check that the naring obtained by adjoining a unity element is again in the class $\mathfrak{R}$. For example, if $\mathfrak{R}$ is the class of all Lie narings, then adjoining a unity gives a naring which is not a Lie naring, i.e. is outside of $\mathfrak{R}$.

Lemmas 1 and 2, with their proofs, hold in their entirety, for the class of all narings.

Theorem 1, however, presents three associative difficulties. The first is the statement that $x \cdot S(I) + S(I)$ is an ideal of I. The second is the statement that

$$\theta(y_1 y_2) = x \cdot y_1 y_2 + S(I) = 0 + S(I).$$

The third is the statement that

$$\theta(y_1) \cdot \theta(y_2) = xy_1 \cdot xy_2 + S(I) = 0 + S(I).$$

At the present time we are unable to overcome these difficulties for the class of all narings, and we suspect they are false.

3. Alternative narings. Following common practice we shall call alternative narings, alternative rings.

We propose to show that the three associative difficulties of Theorem 1 can be overcome for the class of alternative rings.

We remind the reader that a naring R is said to be an alternative ring if $xx \cdot y = x \cdot xy$ and $y \cdot xx = yx \cdot x$, for every x and y of R. The associator (x, y, z)

is defined to be $xy \cdot z - x \cdot yz$, and for alternative rings it can be shown **(2)** that for every x, y, z, a, and b of R we have:

$$(1)\qquad (x, y, z) = (z, x, y) = (y, z, x) = -(x, z, y) = -(y, x, z) = -(z, y, x).$$

$$(2)\qquad (x, xy, z) = (x, y, z) \cdot x \qquad \textbf{(2}, \text{ p. 880, formula 2.13}\textbf{)}.$$

$$(3)\qquad (x, yx, z) = x \cdot (x, y, z) \qquad \textbf{(2}, \text{ p. 880, formula 2.14}\textbf{)}.$$

(Note that formula 2.14 referred to in (3) appears in **(2)** with a misprint. A correct version appears in *Studies in Modern Algebra* (ed. A. A. Albert), (Menasha, Wis., 1963), p. 130, formula (5).)

$$(4)\qquad xy \cdot x = x \cdot yx.$$

$$(5)\qquad a(xy)a = ax \cdot ya \qquad \textbf{(3}, \text{ Lemma 3}\textbf{)}.$$

By letting $x = a + b$ in (2) and using (2) we obtain:

$$(6)\qquad (a, by, z) + (b, ay, z) = (a, y, z) \cdot b + (b, y, z) \cdot a.$$

By letting $x = a + b$ in (3) and using (3) we obtain:

$$(7)\qquad (a, yb, z) + (b, ya, z) = b \cdot (a, y, z) + a \cdot (b, y, z).$$

We now require:

LEMMA 3. *If I is an ideal of an alternative ring R and if M is an ideal of I, then:*

(8) $MM \cdot R \subseteq M$.

(9) $(m, x, I) \subseteq M + Mx$ *for every x in R.*

(10) $M + Mx$ *is an ideal of I for every x in R.*

(11) $M + Mx \cdot Mx$ *is an ideal of I for every x in R, and*

$$M + Mx \cdot Mx \subseteq M + Mx.$$

(12) $MR \cdot II \subseteq M$.

(13) $(Mx \cdot Mx)(Mx \cdot Mx) \subseteq M$ *for every x in R.*

Remark. $Mx \cdot Mx$ is of course defined as the set of all finite sums of the form

$$\sum_{i=1}^{t} m_i x \cdot n_i x,$$

for m_i and n_i in M for all $i = 1, \ldots, t$.

Generally speaking this causes no difficulty in the proof. We may often just consider a single term of the form $mx \cdot nx$ to establish properties about $Mx \cdot Mx$, when it is clear that the property extends to finite sums.

Proof. Let m, n be in M and x in R. Then

$$mn \cdot x = m \cdot nx + (m, n, x) = m \cdot nx - (m, x, n)$$

by (1) and this equals $m \cdot nx - mx \cdot n + m \cdot xn$. Since m is in I, mx is in I and therefore $mx \cdot n$ is in M. Similarly $m \cdot nx$ and $m \cdot xn$ are both in M. Thus $mn \cdot x$ is in M and therefore (8) is established.

Let m be in M, x in R, and i in I. Then $(m, x, i) = -(m, i, x)$ by (1) and this is equal to $-mi\cdot x + m\cdot ix$. Now $m\cdot ix$ is in M and $-mi\cdot x$ is in Mx and this proves (9).

A typical element of $M + Mx$ is $m + nx$. Now

$$(m + nx)i = mi + nx\cdot i = mi + n\cdot xi + (n, x, i).$$

Since (n, x, i) is in $M + Mx$ by (9), and both mi and $n\cdot xi$ are in M, $M + Mx$ is a right ideal of I. Similarly

$$\begin{aligned} i(m + nx) &= im + i\cdot nx = im + in\cdot x - (i, n, x) \\ &= im + in\cdot x - (n, x, i) \text{ by (1)}. \end{aligned}$$

Again by (9) this triple sum is in $M + Mx$, and thus $M + Mx$ is an ideal of I. Thus we have (10).

To establish (11) we consider:

$$\begin{aligned} (M + Mx\cdot Mx)I &\subseteq M + (Mx\cdot Mx)I \\ &\subseteq M + Mx\cdot(Mx\cdot I) + (Mx, Mx, I) \\ &\subseteq M + Mx\cdot(Mx\cdot I) + (Mx, I, Mx) \qquad \text{by (1)}, \\ &\subseteq M + Mx\cdot(Mx\cdot I) + (Mx\cdot I)(Mx) + Mx\cdot(I\cdot Mx). \end{aligned}$$

Since $M + Mx$ is an ideal of I, $Mx\cdot I \subseteq M + Mx$ and $I\cdot Mx \subseteq M + Mx$. Thus

$$\begin{aligned} (M + Mx\cdot Mx)I &\subseteq M + Mx\cdot(M + Mx) + (M + Mx)(Mx) + Mx\cdot(M + Mx) \\ &\subseteq M + Mx\cdot Mx. \end{aligned}$$

Similarly $I\cdot(M + Mx\cdot Mx) \subseteq M + Mx\cdot Mx$ and thus $M + Mx\cdot Mx$ is an ideal of I. Since $Mx \subseteq I$ and $M + Mx$ is an ideal of I, $(M + Mx)\cdot Mx \subseteq M + Mx$ and therefore $M + Mx\cdot Mx \subseteq M + Mx$. This establishes (11).

To establish (12) we shall make use of (6). Take in (6), a in M, b and z in I, and y in R. Then (a, by, z) is in $(M, I, I) \subseteq M$. Also $(b, y, z)\cdot a$ is in

$$(I, R, I)\cdot M \subseteq IM \subseteq M.$$

Therefore by (6), $(b, ay, z) - (a, y, z)\cdot b$ is in M. By (1) we get

$$(ay, z, b) - (a, y, z)\cdot b$$

in M and this means that

$$(ay\cdot z)b - ay\cdot zb - (ay\cdot z)b + (a\cdot yz)b = (a\cdot yz)b - ay\cdot zb$$

is in M. However, yz is in I, $a\cdot yz$ is in M, and $(a\cdot yz)b$ is in M. Thus $ay\cdot zb$ is in M, for every a in M, y in R, z, b in I. Therefore $MR\cdot II \subseteq M$ and we have (12).

Finally

$$\begin{aligned} (Mx\cdot Mx)(Mx\cdot Mx) &\subseteq (M + Mx)(Mx\cdot Mx) \qquad \text{by (11)} \\ &\subseteq M(Mx\cdot Mx) + Mx\cdot(Mx\cdot Mx) \\ &\subseteq M + MR\cdot II \\ &\subseteq M \qquad \text{by (12)}. \end{aligned}$$

This gives us (13) and completes the proof of Lemma 3.

LEMMA 4. *If I is an ideal of an alternative ring R and if M is an ideal of I, then:*

(8′) $R \cdot MM \subseteq M$.
(9′) $(I, x, M) \subseteq M + xM$ *for every x in R.*
(10′) $M + xM$ *is an ideal of I for every x in R.*
(11′) $M + xM \cdot xM$ *is an ideal of I for every x in R, and* $M + xM \cdot xM \subseteq M + xM$.
(12′) $II \cdot RM \subseteq M$.
(13′) $(xM \cdot xM)(xM \cdot xM) \subseteq M$ *for every x in R.*

Proof. The proof is almost a mirror image of the proof of Lemma 3, where (7) is used instead of (6).

Before we make use of Lemmas 3 and 4, we need

LEMMA 5. *If S is any radical property defined for the class of all alternative rings, if R is any alternative ring and I is an ideal of R such that $II = 0$, then $S(I)$ is an ideal of R.*

Proof. In this case, when $II = 0$, the three associative difficulties do not pose a problem. The set $x \cdot S(I) + S(I)$ is an additive subgroup, and since it is in I, with $II = 0$, it is obviously an ideal of I.

Secondly,

$$\theta(y_1 y_2) = x \cdot y_1 y_2 + S(I) = 0 + S(I)$$

since $y_1 y_2$ is in $II = 0$.

Finally,

$$\theta(y_1) \cdot \theta(y_2) = xy_1 \cdot xy_2 + S(I) = 0 + S(I)$$

since xy_1 and xy_2 are both in I and $II = 0$.

Thus the associative proof of Theorem 1 goes through and Lemma 5 is true.

Remark. Since nothing alternative was used, Lemma 5 holds in the class of all narings.

THEOREM 2. *If S is any radical property defined for the class of all alternative rings, if R is any alternative ring, and I any ideal of R, then $S(I)$ is an ideal of R.*

Proof. Let $S(I)$ be called M. If M is not an ideal of R, then there exists an element x of R such that either xM or Mx is not contained in M. Assume first that Mx is not contained in M. Then $M + Mx$ properly contains M and is an ideal of I by (10).

Since I/M is, by (c), S-semi-simple, the non-zero ideal $(M + Mx)/M$ cannot be an S-ring. We plan to proceed as in the proof of Theorem 1.

We shall attempt to set up a homomorphism from M onto $(M + Mx)/M$, and if we can do this, then $(M + Mx)/M$ will have to be an S-ring, by (a). This contradiction will ensure that $Mx \subseteq M$.

Let y be any element of M and define

$$\theta(y) \equiv yx + M.$$

Then θ is a mapping from M to $(M + Mx)/M$. It is clearly an onto mapping and preserves addition. To show that $\theta(y_1 y_2) = \theta(y_1)\cdot\theta(y_2)$, we shall again show that they are both the zero coset.

Since $\theta(y_1 y_2) = y_1 y_2 \cdot x + M$, using (8), we can conclude that $y_1 y_2 \cdot x$ is in M and thus $\theta(y_1 y_2) = 0 + M$. On the other hand

$$\theta(y_1)\cdot\theta(y_2) = y_1 x \cdot y_2 x + M.$$

If $Mx \cdot Mx \subseteq M$, then $y_1 x \cdot y_2 x$ is in M and

$$\theta(y_1)\cdot\theta(y_2) = 0 + M = \theta(y_1 y_2).$$

In that case θ is a homomorphism, and the contradiction yields the fact that $Mx \subseteq M$.

We shall therefore assume that $Mx \cdot Mx$ is not contained in M. Then $M + Mx \cdot Mx$ properly contains M, and by (11) it is an ideal of I. Thus $(M + Mx \cdot Mx)/M$ is a non-zero ideal in the S-semi-simple ring I/M. By (13) we see that this non-zero ideal is a zero ring, i.e.,

$$(M + Mx \cdot Mx)/M \cdot (M + Mx \cdot Mx)/M = 0 + M.$$

By Lemma 5, $S[(M + Mx \cdot Mx)/M]$ is an ideal of I/M. Since I/M is S-semi-simple, $S[(M + Mx \cdot Mx)/M]$ must be 0 and thus $(M + Mx \cdot Mx)/M$ is an S-semi-simple ring.

Since $Mx \cdot Mx$ is not contained in M, there exist elements m_i and p_i in M, for $i = 1, \ldots, n$, such that

$$\sum_{i=1}^{n} m_i x \cdot p_i x$$

is not in M, and in fact for some k, $1 \leqslant k \leqslant n$, $m_k x \cdot p_k x$ is not in M.

We now define a mapping, for any y in M, as

$$h(y) \equiv yx \cdot p_k x + M.$$

This is a map from M onto $(Mx \cdot p_k x + M)/M$, and it clearly preserves addition. Furthermore,

$$h(y_1 y_2) = (y_1 y_2 \cdot x) \cdot p_k x + M = 0 + M$$

by (8). And

$$h(y_1)\cdot h(y_2) = (y_1 x \cdot p_k x)\cdot(y_2 x \cdot p_k x) + M = 0 + M$$

by (13). Therefore h is a homomorphism from the S-ring M onto the ring $(Mx \cdot p_k x + M)/M$. Since $m_k x \cdot p_k x$ is not in M, the image is a non-zero ring. Also the image is an ideal of $(Mx \cdot Mx + M)/M$, since it is closed under addition, and multiplication is always zero. Thus, by (a), the image is a non-zero S-ideal in the S-semi-simple ring $(Mx \cdot Mx + M)/M$. This is a contradiction and thus $Mx \cdot Mx \subseteq M$, in which case $Mx \subseteq M$

Similarly, using Lemma 4, we obtain $xM \subseteq M$. This proves the theorem.

COROLLARY 1. *If S is any radical property defined for the class of all alternative rings, then for any ideal I of an alternative ring R, we have*

$$S(I) \subseteq I \cap S(R).$$

Proof. Since $S(I)$ is an ideal of R, and it is an S-ring, it must, by (b), be contained in $S(R)$. It is certainly contained in I and thus $S(I) \subseteq I \cap S(R)$.

COROLLARY 2. *If S is any radical property defined for the class of all alternative rings, then every ideal of an S-semi-simple alternative ring is itself S-semi-simple.*

This follows from Corollary 1 and Lemma 2.

Kaplansky (**3**, Theorem 1) has shown that the Jacobson radical property, for alternative rings, satisfies condition (d), i.e. is hereditary. We shall now show that several other radical properties, including the Brown–McCoy radical property (**5**), are also hereditary.

LEMMA 6. *Let $\mathfrak{Q}$ be any class of simple alternative rings with unity elements. If I is an alternative ring which can be mapped homomorphically onto a ring A in $\mathfrak{Q}$, and if I is an ideal of an alternative ring R, then R can be mapped homomorphically onto A.*

Proof. Let h be the homomorphism from I onto A. Let i be any element in I such that $h(i) = e$, the unity element of A. Then for any x in R we define the mapping

$$\bar{h}(x) \equiv h(ixi).$$

Note that ixi is uniquely defined, by (4). Then $\bar{h}$ preserves addition because

$$\bar{h}(x + y) = h(i[x + y]i) = h(ixi + iyi) = h(ixi) + h(iyi),$$

since h preserves addition. Thus $\bar{h}(x + y) = \bar{h}(x) + \bar{h}(y)$.

To show that $\bar{h}$ preserves multiplication, we note by (5) that

$$\begin{aligned}\bar{h}(xy) &= h(i(xy)i) = h(ix \cdot yi) = h(ix) \cdot h(yi) \\ &= [h(ix) \cdot e] \cdot [e \cdot h(yi)] = [h(ix) \cdot h(i)] \cdot [h(i) \cdot h(yi)] \\ &= h(ixi) \cdot h(iyi) = \bar{h}(x) \cdot \bar{h}(y).\end{aligned}$$

Therefore $\bar{h}$ is a homomorphism.

For x in I,

$$\bar{h}(x) = h(ixi) = h(i)h(x)h(i) = eh(x)e = h(x).$$

Thus the image of $\bar{h}$ is precisely A. Therefore $\bar{h}$ is a homomorphism from R onto A and the lemma is established.

THEOREM 3. *If $\mathfrak{Q}$ is any class of simple alternative rings with unity elements, then the upper radical property determined by $\mathfrak{Q}$ is hereditary.*

Proof. Let R be an $S_{\mathfrak{Q}}$-ring and I a non-zero ideal of R. If I is not an $S_{\mathfrak{Q}}$-ring, then it can be mapped homomorphically onto a ring in $\mathfrak{Q}$. By Lemma 6, R

can be mapped homomorphically in this way. But this is impossible since R is an $S_{\mathfrak{Q}}$-ring. Therefore I is an $S_{\mathfrak{Q}}$-ring. By Lemma 1, we then have

$$S_{\mathfrak{Q}}(I) \supseteq I \cap S_{\mathfrak{Q}}(R)$$

for any ideal I of any alternative ring R.

Then $S_{\mathfrak{Q}}$ is hereditary by Theorem 2, Corollary 1.

If we now take $\mathfrak{Q}$ to be the class of *all* simple alternative rings with unity elements, and define the upper radical property determined by $\mathfrak{Q}$, i.e. a ring R is an $S_{\mathfrak{Q}}$-ring if it cannot be mapped homomorphically onto a ring in $\mathfrak{Q}$, then it is known **(5)** that this is the Brown–McCoy radical property for alternative rings. We then have:

COROLLARY. *The Brown–McCoy radical property for alternative rings is hereditary.*

We can also use Lemma 6 to obtain:

THEOREM 4. *Every $S_{\mathfrak{Q}}$-semi-simple alternative ring can be subdirectly embedded into a direct sum of rings from $\mathfrak{Q}$.*

Proof. Let R be an $S_{\mathfrak{Q}}$-semi-simple alternative ring, and let A be the intersection of all ideals I of R such that R/I is in $\mathfrak{Q}$. We shall show that $A = 0$ and this will prove the theorem.

If $A \neq 0$, then A is not an $S_{\mathfrak{Q}}$-ring, because R is $S_{\mathfrak{Q}}$-semi-simple. Therefore there exists a homomorphism h of A onto a ring T in $\mathfrak{Q}$. By Lemma 6, there must exist a homomorphism $\bar{h}$ from R onto T. Let K be the kernel of $\bar{h}$. Then K is an ideal of R such that R/K is in $\mathfrak{Q}$. Thus $A \subseteq K$. However, A cannot be contained in the kernel K because $\bar{h}$ coincides with h on A and if $\bar{h}(A) = 0$, then $h(A) = 0$ and thus $T = 0$, a contradiction. Therefore $A = 0$ and the theorem is true.

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