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A NOTE ON QUADRATIC FORMS OVER ARBITRARY SEMI-LOCAL RINGS

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1. Introduction and preliminaries. Let R be a commutative ring. A *bilinear space* (E, B) over R is a finitely generated projective R-module E together with a symmetric bilinear mapping $B:E \times E \rightarrow R$ which is non-degenerate (i.e. the natural mapping $E \rightarrow \mathrm{Hom}_R(E, R)$ induced by B is an isomorphism). A *quadratic space* (E, B, ϕ) is a bilinear space (E, B) together with a quadratic mapping $\phi:E \rightarrow R$ such that $B(x,y) = \phi(x + y) - \phi(x) - \phi(y)$ and $\phi(rx) = r^2\phi(x)$ for all x, y in E and r in R. If 2 is a unit in R, then $\phi(x) = \frac{1}{2}\cdot B(x, x)$ and the two types of spaces are in obvious $1 - 1$ correspondence.

We will write $W(R)$ for the *Witt ring* based on bilinear spaces (as defined in [**13**, p. 123]), and denote the corresponding group based on quadratic spaces as defined in [**3**, p. 144] by $W_q(R)$. A hyperbolic plane H is the free rank 2 quadratic (respectively bilinear) space with basis $\{x, y\}$ such that $\phi(x) = \phi(y) = 0$ and $B(x, y) = 1$ (respectively $B(x, x) = B(y, y) = 0$ and $B(x, y) = 1$). We call a space hyperbolic if it is an orthogonal sum of hyperbolic planes. Let R be a commutative semi-local ring. Then R can be written as $\prod_{i=m}^{n} e_iR$ where each e_iR is a connected semi-local ring (i.e. no non-trivial idempotents), and the $\{e_i\}$ are a set of orthogonal idempotents. This ring decomposition induces decompositions $W_q(R) = \prod W_q(e_iR)$ and $W(R) = \prod W(e_iR)$ as group and rings respectively (cf. [**13**, Lemma 1.9]). A quadratic space (E, B, ϕ) is then trivial in $W_q(R)$ if and only if (e_iE, B, ϕ) is trivial in each $W_q(e_iR)$. But over e_iR, projectives are free, thus by the definition of $W_q(e_iR)$ and [**11**, Kürzungssatz, p. 256] it is easy to deduce that (e_iE, B, ϕ) is trivial in $W_q(e_iR)$ if and only if it is hyperbolic. Therefore, (E, B, ϕ) is trivial in $W_q(R)$ if and only if each e_iE is hyperbolic over e_iR, or, if E is free, if and only if E is hyperbolic over R. If 2 is a unit in R, the analogous statement holds for $W(R)$.

Witt [**17**] has shown that a quadratic space of rank 3 or less over a field K of characteristic different from 2 is determined up to isometry by its rank, determinant, and Hasse invariant. In [**1**], Arf proved a corresponding result when the characteristic of K is 2. In Section 2 we simultaneously extend these results to quadratic spaces over an arbitrary semi-local ring R, using the theory of the Clifford algebra and graded Brauer group in [**3**] and [**16**]. In Section 3 we use the results of Section 2 together with some results in [**13**] to prove that if R and S are semi-local rings in which 2 is a unit, then $W(R) \simeq W(S)$ if and only if

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$W(R)/I(R)^3 \simeq W(S)/I(S)^3$, where $I(R)$ and $I(S)$ denote the ideals generated by the free forms of even rank.

We wish to acknowledge the aid we received from M. Knebusch's unpublished paper [**11**]. After seeing it we decided to alter our original version of Section 2 (which assumed 2 was a unit) to the present more general situation. We have also learned in recent communication with J. S. Hsia that he and Roger Peterson have also proved the results of Section 2 when 2 is a unit [**9**].

We conclude this section with some notation and some general facts about quadratic forms over semi-local rings.

If R is a commutative ring, we denote the group of multiplicative units of R by $U(R)$. If (E, B) is a free rank n bilinear space with orthogonal basis $\{x_1, x_2, \ldots, x_n\}$, we will write $\langle a_1, \ldots, a_n \rangle$ for (E, B) where the $a_i = B(x_i, x_i)$ are in $U(R)$. In general, if (E, B, ϕ) is a free rank n quadratic space with basis $\{x_1, \ldots, x_n\}$, we will denote the space by the table $[a_{ij}]$ where $a_{ij} = B(x_i, x_j)$ if $i \neq j$ and $a_{ii} = \phi(x_i)$. In the case $n = 2$ and $B(x_1, x_2) = 1$ we reserve the notation $[a_{11}, a_{22}]$ for the quadratic space. If E is a form and m is an integer, mE will denote the m-fold orthogonal sum of E. If u is in $U(R)$, (E^u, B^u, ϕ^u) will denote the space obtained when B and ϕ are simply multiplied by u.

We now prove a decomposition theorem for quadratic spaces over a semi-local ring. A similar theorem in the special case of a valuation ring can be found in [**10**, Theorem 3.8, p. 47].

PROPOSITION 1.1. *Let (E, B, ϕ) be a free non-degenerate quadratic space of rank n over a semi-local ring R. Then*:

(1) *if 2 is a unit in R, $(E, B) \simeq \langle a_1, \ldots, a_n \rangle$ where each a_i is a unit in R;*

(2) *if 2 is not a unit in R, n is even and*

$$E \simeq \perp_{i=1}^{n/2} [b_i, c_i],$$

where b_i and $1 - 4b_ic_i$ are units in R for each i. In any case, if E has even rank it decomposes as in (2).

Proof. Part (1) follows from [**13**, Lemma 1.12]. For part (2) we note that the non-degeneracy of E forces $1 - 4b_ic_i$ to be a unit once we have the decomposition. Furthermore, if d is a unit

$$\begin{bmatrix} b & d \\ d & c \end{bmatrix} \simeq [b/d^2, c]$$

by an easy change of basis. Thus, it will suffice to decompose E as

$$\perp_{i=1}^{n/2} \begin{bmatrix} b_i & d_i \\ d_i & c_i \end{bmatrix}$$

where the b_i and d_i are in $U(R)$.

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Let $m_1, m_2, \ldots, m_t$ be the maximal ideals of R. Using the fact that orthogonal decompositions lift modulo the radical [**11**, Lemma 1.1.4] as must units, we may assume Rad $R = m_1 \cap \ldots \cap m_t = 0$. Then by the Chinese Remainder Theorem, we can write $R = \sum_{i=1}^{t} e_i R$ where the e_i are orthogonal idempotents with $e_i = \delta_{ij} (\text{mod } m_j)$, so that $e_i R \simeq R/m_i$ is a field.

Since we are assuming 2 is not a unit in R, there is some i for which $e_i R$ is a field of characteristic 2. Then by [**1**, Satz 2, p. 150] we can decompose $e_i E$ into an orthogonal sum of rank 2 spaces with array of the form $\begin{bmatrix} b & d \\ d & c \end{bmatrix}$ with $d \neq 0$ in $e_i R$. To be of the required form, we must also obtain that b is not 0 in $e_i R$. Of course, if b is 0 while c is not, we simply interchange basis elements. If both b and c are 0 we note that

$$\begin{bmatrix} 0 & d \\ d & 0 \end{bmatrix} \simeq \begin{bmatrix} d & d \\ d & 0 \end{bmatrix}.$$

Besides giving us the required decomposition when the characteristic is 2, this implies that n is even. Then if $e_j R$ is of characteristic different from 2, $e_j E$ has an orthogonal basis and can be decomposed into rank 2 spaces which are diagonal. But by an easy basis change we see

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \simeq \begin{bmatrix} a & 2a \\ 2a & a+b \end{bmatrix}.$$

So no matter what the characteristic of $e_j R$, we can write $e_j E \simeq \perp_{k=1}^{n/2} M_k^j$ where M_k^j has basis $\{x_k^j, y_k^j\}$ over $e_j R$ and both $B(x_k^j, y_k^j)$ and $\phi(x_k^j)$ are not 0 in $e_j R$. Now letting

$$x_k = \sum_{j=1}^{t} x_k^j, \quad y_k = \sum_{j=1}^{t} y_k^j$$

and M_k be the submodule of M with basis $\{x_k, y_k\}$ we have $E = \perp_{k=1}^{n/2} M_k$ in the required form, completing the proof.

A free quadratic space (E, B, ϕ) is said to be *isotropic* if there is a basis vector x with $\phi(x) = 0$, otherwise it is called *anisotropic*. Clearly hyperbolic spaces are isotropic. The next proposition provides a partial converse.

PROPOSITION 1.2. *Let (E, B, ϕ) be a free non-degenerate quadratic space over a semi-local ring R. Then E has a decomposition $E = K \perp F$, where K is hyperbolic and F is anisotropic.*

Proof. By induction, it will suffice to show that if E is isotropic then $E = H \perp E_1$. Thus, suppose $\phi(x) = 0$ for some basis vector x. We first assume there is a y in E such that $B(x, y) = a$ is a unit and $\{x, y\}$ can be extended to a basis of E. By an easy change of basis, we see that E_0, the span of $\{x, y\}$, is isometric to a hyperbolic plane. But then by [**13**, Lemma 1.1],

$$E = E_0 \perp E_1 \simeq H \perp E_1$$

for some E_1.

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To pick y as required we may assume Rad $R = 0$ by [**4**, Proposition 5, p. 106]. Thus $R = \prod_1^k Re_i$ with the Re_i fields. Then e_iE is a non-degenerate free space over the field e_iR, with e_ix part of a basis. Hence, there must be a y_i in e_iE with $B(e_ix, y_i) \neq 0$ in e_iR, for otherwise the non-degeneracy is violated. Furthermore, y_i is not a multiple of e_ix, since if $r \in e_iR$, $B(e_ix, re_ix) = re_iB(x,x) = 2re_i\phi(x) = 0$. Hence e_ix and y_i can be extended to a basis of e_iE. It then follows that $y = \sum y_i$ is as required.

2. The Clifford algebra. Let R be a commutative ring. In this section we will draw heavily on the theory of graded central separable R-algebras in [**3**, Chapter 4] and [**16**]. Before proceeding we recall some of the definitions and theorems we will need from these sources.

Let $\mathbf{F}_2$ be the field with 2 elements. An $(\mathbf{F}_2)$-*graded R-algebra* A is an R-algebra together with an R-module decomposition $A = A_0 \oplus A_1$, where $A_i \cdot A_j \subseteq A_{i+j}$ (i, j in $\mathbf{F}_2$). If x is in either A_i we say it is a homogeneous element and write $i = \deg x$. Clearly, A_0 is an ungraded algebra, and conversely any ungraded algebra B can be thought of as graded by setting $A_0 = B$ and $A_1 = 0$. Graded algebras of this last type are said to be *concentrated* in degree 0. When A is a graded algebra we will write $|A|$ for the algebra considered as an ungraded algebra. If A and B are graded R-algebras we define the *skew tensor product* $A \otimes' B$ to be the graded R-algebra C, where $C_0 = (A_0 \otimes B_0) \oplus (A_1 \otimes B_1)$, $C_1 = A_0 \otimes B_1 \oplus A_1 \otimes B_0$ and the multiplication is extended by linearity from $(a \otimes b)(c \otimes d) = (-1)^{\deg b \cdot \deg c} ac \otimes bd$, where a, b, c, d are homogeneous elements of A and B. We will write $A \otimes B$ for the algebra with the ordinary multiplication of the tensor product and the above grading. If S is a subset of A we write A^S for the graded subalgebra of A generated by

$$\{a \in A_0 \text{ or } A_1 | as = -1^{\deg a \cdot \deg s} sa \text{ for all homogeneous } s \text{ in } S\}.$$

By a graded R-module we will mean an R-module M together with an R-module decomposition $M = M_0 \oplus M_1$. We will then write $A = \mathbf{End}_R\, M$ for the R-algebra $\mathrm{End}_R\, M$ graded by $A_0 = \mathrm{End}_R\, M_0 \oplus \mathrm{End}_R\, M_1$ and $A_1 = \mathrm{Hom}_R(M_0, M_1) \oplus \mathrm{Hom}_R(M_1, M_0)$.

A graded R-algebra is called central if A^A is just R. For a definition of separability for graded R-algebras see [**16**, 2.10, p. 464]. We will say that two graded central separable R-algebras A and B are equivalent if $A \otimes' \mathbf{End}\, P \simeq B \otimes' \mathbf{End}\, Q$ where P and Q are graded R-module which are R-progenerators, i.e. finitely generated, faithful and projective, when considered as ungraded R-modules. The set of equivalence classes of graded central separable R-algebras then forms a group where the multiplication is induced by skew tensor product [**16**, p. 485 and 3.12, 3.14, p. 467]. We will write $Br_2(R)$ for this group, and refer to it as "the graded Brauer group of R", and reserve $Br(R)$ for the ordinary Brauer group.

LEMMA 2.1. *Let A and B be graded central separable R-algebras which are in*

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the same class of $Br_2(R)$. Then $|A|$ and $|B|$ are separable R-algebras, and if they are both central R-algebras they will be in the same class of $Br(R)$.

Proof. That $|A|$ and $|B|$ are separable R-algebras follows from [**16**, Corollary 5.13, p. 482]. Since A and B are in the same class of $Br_2(R)$, we may write $A \otimes' \mathbf{End}\ P \simeq B \otimes' \mathbf{End}\ Q$ where P and Q are graded R-modules which are R-progenerators as ungraded R-modules. But then by [**5**, Proposition 1.5, p. 10], $A \otimes \mathbf{End}_R\ P \simeq B \otimes \mathbf{End}_R\ Q$. Then simply ignoring the grading this yields $|A| \otimes \mathrm{End}_R\ P \simeq |B| \otimes \mathrm{End}_R\ Q$ as R-algebras. But now if we assume $|A|$ and $|B|$ are central R-algebras, it follows by definition that they are in the same class of $Br(R)$.

Let $Q_2(R)$ be the set of isomorphism classes of graded separable R-algebras which are projective R-modules of rank two. When any such algebra is considered without its grading it is a Galois extension of R with a cyclic Galois group of order two [**16**, Corollary 7.4, p. 487]. An abelian group multiplication $*$ is then defined on Q_2 as follows: let L_1 and L_2 be two algebras representing classes in Q_2. Then if σ_1, σ_2 are their non-trivial automorphisms we define $L_1 * L_2$ to be the isomorphism class of the subalgebra of $L_1 \otimes' L_2$ fixed by $\sigma_1 \otimes \sigma_2$ [**16**, p. 488]. Furthermore, there is an exact sequence of abelian groups

$$0 \to Br(R) \to Br_2(R) \to Q_2(R) \to 0.$$

Here, the first map is the one that takes the class of an ungraded algebra to the class of that algebra concentrated in degree 0, and the second takes the class of a non-degenerate algebra A (i.e. A_0 and A_1 have positive rank) to $L(A)$, where $L(A) = \text{class } A^{A_0}$ [**16**, Theorem 7.10, p. 490].

Suppose (M, B, ϕ) is a non-degenerate quadratic space over R. Let $T(M)$ be the tensor algebra of M, with the obvious $\mathbf{F}_2$-grading. We denote by $J(M)$ the ideal of $T(M)$ generated by the homogeneous elements $x \otimes x - \phi(x)$ for all x in M. Then Cliff M is defined to be the graded quotient algebra $T(M)/J(M)$. By [**3**, Corollary 3.8, p. 152] the R-module map $M \to T(M) \to \text{Cliff } M$ is an injection. Therefore, we can identify M with its image in Cliff M, where it, together with R, generates Cliff M. Cliff M is actually a graded central separable R-algebra and Cliff induces a natural group homomorphism $W_q(R) \to Br_2(R)$ by [**3**, Theorem 3.9 and Corollary 3.10, p. 154].

Example 2.2. Let (M, B, ϕ) be a free quadratic space over a commutative ring R, with Cliff $M = C = C_0 \oplus C_1$. Then:

(i) If $(M, B) = \langle 2a \rangle$, then $(\text{Cliff } M)^{C_0} = \text{Cliff } M = R \oplus Rx$ with $x^2 = a$, where the grading is given by $C_0 = R$, $C_1 = Rx$.

(ii) If $(M, B) = \langle 2a_1, \ldots, 2a_n \rangle$ with respect to a basis $\{x_1, \ldots, x_n\}$, then as an ungraded algebra Cliff M is a free R-algebra of rank 2^n with homogeneous basis $\{x_1^{r_1}, \ldots, x_n^{r_n} | r_i = 0, 1\}$ where $x_i^2 = a_i$ and $x_i x_j = -x_j x_i$ when $i \neq j$. The degree of $x_1^{r_1}, \ldots, x_n^{r_n}$ is congruent to $\sum r_i$ modulo 2.

(iii) If $(M, B, \phi) = [a, b]$ then Cliff M is the free rank 4 R-algebra with basis

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$\{1, x, y, xy\}$ where $x^2 = a$, $y^2 = b$, $xy + yx = 1$, and the grading is given by $C_0 = R \oplus Rxy$, $C_1 = Rx \oplus Ry$. $(\text{Cliff } M)^{C_0}$ is C_0 concentrated in degree 0.

Proof. (i). Since $T(M) \simeq R[x]$ where $M = Rx$, the description of Cliff M is immediate from the definition. Moreover, since $C_0 = R$, we obtain $(\text{Cliff } M)^{C_0} = \text{Cliff } M$.

(ii). By [**3**, Lemma 3.1, p. 147] we have Cliff $\langle 2a_1, \ldots, 2a_n \rangle = \otimes'^{n}_{i=1}$ Cliff $\langle 2a_i \rangle$. The description of Cliff M now follows immediately from the definition of $\otimes'$ and part (i).

(iii). Let $M = [a, b]$ with respect to a basis $\{x, y\}$. Then in Cliff M we have $x^2 = \phi(x) = a$ and $y^2 = \phi(y) = b$. Furthermore, in Cliff M, $\phi(x + y) = \phi(x) + \phi(y) + B(x, y)$ as well as $\phi(x + y) = (x + y)^2 = x^2 + y^2 + xy + yx = \phi(x) + \phi(y) + xy + yx$. Hence $xy + yx = B(x, y) = 1$. Since M and R generate Cliff M as an R-algebra, these relations imply that $\{1, x, y, xy\}$ span Cliff M over R. In fact $\{x, y\}$ span C_1 and $\{1, xy\}$ span C_0. Now, since M is free on $\{x, y\}$ they are independent and thus a basis of C_1. This implies $\{1, xy\}$ are independent and thus a basis of C_0: For suppose $r \cdot 1 + s \cdot xy = 0$. Then $r \cdot x + s \cdot x^2 y = 0$ or $r \cdot x + sa \cdot y = 0$, which yields $r = 0$ and $sa = 0$. Then since a is a unit it follows that $s = 0$, which establishes the independence. But this is just the structure we claimed for Cliff M.

Now we compute $(\text{Cliff } M)^{C_0}$. Clearly it contains C_0 since C_0 is commutative. Then $C^{C_0} = C_0 \oplus D_1$ with D_1 contained in C_1. Now $L(\text{Cliff } M)$ is in $Q_2(R)$, hence C^{C_0} is a projective R-module of rank 2. But since C_0 is a free R-module of rank 2, this makes the localization $(D_1)_P = 0$ for every prime ideal P of R. Therefore $D_1 = 0$ and $C^{C_0} = C_0$.

LEMMA 2.3. *Let (M, B, ϕ) be a free quadratic space over a commutative ring R, which has a decomposition as in Proposition 1.1* (2) (which is always true if R is semi-local and the rank of M is even). *Then* $|\text{Cliff } M|$ *is a central separable (ungraded) R-algebra. If (M', B', ϕ') has the same even rank and* Cliff M = Cliff M' *in $Br_2(R)$, then* $|\text{Cliff } M|$ *and* $|\text{Cliff } M'|$ *are isomorphic.*

Proof. The last assertion follows from the first, Lemma 2.1, and [**6**, Corollary 1] (applied after decomposing R into a product of connected rings).

As in Lemma 2.1 we always know $|\text{Cliff } M|$ is separable. Thus, we must only show that $|\text{Cliff } M|$ is central. Let $C = \text{Cliff } M = C_0 \oplus C_1$. By Proposition 1.1 we can write $M = \perp M_i$, where each M_i is free of rank 2. But by [**3**, Lemma 3.1] Cliff $M = \otimes'$ Cliff M_i, hence $L(\text{Cliff } M) = * L(\text{Cliff } M_i)$ since L is a group homomorphism. By Example 2.2 (iii) each representing algebra in $L(\text{Cliff } M_i)$ is concentrated in degree 0, hence the product in Q_2 is concentrated in degree 0. Thus we have $C^{C_0} \subseteq C_0$. Now, let x be in the center of C considered as an ungraded algebra. Then certainly x is in C^{C_0}, hence x is in C_0. But if x is in C_0 and commutes with all of C, it is in C^C. Then since C is central as a graded algebra, $C^C = R$ and x is in R. Therefore C is central in the ungraded sense.

LEMMA 2.4. *Let A be a free rank* 4 *central separable algebra over a commutative*

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ring R and S a commutative R-algebra. If f is an S-algebra isomorphism of $A \otimes S$ *to* $M_2(S)$ *we will write* $N(f, S)$ *for the map* $A \to S$ *defined by* $N(f, S)(a) =$ *determinant* $(f(a \otimes 1))$. *Then*:

(1) *The image of* $N(f, S)$ *lies in R and does not depend on the choice of f and S, i.e. if S' is another commutative R-algebra and* $f': A \otimes' S' \to M_2(S')$ *is an S'-algebra isomorphism, then* $N(f, S) = N(f', S')$.

(2) $N(f, S)$ *defines a quadratic form on A.*

Proof. (1) is just [**7**, Proposition 3.1, p. 237], and (2) follows by a routine calculation with 2×2 matrices.

Under the hypotheses of the Lemma, we will write $N: A \to R$ for the quadratic form $N(S, f)$, and refer to it as the *reduced norm*.

COROLLARY 2.5. *Let* $[a, b]$ *be a non-degenerate quadratic space over a commutative ring R. Then the reduced norm makes* $|\text{Cliff }([a, b])|$ *into a quadratic space isometric to* $[-a, -b] \perp [1, ab]$. *If R is semi-local and* $\text{Cliff }[c, d] = \text{Cliff }[a, b]$ *in* $Br_2(R)$, *it follows that* $[-a, -b] \perp [1, ab] \simeq [-c, -d] \perp [1, cd]$.

Proof. By Example 2.2 (iii), Cliff $([a, b])$ is the rank 4 R-algebra $C = R \oplus Rx \oplus Ry \oplus Rxy$ with $C_0 = R \oplus Rxy$, $C_1 = Rx \oplus Ry$, $x^2 = a$, $y^2 = b$, $xy + yx = 1$, and C^{C_0} is the rank 2 subalgebra $R \oplus Rxy$. But then, $(xy)^2 = x(yx)y = x(1 - xy)y = xy - ab$, hence the free rank 2 R-algebra $S = R \oplus Rz$ with $z^2 = z - ab$ is Galois [**16**, 7.4, p. 287].

We now define an R-module homomorphism $f: C \to M_2(S)$ by letting

$$f(x) = \begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix}, \quad f(y) = \begin{bmatrix} 0 & (1-z)/a \\ z & 0 \end{bmatrix}, \quad f(1) = 1 \quad \text{and}$$

$$f(xy) = \begin{bmatrix} z & 0 \\ 0 & 1-z \end{bmatrix}.$$

It then follows that f is actually an R-algebra homomorphism just by checking the identities $f(xy) = f(x)f(y)$, $f(x)^2 = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, $f(y)^2 = \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}$, and $f(x)f(y) + f(y)f(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Now let $f^*: C \otimes S \to M_2(S)$ be the S-algebra homomorphism induced by f. Since $C \otimes S$ and $M_2(S)$ are free central separable S algebras of the same rank by [**2**, Corollary 3.4], f^* is an isomorphism. Now by directly computing the norm of Lemma 2.3 (i) on the R-basis $\{x, -y, 1, xy\}$ we get the quadratic space $[-a, -b] \perp [1, ab]$.

The final conclusion now follows from Lemma 2.3 together with the uniqueness in Lemma 2.4.

Remark 2.6. In case 2 is a unit, we note that by a change of basis we may restate Corollary 2.5 with $\langle a', b' \rangle$ (a', b' in $U(R)$) replacing $[a, b]$ in the hypothesis, and $\langle -a', -b' \rangle \perp \langle 2, a'b' \rangle$ replacing $[-a, -b] \perp [1, ab]$ in the conclusion (and similarly for c, d).

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PROPOSITION 2.7. *Let* (M, B, ϕ) *and* (N, B', ϕ') *be free quadratic spaces of rank* 2 *over a semi-local ring* R. *Then* M *is isometric to* N *if and only if* Cliff (M) = Cliff (N) *in* $Br_2(R)$.

Proof. Clearly, only the sufficiency need be proved. By Proposition 1.1 we may assume $(M, B, \phi) = [a, b]$ and $(N, B, \phi) = [c, d]$.

As noted in the proof of Corollary 2.5 above, L(Cliff M) is represented by the free rank 2 R-algebra $S = R \oplus Rz$ with $z^2 = z - ab$, which is a Galois extension of R with a 2 element Galois group by [**16**, 7.4, p. 281]. Then since $(1 - z)^2 = (1 - z) - ab$, S has a unique non-trivial automorphism j with $j(z) = 1 - z$. Then defining $\lambda(w) = wj(w)$ for any w in S we define an $S^j = R$ valued quadratic form λ on S. Computing λ on the basis $\{1, z\}$ we see this form is isometric to $[1, ab]$. Therefore, since $L(\text{Cliff } M) = L(\text{Cliff } N)$, we have $[1, ab] \simeq [1, cd]$. Then by Corollary 2.5 and [**12**, Kürzungssatz], $[-a, -b] \simeq [-c, -d]$. It then follows that $M \simeq N$.

If (M, B, ϕ) (respectively, (M, B)) is a free quadratic (respectively, bilinear) space over R, we write Det M for the determinant of B (relative to some basis) and det M for the class of Det M in $U(R)/U(R)^2$.

If 2 is a unit in R, then for each i in $\mathbf{F}_2$ and $a \in U(R)$ we will write $R\{a\}_i$ for the graded algebra $R \oplus Rz$ with $z^2 = a$ and deg $z = i$. From Example 2.2 (1) and 2.2 (3) (with a change of basis), we see $L(\text{Cliff } \langle 2a_1 \rangle) = cl(R\{a_1\}_1)$ and $L(\text{Cliff } \langle 2a_1, 2a_2 \rangle) = \text{cl}(R\{-a_1a_2\}_0)$, thus $R\{a\}_i$ always represents a class in $Q_2(R)$. Clearly, the automorphism σ with $\sigma(z) = -z$ is the unique non-trivial automorphism of $R\{a\}_i$.

LEMMA 2.8. *Let* R *be a semi-local ring in which* 2 *is a unit.*

(1) *If* (M, B, ϕ) *is a free rank n quadratic space, then* L(Cliff M) *is represented by* $R\{(\frac{1}{2})^n(-1)^{n(n-1)/2}\text{Det } M\}_{\bar{n}}$.

(2) *If* (M, B, ϕ) *and* (M', B', ϕ') *are two free spaces of the same rank, then* L(Cliff M) = L(Cliff N) *if and only if* det M = det N.

Proof. (1) If σ_1 and σ_2 are the unique non-trivial automorphism of $R\{a_1\}_i$ and $R\{a_2\}_j$, it is not hard to show that the subalgebra of $(R\{a_1\}_i \otimes' R\{a_2\}_j)$ fixed by $\sigma_1 \otimes \sigma_2$ is isomorphic to $R\{(-1)^{ij}a_1a_2\}_{i+j}$. The result then follows easily by induction using Proposition 1.1 (1) and Example 2.2 (1).

(2) Clearly, by part (1) it will suffice to show that if an R-algebra has two R-bases $\{1, t\}$ and $\{1, y\}$ with t^2 and u^2 in R, then $t = bu$ for some unit b of R. But since $\{1, u\}$ is a basis, we can write $t = a + bu$ for some a and b in R. Then $t^2 = (a^2 + b^2u^2) + 2abu$, hence $ab = 0$. But then $-a^2 \cdot 1 + a \cdot t = 0$, thus by the independence of 1 and t we obtain $a = 0$ and $t = bu$. Furthermore, since $\{1, t\}$ is a basis b must be a unit of R.

Let R be a commutative ring in which 2 is a unit. If a, b are units in R, we define the *quaternion algebra* $[(a, b)/R]$ to be the (ungraded) free rank 4

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R-algebra with basis $\{1, x, y, xy\}$ subject to $x^2 = a$, $y^2 = b$ and $xy = -yx$. By Example 2.2 (2) we see that $[(a, b)/R] = |\text{Cliff } \langle 2a, 2b\rangle|$, and is thus central separable by Lemma 2.3.

LEMMA 2.9. *Let R be a commutative ring in which 2 is a unit. If a, b, c are units of R, then* Cliff $\langle 2a, 2b, 2c, 2abc\rangle = [(-ab, -ac)/R]$ *in $Br_2(R)$, and are in fact equal as ungraded algebras in $Br(R)$.*

Proof. Let Cliff $\langle 2a, 2b, 2c\rangle = C = C_0 \oplus C_1$. By Example 2.2 (2) we see that $C_0 \simeq [(-ab, -ac)/R]$. But by [**16**, Corollary 6.4, p. 483] $C \simeq C_0 \otimes C^{C_0}$, thus by Lemma 2.7 (1) $C \simeq [(-ab, -ac)/R] \otimes' R\{-abc\}_1$. Then by Example 2.2 (1), $C \simeq [(-ab, -ac)/R] \otimes \text{Cliff } \langle -2abc\rangle$. Therefore,

$$\text{Cliff } \langle 2a, 2b, 2c, 2abc\rangle \simeq C \otimes' \text{Cliff } \langle 2abc\rangle \simeq [(-ab, -ac)/R] \otimes \text{Cliff } \langle -2abc, 2abc\rangle.$$

Since $\langle -2abc, 2abc\rangle$ is hyperbolic this implies that Cliff $\langle 2a, 2b, 2c, 2abc\rangle = [(-ab, -ac)/R]$ in $Br_2(R)$. The final conclusion now follows from Lemmas 2.1 and 2.3.

THEOREM 2.10. *Let (M, B, ϕ) and (N, B', ϕ') be free non-degenerate quadratic spaces of rank less than or equal to 3, over a semi-local ring R. Then M is isometric to N if and only if* Cliff M = Cliff N *in $Br_2(R)$.*

Proof. Clearly, only the sufficiency need be proved. Let n be the common rank. We already have the conclusion if $n = 2$ by Proposition 2.7. Thus n must be odd, and by Proposition 1.1, 2 is a unit in R. But then the $n = 1$ case is resolved by Example 2.2 (1) and Lemma 2.8. Therefore we may assume $n = 3$.

By Proposition 1.1 we may write $M = \langle 2a, 2b, 2c\rangle$ and $N = \langle 2d, 2e, 2f\rangle$. But, by Lemma 2.8 we know $abc \equiv def (\text{mod } U(R)^2)$, thus Cliff $\langle 2a, 2b, 2c, 2abc\rangle$ = Cliff $\langle 2d, 2e, 2f, 2def\rangle$ in $Br_2(R)$, which yields

$$|\text{Cliff } \langle -2ab, -2ac\rangle| = |\text{Cliff } \langle -2de, -2df\rangle|$$

in $Br_2(R)$ by Lemma 2.9 and Example 2.2 (2). Thus by Remark 2.6 we can conclude that $\langle 2ab, 2ac, 2, 2bc\rangle \simeq \langle 2de, 2df, 2, 2ef\rangle$. After cancelling $\langle 2\rangle$ from each side and scaling by $abc \equiv def$, the required result follows.

Theorem 2.10 does not hold in general for spaces of higher rank even when 2 is a unit. For example, it is easy to check that $4\langle 1\rangle$ and $4\langle -1\rangle$ have the same invariants over the rational numbers although they are not isometric [**14**, Example 58:5, p. 152]. After proving a technical Lemma we will give a condition on R under which Theorem 2.10 will hold for spaces of arbitrary rank.

LEMMA 2.11. *Let R be a semi-local ring and (M, B, ϕ) a free rank 4 non-*

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degenerate quadratic space over R. Then M is hyperbolic if and only if Cliff M *is trivial in* $Br_2(R)$.

Proof. The necessity is immediate since we know that Cliff induces a group homomorphism $W_q(R) \to Br_2(R)$. Thus, suppose Cliff M is trivial in $Br_2(R)$. By Proposition 1.1 we can write $M = M_1 \perp M_2$ where each M_i is of rank 2. But then cl(Cliff M_1) = cl(Cliff $M_2)^{-1}$ in $Br_2(R)$. However, $M_1 \perp M_1^{-1}$ is hyperbolic where M_1^{-1} is the space M_1 scaled by -1, hence cl(Cliff M_1) = cl(Cliff $M_1^{-1})^{-1}$ as well. Therefore, Cliff M_1^{-1} and Cliff M_2 are in the same class of $Br_2(R)$. Then by Theorem 2.10, $M_1^{-1} \simeq M_2$, and thus $M \simeq M_1 \perp M_2 \simeq M_1 \perp M_1^{-1}$. But this makes M hyperbolic.

THEOREM 2.12. *Let R be a semi-local ring over which all free non-degenerate spaces of rank greater than* 4 *are isotropic. Then any free non-degenerate quadratic space (M, B, ϕ) is determined up to isometry by its dimension and by the class of* Cliff M *in* $Br_2(R)$.

Proof. If the statement is false, let (M, B, ϕ) be a counterexample of minimal rank n.

Suppose n is greater than 4. Then by hypothesis, M is isotropic and so by Proposition 1.2, we have $M \simeq M' \perp H$ where M' is now of rank $n - 2$. But since Cliff induces a homomorphism from $W_q(R)$ to $Br_2(R)$, Cliff M is in the same class as Cliff M'. Therefore, by the minimality of n, the isometry class of M' is completely determined by Cliff M. Since M' then determines M we have a contradiction.

By Theorem 2.10, n cannot be less than 4, hence it is exactly 4. Therefore, there exist two rank 4 quadratic spaces (M, B, ϕ) and (N, B', ϕ') with Cliff M and Cliff N in the same class of $Br_2(R)$, but M not isometric to N. By Proposition 1.1 we can write $N = N_1 \perp N_2$ where N_1, N_2 are rank 2 spaces. Then $M^{-1} \perp N_1$ is a rank 6 space, which by hypothesis must be isotropic. By Proposition 1.2 we can write $M^{-1} \perp N_1 \simeq E_1 \perp H$ for some E_1 of rank 4. Then since $E_1 \perp N_2$ is also of rank 6, it too can be written as $E_2 \perp H$ for some rank 4 space E_2. Combining the isometries we obtain $M^{-1} \perp N \simeq M^{-1} \perp N_1 \perp N_2 \simeq E_1 \perp H \perp N_2 \simeq E_2 \perp 2H$. But then $4H \perp N \simeq M \perp M^{-1} \perp N \simeq M \perp E_2 \perp 2H$, which implies class (Cliff N) = class (Cliff M)$\cdot$class (Cliff E_2). Therefore class (Cliff E_2) is trivial in $Br_2(R)$, and Lemma 2.11 implies that E_2 is hyperbolic. Substituting for E_2 above yields $4H \perp N \simeq M \perp 4H$, hence by [**11**, Kürzungssatz] $M \simeq N$. This is a contradiction.

We conclude this section by reformulating our results when 2 is a unit in terms of the easier to compute Hasse invariant.

Let M be a free rank m quadratic space over a commutative ring R in which 2 is a unit. We define $H(M)$ as the class of

$$\text{Cliff } (M \perp m\langle -\tfrac{1}{2}\rangle \perp \langle \tfrac{1}{2}, -(\tfrac{1}{2})^{m-1} \text{ Det } M\rangle) \text{ in } Br_2(R).$$

By direct calculation we see that $H(\langle a\rangle) = \text{cl(Cliff } \langle a, -\tfrac{1}{2}, \tfrac{1}{2}, -a\rangle) = 0$, and

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if N is another free quadratic space of rank n then

$$\begin{aligned} &H(M) + H(N) - H(M \perp N) \\ &= \mathrm{cl}(\mathrm{Cliff}\ \langle \tfrac{1}{2}, -(\tfrac{1}{2})^{m-1} \operatorname{Det} M, -(\tfrac{1}{2})^{n-1} \operatorname{Det} N, (\tfrac{1}{2})^{m+n-1} \operatorname{Det} M \cdot \operatorname{Det} N\rangle) \\ &= \mathrm{cl}\left(\left(\frac{(\frac{1}{2})^m \operatorname{Det} M, (\frac{1}{2})^n \operatorname{Det} N}{R}\right)\right) \end{aligned}$$

by Lemma 2.9. Using these two observations and the bilinearity of the quaternion algebra in $Br(R)$, we easily obtain by induction that

$$H(\langle 2a_1, \ldots, 2a_n\rangle) = \mathrm{cl}\left(\prod_{i<j} [(a_i, a_j)/R]\right) \text{ in } Br_2(R).$$

Definition 2.13. If $(M, B) = \langle b_1, \ldots, b_n \rangle$ is a bilinear space we define the Hasse invariant

$$\text{Hasse } M = \mathrm{cl}\left(\prod_{i<j} [(b_i, b_j)/R]\right)$$

in $Br(R)$. By the bilinearity and the above formula for $H(M)$ it is easy to check that

$$\text{Hasse } M = H(M) \cdot \mathrm{cl}\left(\left(\frac{2, (\operatorname{Det} M)^{n-1} \cdot 2^{n(n-1)/2}}{R}\right)\right) \text{ in } Br_2(R).$$

Since the map $Br(R) \to Br_2(R)$ is injective, this simultaneously shows that Hasse (M) is independent of the diagonalization of M and, if M and N have the same rank and determinant then Hasse M = Hasse N if and only if $H(M) = H(N)$.

THEOREM 2.14. *Let R be a semi-local ring in which 2 is a unit, and suppose (M, B) and (N, B) are two free spaces of the same rank. Then* Cliff M = Cliff N *in $Br_2(R)$ if and only if* Hasse M = Hasse N *and det M* = det N. *Consequently, Theorems* 2.10 *and* 2.12 *remain true if the class of the Clifford algebra is replaced by the Hasse invariant and determinant.*

Proof. By Lemma 2.8 (2), Cliff M = Cliff N in $Br_2(R)$ implies det M = det N, thus by the definition of $H(M)$, Cliff M = Cliff N if and only if $H(M) = H(N)$. The proof is then complete by the observation preceding the theorem.

3. The quotient $W(R)/I(R)^3$. In this section we will prove that $W(R)/I(R)^3$ determines $W(R)$, where $I(R)$ is the ideal of $W(R)$ generated by the classes of even dimensional forms, and R is a semi-local ring with 2 a unit. The first step will be to prove that for any semi-local ring R, $I(R)/I(R)^2$ is isomorphic to $U(R)/U(R)^2$, which we do by using the following somewhat more abstract considerations:

Let G be an abelian group of exponent 2. We will write $\mathbf{Z}[G]$ for its group ring and $\{g\}$ for the image in $\mathbf{Z}[G]$ of an element g of G. The group will be written

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multiplicatively with identity e. Let M_0 be the kernel of the ring homomorphism $\mathbf{Z}[G] \to \mathbf{Z} \to \mathbf{Z}/2\mathbf{Z}$ defined by sending each group element to 1 and reducing mod $2\mathbf{Z}$. That is, M_0 is the ideal of $\mathbf{Z}[G]$ consisting of elements of the form $\sum n_i\{g_i\}$ with $\sum n_i$ even, and is easily seen to be generated additively by the elements $\{e\} + \{g\}$. If g is an element of G we define $\mathrm{disc}_g : M_0 \to G$ by

$$\sum n_i\{g_i\} \to (\prod g_i^{n_i}) \cdot g^{\Sigma n_i/2}.$$

From the formula it is clear that disc_g is a group homomorphism using the additive structure on M_0 and the multiplicative structure on G. If K is an ideal of $\mathbf{Z}[G]$ contained in M_0 we will write $\overline{M_0}$ for the ideal M_0/K in $\mathbf{Z}[G]/K$, and use bars to indicate reduction modulo K.

LEMMA 3.1. *Let K be an ideal of $\mathbf{Z}[G]$ contained in M_0 with $\{e\} + \{g\}$ in K and* $\mathrm{disc}_g(K) = e$. *Then* disc_g *induces an isomorphism $\overline{M_0}/\overline{M_0}^2 \to G$ of groups.*

Proof. Since $\mathrm{disc}_g(K) = e$ by assumption, we at least know disc_g induces a well defined homomorphism $\overline{M_0} \to G$. To actually define the map from $\overline{M_0}/\overline{M_0}^2$ it will suffice to show $\mathrm{disc}_g(M_0^2) = e$. But M_0^2 is additively generated by elements of the form $(\{e\} + \{h\})(\{e\} + \{k\})$, hence it is enough to show disc_g vanishes on these. Now

$$\begin{aligned}\mathrm{disc}_g((\{e\} + \{h\})(\{e\} + \{k\})) &= \mathrm{disc}_g(\{e\} + \{h\} + \{k\} + \{hk\}) \\ &= e \cdot h \cdot k \cdot hk \cdot g^2 = e,\end{aligned}$$

since G has exponent 2.

We now define an inverse map $f : G \to \overline{M_0}/\overline{M_0}^2$ by $h \to \mathrm{cl}(\{e\} + \{hg\})$ where cl denotes the map $M_0 \to \overline{M_0} \to \overline{M_0}/\overline{M_0}^2$. Then by direct calculation $f(hk) - f(h) - f(k) = \mathrm{cl}(-\{e\} - \{hg\} - \{kg\} + \{hkg\})$. But $-\mathrm{cl}(\{e\}) = \mathrm{cl}\{g\}$ by hypothesis, hence $f(hk) - f(h) - f(k) = \mathrm{cl}\{g\} \cdot \mathrm{cl}(\{e\} - \{h\} - \{k\} + \{hk\}) = \mathrm{cl}\{g\} \cdot \mathrm{cl}(\{e\} - \{h\}) \cdot \mathrm{cl}(\{e\} - \{k\}) = 0$. Hence f is a group homomorphism.

If is now easy to check $\mathrm{disc}_g \circ f$ and $f \circ \mathrm{disc}_g$ are the identity (the latter need only be checked on generators).

We now specialize to the case where G is the group $U(R)/U(R)^2$ for a commutative ring R. An upper dot will be used to indicate reduction mod $U(R)^2$.

COROLLARY 3.2 (cf. [**15**, Korollar, p. 122]). *If R is a semi-local ring then there is an abelian group isomorphism from $I(R)/I(R)^2$ onto $U(R)/U(R)^2$. This isomorphism is given by*

$$[\langle a_1, \ldots, a_{2n}\rangle] + I(R)^2 \mapsto (\prod \dot{a}_i) \cdot (-\dot{1})^n \text{ with inverse } \dot{a} \mapsto [(1, -a)] + I(R)^2.$$

Proof. First suppose R is connected. Then if we let $G = U(R)/U(R)^2$, the ring homomorphism which takes $\{\dot{a}\} \mapsto [\langle a \rangle]$ is a surjection of $\mathbf{Z}[G]$ onto $W(R)$ [**13**, Theorem 1.16] whose kernel we denote by K. The ideal K is generated by $\{\dot{1}\} + \{-\dot{1}\}$ and the elements of the form $\sum_{i=1}^{4} (\{\dot{a}_i\} - \{\dot{b}_i\})$ with $\langle a_1, a_2, a_3, a_4\rangle \simeq \langle b_1, b_2, b_3, b_4\rangle$ [**13**, Corollary 1.17 (i)]. We now wish to apply Lemma 3.1 to $W(R) \simeq \mathbf{Z}[G]/K$, with $g = -\dot{1}$. Certainly $\{\dot{1}\} + \{g\} = \{\dot{1}\} +$

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$\{-\dot{1}\}$ is in K. Furthermore,

$$\mathrm{disc}_{-\dot{1}}\left(\sum_{i=1}^{4} (\{\dot{a}_i\} - \{\dot{b}_i\})\right) = \left(\prod_{i=1}^{4} \dot{a}_i\right) \cdot \left(\prod_{i=1}^{4} \dot{b}_i\right) = \dot{1},$$

the last equality following since isometric forms have the same determinant. Thus Lemma 3.1 applies and $\overline{M_0}/\overline{M_0^2}$ is isomorphic to G. The result now follows by composing the explicit isomorphism of Lemma 3.1 with the induced isomorphism of $M_0/M_0^2 \to I(R)/I(R)^2$ given by

$$\mathrm{cl}(\sum n_i\{g_i\}) \mapsto (\sum n_i[(g_i)]) + I(R)^2.$$

In the general case the result is obtained by decomposing R into a product of connected rings, say $R = \prod R_i$. Then $W(R) \simeq \prod (W(R_i))$ with $I(R)$ breaking up into $\prod I(R_i)$ in this natural correspondence [**13**, Lemma 1.9]. Then $I(R)/I(R)^2 \simeq \prod I(R_i)/I(R_i)^2 \simeq \prod U(R_i)/U(R_i)^2 \simeq U(R)/U(R)^2$.

Remark 3.3. The same proof also shows that for the Witt ring $W(R, J)$ of hermitian forms (see [**13**] for definitions), $I(R)/I(R)^2$ is isomorphic to $U(R)/N(R)$, as long as (R, J) can be factored into a product of pairs (R_i, J_i) to which [**13**, Corollary 1.17 (i)] can be applied as above, i.e. where R_i is a connected ring. However, the result fails in general for semi-local rings with involution. For let $R = A \times A$ where A is a connected semi-local ring, and then define J to be the involution which switches coordinates. Then by [**13**, Corollary 1.17 (ii)], $W(R, J) = 0$ which implies $I(R)/I(R)^2 = 0$, yet $U(R)/N(R) \simeq U(A)$.

LEMMA 3.4. *Let R be a connected semi-local ring where 2 is a unit Then* Cliff $(I^3(R)) = 1$ *in* $Br_2(R)$. *If (E_1, B_1) and (E_2, B_2) are bilinear spaces of rank $r \leqq 3$ with* $\mathrm{cl}(E_1) \equiv \mathrm{cl}(E_2) (\mathrm{mod}\ I^3(R))$, *then* $E_1 \simeq E_2$.

Proof. Since $I(R)$ is generated additively by the forms $\langle 1, a\rangle$, $I^3(R)$ must be generated additively by the forms $\langle 1, a\rangle\langle 1, b\rangle\langle 1, c\rangle$. But

$$\begin{aligned}\mathrm{Cliff}\,(\langle 1, a\rangle\langle 1, b\rangle\langle 1, c\rangle) &= \mathrm{Cliff}\,(\langle 1, a, b, ab\rangle)\cdot \mathrm{Cliff}\,(\langle c, ac, bc, abc\rangle)\\ &= \mathrm{cl}[((-a, -b)/R)]^2 = 1,\end{aligned}$$

by Lemma 2.9. Thus Cliff $(I^3(R)) = 1$ in $Br_2(R)$. Our last assertion now follows from Theorem 2.10.

LEMMA 3.5. *Let R and S be semi-local rings in which 2 is a unit. Further suppose that $R = \prod_{i=1}^m R_i$ and $S = \prod_{i=1}^n S_i$ are decompositions of R and S into connected rings. Then if $W(R)/I(R)^k \simeq W(S)/I(S)^k$ for any integer $k \geqq 0$, it follows that $m = n$ and $W(R_i)/I(R_i)^k \simeq W(S_i)/I(S_i)^k$* (after renumbering).

Proof. Clearly the ring decompositions induce an isomorphism

$$\prod_{i=1}^{m} W(R_i)/I(R_i)^k \simeq \prod_{i=1}^{n} W(S_i)/I(S_i)^k$$

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as in [**13**, Lemma 19]. The result now follows if we show that each $W(T)/I(T)^k$ is connected when T is a connected semi-local ring. If $k = 0$ this is immediate from [**13**, Example 3.11 and Corollary 3.10]. If $k > 0$, 2^k is in $I^k(T)$ which implies $2^k(W(T)/I^k(T)) = 0$, and the connectivity follows from [**13**, Theorem 2.9 (16) and Corollary 3.10].

From now on we will write disc for $\text{disc}_{-\dot{1}}$ and any of its induced maps.

THEOREM 3.6 (cf. [**8**, Theorem, p. 21]). *Let R and S be semi-local rings in which 2 is a unit. Then $W(R)/I(R)^3$ is isomorphic to $W(S)/I(S)^3$ if and only if $W(R)$ is isomorphic to $W(S)$.*

Proof. By Lemma 3.5, we may assume R and S are connected. Clearly, since $I(R)$ and $I(S)$ are the unique maximal ideals of $W(R)$ and $W(S)$ containing 2 [**13**, Lemma 2.13], $W(R)$ isomorphic to $W(S)$ implies $W(R)/I(R)^3$ is isomorphic to $W(S)/I(S)^3$.

Now suppose f is a ring isomorphism $W(R)/I(R)^3 \to W(S)/I(S)^3$. By the characterization of $I(R)$ and $I(S)$ mentioned above, we see that f induces an isomorphism of $I(R)/I(R)^3 \to I(S)/I(S)^3$ and consequently an isomorphism $I(R)/I(R)^2 \to I(S)/I(S)^2$. Then by Corollary 3.2 we get an isomorphism $f' : U(R)/U(R)^2 \to I(R)/I(R)^2 \to I(S)/I(S)^2 \to U(S)/U(S)^2$, with

$$\dot{a} \mapsto [\langle 1, -a \rangle] + I(R)^2 \mapsto f([\langle 1, -a \rangle] + I(R)^3) + I(S)^2$$
$$\mapsto \text{disc}(f([\langle 1, -a \rangle] + I(R)^3)).$$

If we write G_1 for $U(R)/U(R)^2$ and G_2 for $U(S)/U(S)^2$, then f induces a ring isomorphism of $\mathbf{Z}[G_1] \to \mathbf{Z}[G_2]$. Now, by [**13**, Corollary 1.17 (i)] this in turn induces a ring homomorphism $f^* : W(R) \to W(S)$, if we can show $\langle f'(\dot{1}), f'(-\dot{1}) \rangle \simeq \langle \dot{1}, -\dot{1} \rangle$ and $\langle f'(\dot{a}), f'(\dot{b}) \rangle \simeq \langle f'(\dot{c}), f'(\dot{d}) \rangle$ when $\langle a, b \rangle \simeq \langle c, d \rangle$. In fact, this actually proves f^* is an isomorphism, since the same argument applied to $(f')^{-1}$ produces $(f^*)^{-1}$.

Since f' is a group isomorphism $f'(\dot{1}) = \dot{1}$, hence to prove the first assertion it is enough to prove $f'(-\dot{1}) = -\dot{1}$. Now, $f'(-\dot{1}) = \text{disc}(f([\langle 1, 1 \rangle] + I(R)^3)) = \text{disc}(f([\langle 1 \rangle] + I(R)^3) + f([\langle 1 \rangle] + I(R)^3))$. But f is a ring isomorphism, hence $f([\langle 1 \rangle] + I(R)^3) = [\langle 1 \rangle] + I(S)^3$. Then $f'(-\dot{1}) = \text{disc}(([\langle 1 \rangle] + I(S)^3) + ([\langle 1 \rangle] + I(S)^3)) = \text{disc}([\langle 1, 1 \rangle] + I(S)^3) = -\dot{1}$.

Let x_1 and x_2 be in $I(R)$. Then $f(x_i + I(R)^3)$, $i = 1, 2$, are in $I(S)/I(S)^3$, hence we may write $f(x_i + I(R)^3) = y_i + I(S)^3$, where each y_i is in $I(S)$. Now if we write $\dot{c}_i = \text{disc}\, y_i$, we obtain $\langle 1, -c_i \rangle + I(S)^2 = y_i + I(S)^2$ since both sides have the same image under $\text{disc} : I(S)/I(S)^2 \to U(S)/U(S)^2$. Therefore we can write $y_i = [\langle 1, -c_i \rangle] + z_i$ for some z_i in $I(S)^2$. Now,

$$f(x_1x_2 + I(R)^3) = f(x_1 + I(R)^3)f(x_2 + I(R)^3) = (y_1 + I(S^3)(y_2 + I(S)^3)$$
$$= ([\langle 1, -c_1 \rangle] + z_1 + I(S)^3)([(1, -c_2)] + z_2 + I(S)^3).$$

But expanding this last product all terms except $[(1, -c_1)] \cdot [(1, -c_2)]$ lie in $I(S)^3$, thus $f(x_1x_2 + I(R)^3) = [\langle 1, -c_1 \rangle] \cdot [\langle 1, -c_2] + I(S)^3$.

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Now, we substitute $x_1 = [\langle 1, -a\rangle]$ and $x_2 = [\langle 1, -b\rangle]$ into this last formula. By the definition of f' we have $\dot{c}_1 = f'(\dot{a})$ and $\dot{c}_2 = f'(\dot{b})$, hence $f([\langle 1, -a\rangle]\cdot[\langle 1, -b\rangle] + I(R)^3) = [\langle 1, -f'(\dot{a})\rangle][\langle 1, -f'(\dot{b})\rangle] + I(S)^3$. But $\langle 1, -a\rangle\langle 1, -b\rangle = \langle 1, ab\rangle \perp \langle -a, -b\rangle$, thus $\langle a, b\rangle \simeq \langle c, d\rangle$ implies $\langle 1, -a\rangle\langle 1, -b\rangle \simeq \langle 1, -c\rangle\langle 1, -d\rangle$ since $\dot{a}\dot{b} = \dot{c}\dot{d}$. This then yields $[\langle 1, -f'(\dot{a})\rangle][\langle 1, -f'(\dot{b})\rangle] \equiv [\langle 1, -f'(\dot{c})\rangle][\langle 1, -f(\dot{d})\rangle] (\text{mod } I(S)^3)$ or

$$[\langle 1, f'(\dot{a})f'(\dot{b})\rangle] + [\langle -f'(\dot{a}), -f'(\dot{b})\rangle] \equiv [\langle 1, f'(\dot{c})f'(\dot{d})\rangle] + [\langle -f'(\dot{c}), -f'(\dot{d})\rangle] (\text{mod } I(S)^3).$$

We know f' is a group isomorphism hence $\dot{a}\dot{b} = \dot{c}\dot{d}$ implies $f'(\dot{a})f'(\dot{b}) = f'(\dot{c})f'(\dot{d})$, and the last congruence becomes $[\langle f'(\dot{a}), f'(\dot{b})\rangle] \equiv [\langle f'(\dot{c}), f'(\dot{d})\rangle] (\text{mod } I(S)^3)$. Now, an application of Lemma 3.4 yields $\langle f'(a), f'(b)\rangle \simeq \langle f'(c), f'(d)\rangle$ which is the required conclusion.

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