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A VARIATIONAL METHOD FOR THE CONSTRUCTION OF CONVERGENT ITERATIVE SEQUENCES

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Abstract

Convergent iterative sequences are constructed for the polynomials $f_m = z + z^m$, $m \geqslant 2$, with initial point the lemniscate $\{z: |f_m'(z)| \leqslant 1\}$. In the particular case $m = 2$ convergent iterative sequences are constructed also for $f_m^{-1}(z)$ with an arbitrary initial point. The method is based on a certain variational principle which allows reducing the problem to the well known situation of an analytic function mapping a simply connected domain into a proper subset of itself and possessing a fixed point in the domain.

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1. Introduction

The following easy consequence of Schwarz's lemma and the Riemann mapping theorem was applied in [3] for the construction of convergent iterative radicals.

LEMMA 1. *Let f be an analytic mapping of a simply connected region G of the complex plane into one of its proper subsets. If f has a fixed point $p \in G$, then for every $z_0 \in G$ the sequence $z_{n+1} = f(z_n)$, $n = 0, 1, \ldots,$ converges to p as $n \to \infty$.*

The restriction of $p \in G$ is essential for the proof of Lemma 1. However, in many applications it appears that p is on the boundary of G. We then apply Lemma 1 to a perturbed function f_ε which depends on a positive parameter ε and

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show that the perturbed sequences $\{w_n(\varepsilon)\}_{n=0}^{\infty}$, $w_n(\varepsilon) = f_\varepsilon(w_{n-1}(\varepsilon))$, converge to $\{z_n\}_{n=0}^{\infty}$ as $\varepsilon \to 0$ eventually uniformly in n (see Lemma 3). We apply this procedure to the construction of a convergent iterative polynomial sequence of arbitrary degree $m (m \geqslant 2)$, where $f = z + z^m$, and where G is a component of the lemniscate $\{z\colon |f'(z)| < 1\}$. In the particular case $m = 2$ the variational method is also applied to an analytic branch of f^{-1} in a suitable G to construct convergent sequences with an arbitrary z_0. Certain open problems are mentioned. For a general existence theorem of convergent cyclic sequences formed by f and f^{-1}, see [1].

2. Several lemmas

Throughout this note $f(z) = z + z^m$, $m \geqslant 2$, and $R = \{z\colon |f'(z)| < 1\}$; R consists of $m - 1$ simply connected components having a joint boundary point at the origin. Each component has two tangents at the origin which make an angle of $\alpha = \pi/(m-1)$. Two adjacent components are separated by a sector of aperture α. Let R_m be the component of R which is symmetric with respect to the ray $\arg z = \alpha$.

LEMMA 2. $f(z)\colon R_m \to R_m$.

PROOF. (a) We show first $f(z)\colon R \to R$. We have to show that for $z \in R$

$$\left|(z + z^m)^{m-1} + \frac{1}{m}\right| < \frac{1}{m},$$

or, setting $w = 1/m + z^{m-1}$, that

$$h(w) = \left(w - \frac{1}{m}\right)\left(w + 1 - \frac{1}{m}\right)^{m-1} + \frac{1}{m}$$

has modulus less than $1/m$ for $|w| < 1/m$. Now

$$h(w) = \sum_{k=2}^{m} \frac{(m-1)^{m-k-1}}{m^{m-k}}\left[(m-1)\binom{m-1}{k-1} - \binom{m-1}{k}\right]w^k + \frac{1}{m}\left[1 - \left(\frac{m-1}{m}\right)^{m-1}\right].$$

Denote the first sum by $h_1(w)$. $h_1(w)$ has positive coefficients, so that $|h_1(w)| < h_1(1/m)$ for $|w| < 1/m$. Now a direct calculation shows that

$$h_1\left(\frac{1}{m}\right) = \frac{(m-1)^{m-1}}{m^m}.$$

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Therefore,

$$(1)\qquad |h(w)| < h_1\left(\frac{1}{m}\right) + \frac{1}{m} - \frac{(m-1)^{m-1}}{m^m} = \frac{1}{m}.$$

(b) Now let $z \in R_m$. Then

$$\arg f(z) = \arg z + \arg(1 + z^{m-1}).$$

Also,

$$1 + z^{m-1} = \frac{m-1}{m} + \rho e^{i\phi}, \qquad 0 \leqslant \rho \leqslant \frac{1}{m},$$

so that $\theta = \arg(1 + z^{m-1})$ satisfies $|tg\,\theta| \leqslant 1/(m(m-2))^{1/2} < \pi/(m-1)$ for $m \geqslant 3$. Now the various components of R are separated by angles of $\pi/(m-1)$, so that $f(z) \in R_m$. For $m = 2$, we have $R_m = R$, so that part (a) of the proof is sufficient.

Remark. It is clear from the inequality $|h(w)| \leqslant 1/m - h_1(1/m) + |h_1(w)|$ that $|h(w)| = 1/m$ can occur only when $|h_1(w)| = h_1(1/m)$, or $w = 1/m$, if $m > 2$, that is, at $z = 0$. If $S_m = f(R_m)$, then $S_m \subset R_m$ and the boundaries of S_m and R_m intersect only at the origin. For $m = 2$ this can be verified directly. Indeed the above equality occurs for $w = \pm 1/m$, which values correspond to $z = 0$ and $z = -1$. Both of these points are mapped by f to the origin. One concludes that a sufficiently small translation of S_m in the direction of the axis of symmetry of R_m will still be a subset of R_m; that is, if

$$(2)\qquad f_\varepsilon = f + \varepsilon \exp\left(\frac{\pi i}{m-1}\right),$$

then $f_\varepsilon(\overline{R}_m) \subset R_m$ for all sufficiently small $\varepsilon > 0$. f_ε has a single fixed point $p_m = \varepsilon^{1/m}\exp(\pi i/(m-1))$ in R_m.

For a fixed $\varepsilon > 0$, let z_0, $w_0 \in R_m$, $z_n = f(z_{n-1})$, and $w_n = f_\varepsilon(w_{n-1})$, $n = 1, 2, \ldots$.

Lemma 3. *For all sufficiently small* $\varepsilon > 0$, *there is an integer* N *such that, for all* $n \geqslant N$,

$$(3)\qquad |w_{n+1} - z_{n+1}| \leqslant |w_n - z_n|\left(1 - \tfrac{1}{2}\varepsilon^{(m-1)/m}\right) + \varepsilon.$$

Proof. By Lemma 1, $w_n \to p_m$ as $n \to \infty$. Choose N such that

$$(4)\qquad w_n = p_m + t_n$$

with $|t_n| < \varepsilon$ for $n > N$. Also, since $z_n \in R_m$, we have $z_n^{m-1} = -1/m + r_m$, $|r_m| < 1/m$, and

$$(5)\qquad \frac{\pi}{2(m-1)} < \arg z_n < \frac{3\pi}{2(m-1)}.$$

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By (4), for $j \geqslant 1$, we have

(6) $$w_n^j = p_m^j + O(\varepsilon^{1+(j-1)/m})$$

as $\varepsilon \to 0$. Since $\arg p_m = \pi/(m-1)$, we have

$$\arg w_n^j = \frac{\pi j}{(m-1)} + O(\varepsilon^{1-1/m}).$$

By (5) and (6), for $1 \leqslant k \leqslant m-1$, we have

$$\frac{\pi}{2}\left[\frac{m+k-1}{m-1} + O(\varepsilon^{1-1/m})\right] \leqslant \arg(z_n^{m-k-1} w_n^k)$$
$$\leqslant \frac{3\pi}{2}\left[\frac{3(m-1)-k}{3(m-1)} + O(\varepsilon^{1-1/m})\right].$$

It follows that for sufficiently small $\varepsilon > 0$, $\arg(z_n^{m-k-1} w_n^k)$ and hence also

$$\arg\left(\sum_{k=1}^{m-2} z_n^{m-k-1} w_n^k\right) = \arg \zeta_m$$

satisfy

(7) $$\frac{\pi}{2} + \delta \leqslant \arg \zeta_m \leqslant \frac{3\pi}{2} - \delta, \qquad \delta > 0.$$

Now by (6),

(8)

$$|f(w_n) - f(z_n)| = |w_n - z_n|\,|1 + w_n^{m-1} + w_n^{m-2} z_n + \cdots + z_n^{m-1}|$$
$$\leqslant |w_n - z_n| \left\{\left|1 - \frac{1}{m} - \varepsilon^{(m-1)/m} + r_m + \zeta_m\right| + O(\varepsilon^{1+(m-2)/m})\right\}$$

and, for sufficiently small ε, by (4) aand (7), we have

(9) $$\left|\left(1 - \frac{1}{m} - \varepsilon^{(m-1)/m}\right) + \zeta_m\right| + |r_m| \leqslant \left|1 - \frac{1}{m} - \varepsilon^{(m-1)/m}\right| + |r_m|$$
$$\leqslant 1 - \varepsilon^{(m-1)/m}.$$

So by (8) and (9),

$$|f(w_n) - f(z_n)| \leqslant |w_n - z_n| \{(1 - \varepsilon^{(m-1)/m}) + O(\varepsilon^{1+(m-2)/m})\}$$
$$\leqslant |w_n - z_n| (1 - \tfrac{1}{2}\varepsilon^{(m-1)/m}).$$

The result now follows by the last inequality and the relation

$$|w_{n+1} - z_{n+1}| \leqslant |f(w_n) - f(z_n)| + \varepsilon.$$

We turn now our attention to the reverse sequence

(10) $$\zeta_n = f^{-1}(\zeta_{n-1}), \qquad n = 1, 2, \ldots,$$

where f^{-1} is one of the possible values of the multiple-valued inverse function of f. Wherever necessary the exact choice of f^{-1} will be indicated.

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LEMMA 4. *The sequence* $\{\zeta_n\}_{n=0}^{\infty}$ *is bounded for every choice of* ζ_0. *In particular*

(11) $$|\zeta_n| \leqslant \operatorname{Max}\left(2^{1/(m-1)}, |\zeta_0|\right).$$

PROOF. Write (10) in the form

(12) $$\zeta_n^m + \zeta_n = \zeta_{n-1}.$$

By Cauchy's theorem [2, p. 122], the zeros ζ_n of (12) are bounded in modulus by the only positive zero r_n of the polynomial

$$p(x) = x^m - x - r_{n-1} = 0, \qquad r_{n-1} = |\zeta_{n-1}|.$$

(a) If $r_{n-1} > 2^{1/(m-1)}$, then $x^m - x > x > r_{n-1}$ for $x > r_{n-1}$. Therefore $r_n \leqslant r_{n-1}$.

(b) If $r_{n-1} \leqslant 2^{1/(m-1)}$, then $r_n \leqslant 2^{1/(m-1)}$ because $p(2^{1/(m-1)}) \geqslant 0$. Thus if $|\zeta_0| \leqslant 2^{1/(m-1)}$, then $|\zeta_n| \leqslant 2^{1/(m-1)}$, and if $|\zeta_0| > 2^{1/(m-1)}$, then we have (11).

REMARK. Lemma 4 implies that for $K > 2^{1/(m-1)}$, if $|\zeta_0| < K$, then also $|\zeta_n| < K$ for all n.

Consider the particular case $m = 2$. Let $g(w) = f^{-1}(w) = -\frac{1}{2} + \sqrt{w + \frac{1}{4}}$, where we assume $\operatorname{Im} g(w) \geqslant 0$. If

$$G_0 = \{w: \operatorname{Im} w > 0\} \cap \{w: |w| < K\}, \qquad K > 2^{1/(m-1)},$$

then $g\colon G_0 \to G_0$. The function $g_\varepsilon = f + i\varepsilon$ satisfies also $g_\varepsilon\colon G_0 \to G_0$ for all sufficiently small $\varepsilon > 0$ and has the unique fixed point $w_\varepsilon = i\varepsilon + \sqrt{i\varepsilon}$ in G_0. We shall need the following lemma.

LEMMA 5. *For* $\varepsilon > 0$ *sufficiently small, there is an integer* N *such that, for all* $n \geqslant N$, *the sequences* $w_n = g_\varepsilon(z_{n-1})$, $z_0, w_0 \in G_0$, $z_n = g(z_{n-1})$ *satisfy*

$$|w_{n+1} - z_{n+1}| \leqslant M|w_n - z_n| + \varepsilon,$$

where $M = 2/(2 + \sqrt{\varepsilon})$.

PROOF. Let $\rho_n = |w_n - z_n|$. Then we have

(13) $$\rho_{n+1} = \left|\sqrt{w_n + \tfrac{1}{4}} - \sqrt{z_n + \tfrac{1}{4}} + i\varepsilon\right| \leqslant \frac{\rho_n}{|A|} + \varepsilon,$$

where $A = \sqrt{w_n + \frac{1}{4}} + \sqrt{z_n + \frac{1}{4}}$. Let $\sqrt{z_n + \frac{1}{4}} = a_n + i\alpha_n$, $\sqrt{w_n + \frac{1}{4}} = b_n + i\beta_n$. First we show that $a_n \geqslant \frac{1}{2}$ for all $n \geqslant n_1$. Indeed, if $z_n = x_n + iy_n$ then we have

(14) $$(2x_{n+1} + 1)y_{n+1} = y_n$$

and

(15) $$x_{n+1}^2 + x_{n+1} = x_n + y_{n+1}^2.$$

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Since $y_n \geqslant 0$, $x_{n+1} \geqslant -\frac{1}{2}$. By (15), $x_n \geqslant 0$ implies that $x_{n+1} \geqslant 0$. On the other hand, if $x_n \leqslant 0$ for all n, then by (14), y_n increases to a finite positive limit, say y_0. (14) then implies that $x_n \to 0$, so that $y_n \to 0$ by (15). Thus $y_0 = 0$, and we have a contradiction.

Secondly, we verify that

$$\sqrt{w_\varepsilon + \tfrac{1}{4}} = \tfrac{1}{2} + \sqrt{\tfrac{1}{2}\varepsilon} + i\sqrt{\tfrac{1}{2}\varepsilon}.$$

Since, by Lemma 1, $w_n \to w_\varepsilon$ as $n \to \infty$, it follows that, for $n \geqslant n_2$, we have

$$\sqrt{w_n + \tfrac{1}{4}} = \tfrac{1}{2} + \sqrt{\tfrac{1}{2}\varepsilon} + i\sqrt{\tfrac{1}{2}\varepsilon} + O(\varepsilon).$$

Thus, for $n \geqslant \operatorname{Max}(n_1, n_2)$, we have

$$\operatorname{Re} A \geqslant 1 + \sqrt{\tfrac{1}{2}\varepsilon} + O(\varepsilon) \geqslant 1 + \tfrac{1}{2}\sqrt{\varepsilon}$$

for all sufficiently small ε. By (13),

$$\rho_{n+1} \leqslant M\rho_n + \varepsilon, \tag{16}$$

where $M = 2/(2 + \sqrt{\varepsilon})$ and $n \geqslant N(\varepsilon)$. This completes the proof.

REMARK. Solving inequality (16), we obtain for $k = 1, 2, \ldots,$

$$\rho_{N+k} \leqslant M^k\rho_N + \varepsilon\frac{1 - M^k}{1 - M} \leqslant M^k\rho_N + 3\sqrt{\varepsilon}. \tag{17}$$

3. The main theorems

THEOREM 1. *For every $z_0 \in \overline{R}$ the sequence $z_{n+1} = f(z_n)$ converges to zero.*

PROOF. Assume $z_0 \in \overline{R}_m$. Let $\tau_n = |w_n - z_n|$. By Lemma 3, for $k = 1, 2, \ldots,$ for $N = N(\varepsilon)$, and for ε sufficiently small, we have $\tau_{N+k} \leqslant M_1^k\tau_N + \varepsilon(1 - M_1^k)/(1 - M_1)$, where $M_1 = 1 - \frac{1}{2}\varepsilon^{(m-1)/m}$. This leads to

$$\tau_{N+k} \leqslant M_1^k\tau_N + 2\varepsilon^{1/m}. \tag{18}$$

By Lemmas 1 and 2, $\{w_n\}$ is a convergent sequence, so that $|w_{N_1+k} - w_{N_1+l}| < \varepsilon$ for k, $l = 1, 2, \ldots,$ and for N_1 sufficiently large. Assuming $N_1 \geqslant N$, we now have, by (18),

$$|z_{N_1+k} - z_{N_1+l}| \leqslant \tau_{N_1+k} + \tau_{N_1+l} + \varepsilon \leqslant \tau_{N_1}\left(M_1^k + M_1^l\right) + 4\varepsilon^{1/m} + \varepsilon.$$

Therefore

$$\overline{\lim_{m,n\to\infty}} |z_m - z_n| = \overline{\lim_{k,l\to\infty}} |z_{N_1+k} - z_{N_1+l}| \leqslant 4\varepsilon^{1/m} + \varepsilon. \tag{19}$$

Inequality (19) implies that $\{z_n\}$ is a Cauchy sequence and thus converges to the origin as $n \to \infty$. This completes the proof of Theorem 1.

THEOREM 2. *For every fixed $w_0 \in C$ the sequence $z_{n+1} = g(z_n)$ tends to zero.*

PROOF. It is enough to prove Theorem 2 for $w_0 \in G_0$, since the argument carries over for the reflection of G_0 with respect to the real axis. For real w_0 the result then follows directly.

By Lemmas 4 and 1, and by (17), we have, for $n \geqslant N_2(\varepsilon)$ sufficiently large, and for $k, l = 1, 2, \ldots,$

$$\begin{aligned} |z_{N_2+k} - z_{N_2+l}| &\leqslant \rho_{N_2+k} + \rho_{N_2+l} + |w_{N_2+k} - w_{N_2+l}| \\ &\leqslant \rho_{N_2}(M^k + M^l) + 6\sqrt{\varepsilon} + \varepsilon. \end{aligned}$$

Hence

$$\overline{\lim_{m,n\to\infty}} |z_n - z_m| = \overline{\lim_{l,k\to\infty}} |z_{N_2+k} - z_{N_2+l}| \leqslant 6\sqrt{\varepsilon} + \varepsilon.$$

Therefore $\{z_n\}$ is a convergent sequence and thus tends to the origin.

COROLLARY. *If $m = 2$, then for every $z_0 \in \overline{R}$ there exists a sequence $\{z_n\}_{n=-\infty}^{\infty}$ such that $z_{n+1} = f(z_n)$, and $z_n \to 0$, $z_{-n} \to 0$ as $n \to \infty$. In addition, the sequences $\{z_n\}_{n=0}^{\infty}$ and $\{z_{-n}\}_{n=0}^{\infty}$ are essentially disjoint (except for a finite number of elements).*

PROOF. This is a direct result of Theorems 1 and 2, and of the relations $\operatorname{Re} z_{-n} \geqslant 0$ for $n \geqslant n_1$, and $\operatorname{Re} z_n < 0$ for $n \geqslant 0$, if $z_0 \neq 0$.

We conclude with two conjectures.

CONJECTURE 1. Let $f(z) = z + z^m$, $m \geqslant 2$. There exists a determination of $f^{-1}(z)$ such that for every $z_0 \in C$ the sequence $z_n = f^{-1}(z_{n-1})$ tends to zero as $n \to \infty$.

If this conjecture is true, then by the previous results it would be possible to construct cyclic sequences for a polynomial of arbitrary degree $m \geqslant 2$.

CONJECTURE 2. Let $f(z) = z + a_2 z^2 + \cdots + a_m z^m$ be of degree $m \geqslant 2$, and assume that $a_k \geqslant 0$ for all k. Then for every z_0 such that $|f'(z_0)| \leqslant 1$, the sequence $z_{n+1} = f(z_n)$ converges.

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