THE TEMPERATURE DISTRIBUTION IN A SEMI-INFINITE SOLID WHOSE SURFACE IS MAINTAINED AT AN ARBITRARY TEMPERATURE

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1. The problem discussed is the temperature distribution in a semi-infinite solid, initially at zero, when the surface is suddenly raised to and maintained at a temperature which is an arbitrary function of the distance from some line which lies in that surface.

In the rectangular system of axes (O, xyz) the surface of the solid is the plane $x = 0$ and Ox the inward drawn normal. Oz is the given line in this surface. The surface temperature is a function of y only and may be denoted by $F(y)$. We take $V(x, y, t)$ to be the temperature of the medium, k the thermal conductivity of the medium, ρ the density of the medium, s the specific heat of the medium; $h^2 = \dfrac{k}{\rho . s}$ is the diffusivity of the medium and $F(y)$ is surface temperature. The equation of conduction may be written

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=\frac{1}{h^2}\frac{\partial V}{\partial t}. \qquad \text{.............................(1)}$$

The initial and boundary conditions are:

$$V(x, y, 0) = 0, \qquad (x, y) \notin S \qquad \text{.............................(2)}$$

$$V(0, y, t) = F(y), \qquad t>0 \qquad \text{.................................(3)}$$

$$V(x, y, t) \to 0 \qquad \text{as } x \to \infty,\ t>0. \qquad \text{...................(4)}$$

We also assume that as $t \to \infty$, V tends to a steady finite value at all points.

We solve the set of equations (1)-(4) by means of the operational calculus based on the Laplace transform (**1**)

$$W(x, y, p) = \mathscr{L}\{V(x, y, t);\, p. \qquad \text{..........................(5)}$$

Since $V(x, y, 0) = 0$ (cf. (2) above) we have

$$\mathscr{L}\left\{\Delta^2 V-\frac{1}{h^2}\frac{\partial V}{\partial t};\, p\right\}=(\nabla^2-ph^{-2})W$$

so that equation (1) is equivalent to the Helmholtz equation

$$h^2\nabla^2 W-pW=0. \qquad \text{................................(6)}$$

Also (3) gives the boundary condition

$$W(0, y, p) = p^{-1}F(y) \qquad \text{.....(7a)}$$

and (4) the condition

$$W(x, y, p) \to 0 \text{ as } x \to \infty. \qquad \text{.....(7b)}$$

Once we have solved the set of equations (6), (7) we recover the form of V by means of the

$$V(x, y, t) = \mathscr{L}^{-1}\{W(x, y, p); t\}$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}W(x, y, p)dp \qquad \text{.....(8)}$$

where the path of integration in the contour integral (the standard contour C), goes from $(c-i\infty)$ to $(c+i\infty)$, where c is real and greater than 0, and all the singularities of the integrand are to the left of the path.

2. We now consider the case of a sinusoidal surface temperature

$$F(y) = \cos ny.$$

From (7a) we see that $W(0, y, p) = p^{-1}\cos ny$ and this suggests that we assume a solution of (6) of the form

$$W(x, y, p) = \phi(x, p)\cos ny.$$

It is easily shown that

$$\phi(x, p) = p^{-1}\exp[-x\sqrt{(n^2+p/h^2)}].$$

Hence in this case we find that

$$V(x, y, t) = \cos ny \,.\, \mathscr{L}^{-1}\{p^{-1}\exp[-x\sqrt{(n^2+p/h^2)}]; t\} \qquad \text{.....(9)}$$

so that we have to evaluate the integral

$$\Psi(x, t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} p^{-1}\exp[pt - x\sqrt{(n^2+p/h^2)}]dp. \qquad \text{.....(10)}$$

It is now necessary to transform this contour integral into a real integral. The singularities of $p^{-1}\exp[pt-x\sqrt{(n^2+p/h^2)}]$ are a simple pole at $p = 0$ and a branch point at $p = -n^2h^2$. Fig. 1 shows a contour equivalent to the standard contour C, consisting of the infinite quarter circle from $-i\infty$ to $-\infty$, the real negative axis from $-\infty$ to 0 indented at $x = -n^2h^2$ and at $x = 0$ to pass below the singularities and the similar contour in the upper part of the plane. The contour is bounded on the right hand side by a straight line parallel to the imaginary axis.

Since the contour does not include any singularities, the integral round the closed curve is zero. Let I_1, I_2, etc. denote the values of the integral over various parts of the contour as shown in Fig. 1. Then the sum of these separate integrals is zero.

The integrals I_2, I_2^1 over the infinite quadrants vanish by Jordan's lemma (**1**), since $p^{-1}[\exp -x\sqrt{(n^2+p/h^2)}]$ tends uniformly to zero over the whole semi-circle. The integral, I_6, round the singularity at the origin contributes the residue at this point. Allowing for the direction in which the integral is taken round the contour.

$$I_6 = -2\pi i \exp(-nx).$$

The integrals, I_4, I_4^1 round the infinitesimal semi-circles about $p = -n^2h^2$ vanish since the function is finite there. The two integrals, I_5, I_5^1 from $-n^2h^2$ to the origin are equal and opposite since the origin is not a branch point and the function is unchanged by a circuit round it. There remain the integrals I_3, I_3^1 between $-\infty$ and $-n^2h^2$.

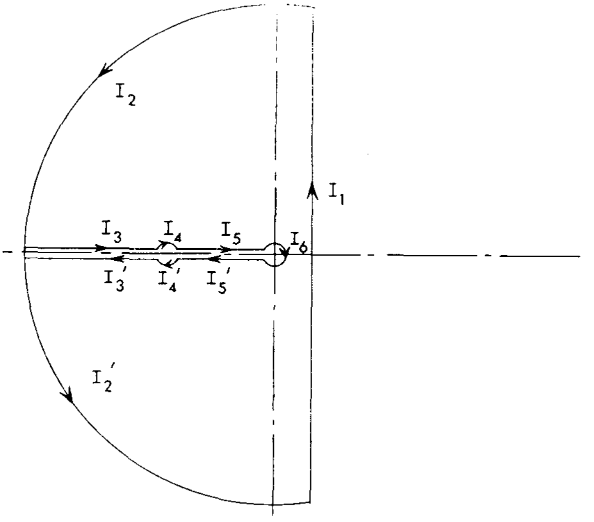

FIG. 1.

When the substitutions $p = z\exp(i\pi)$, $p = z\exp(-i.\pi)$ are made in I_3 and I_3^1 respectively and allowance is made for the change of phase of $\sqrt{(n^2+p/h^2)}$ in the two integrals, then using the radical sign to denote the positive square root, we can easily show that

$$I_3 + I_3^1 = 2i\int_{n^2h^2}^{\infty} z^{-1}e^{-zt}\sin[x\sqrt{(z/h^2-n^2)}]dz.$$

Thus since the sum of the integrals round the closed path is zero, it follows that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} p^{-1}\exp[pt - x\sqrt{(n^2+p/h^2)}]dp$$
$$= \exp(-nx) - \frac{1}{\pi}\int_{n^2h^2}^{\infty} z^{-1}e^{-zt}\sin[x\sqrt{(z/h^2-n^2)}]dz. \quad \text{......(11)}$$

Using (11) in (10), it follows that

$$\Psi(x, t) = \exp(-nx) - \frac{1}{\pi}\int_{n^2h^2}^{\infty} z^{-1}e^{-zt}\sin[x\sqrt{(z/h^2-n^2)}]dz. \quad \text{...(12)}$$

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It may be noted from this equation that as t tends to infinity, the integral tends to zero, and V tends to $\exp(-nx)\cos ny$ which thus represents the final steady state.

The integral $I(t)=\frac{1}{\pi}\int_{n^2h^2}^{\infty} z^{-1}e^{-zt}\sin[x\sqrt{(z/h^2-n^2)}]dz$ may be transformed further. Substituting $\mu^2=\frac{z}{h^2}-n^2$, we get

$$I(t)=\frac{2}{\pi}\int_0^{\infty} e^{-h^2(\mu^2+n^2)t}\sin(\mu x)\frac{\mu d\mu}{\mu^2+n^2}. \qquad \text{.................(13)}$$

But it can easily be shown that $\exp(-nx)=\frac{2}{\pi}\int_0^{\infty}\mu\frac{\sin\mu x}{\mu^2+n^2}d\mu$ so that we have

$$\psi(x,t)=\frac{2}{\pi}\int_0^{\infty}[1-e^{-(\mu^2+n^2)h^2t}]\sin(\mu x)\frac{\mu d\mu}{\mu^2+n^2}. \qquad \text{...........(14)}$$

Now $$\frac{1-\exp[-h^2(\mu^2+n^2)t]}{\mu^2+n^2}=h^2\int_0^t\exp[-h^2(\mu^2+n^2)t]dt$$

so that on substitution in (14) we get

$$\psi(x,t)=\frac{2h^2}{\pi^2}\int_0^{\infty}\int_0^t e^{-h^2(\mu^2+n^2)t}\sin(\mu x)\mu d\mu dt.$$

Inverting the order of integration, we have

$$\psi(x,t)=\frac{2h^2}{\pi}\int_0^t dt\int_0^{\infty} e^{-h^2(\mu^2+n^2)t}\sin(\mu x)\mu d\mu$$

$$=-\frac{2h^2}{\pi}\int_0^t dt\frac{\partial}{\partial x}\int_0^{\infty} e^{-h^2(\mu^2+n^2)t}\cos(\mu x)d\mu \qquad \text{...........(15)}$$

see (2). Now since

$$\int_0^{\infty}\exp(-a^2\mu^2)\cos(b\mu)d\mu=\frac{\sqrt{\pi}}{2a}\exp\left(-\frac{b^2}{4a^2}\right) \qquad \text{...........(16)}$$

it may be seen by substituting this value in (15) that

$$\Psi(x,t)=-\frac{2h^2}{\pi}\int_0^t dt\frac{\partial}{\partial x}\left\{\frac{\sqrt{\pi}}{2h\sqrt{t}}\exp[-h^2n^2t-x^2/(4h^2t)]\right\}$$

so that

$$\psi(x,t)=\frac{1}{2h\sqrt{\pi}}\int_0^t t^{-3/2}xe^{-h^2n^2t-x^2/(4h^2t)}dt. \qquad \text{..............(17)}$$

Writing u for $\frac{x}{2ht^{\frac{1}{2}}}$ we find that (17) can be written in the form

$$\psi(x,t)=\frac{2}{\sqrt{\pi}}\int_{x/(2ht^{\frac{1}{2}})}^{\infty} e^{-u^2-\frac{1}{4}x^2n^2u^{-2}}du. \qquad \text{.................(18)}$$

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Since the steady state must be derivable from (18) by letting $t \to \infty$, we see from (12) that in the steady state

$$\psi(x, t) = \exp(-nx) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2 - \frac{1}{4}n^2x^2u^{-2}} du. \qquad \text{.........(19)}$$

3. We now consider the case of a surface distribution consisting of an uniform temperature H over the range $-l/2 < y < l/2$ and vanishing outside this range. By Fourier's integral theorem we see that $F(y)$ can be represented in the form

$$F(y) = \frac{2H}{\pi} \int_0^\infty \frac{\cos ny \sin(\frac{1}{2}nl) dn}{n}. \qquad \text{.........(20)}$$

The solution of this problem will be found by substituting for $\cos ny$ in the integral in (20), the solution corresponding to a surface temperature $\cos ny$. Thus using (18)

$$V = \frac{4H}{\pi^{3/2}} \int_0^\infty n^{-1} \sin(\tfrac{1}{2}nl) \cos(ny) dn \int_{x/(2ht^{1/2})}^\infty e^{-u^2 - \frac{1}{4}n^2x^2u^{-2}} du. \qquad \text{...(21)}$$

Inverting the order of integration we find that

$$V = \frac{2H}{\pi^{3/2}} \int_{x/(2ht^{1/2})}^\infty du \int_0^\infty n^{-1} e^{-u^2 - \frac{1}{4}n^2x^2u^{-2}} [\sin\{(\tfrac{1}{2}l + y)n\} + \sin\{(\tfrac{1}{2}l - y)n\}dn. \qquad \text{...(22)}$$

Now by integrating (16) with respect to b from 0 to b, it may be seen that

$$\int_0^\infty \mu^{-1} \sin(b\mu) e^{-a^2\mu^2} d\mu = \frac{\sqrt{\pi}}{2a} \int_0^b e^{-\frac{1}{4}b^2a^{-2}} db$$

using this result in (22), we find that

$$V = \frac{2H}{\pi x} \int_{x/(2ht^{1/2})}^\infty u e^{-u^2} du \int_{-\frac{1}{2}l + y}^{\frac{1}{2}l + y} e^{-\lambda^2u^2x^{-2}} d\lambda. \qquad \text{.........(23)}$$

Again inverting the order of integration we find that

$$V = \frac{Hx}{\pi} e^{-\frac{1}{4}x^2h^{-2}t^{-1}} \int_{-\frac{1}{2}l + y}^{\frac{1}{2}l + y} (x^2 + \lambda^2)^{-1} e^{-\frac{1}{4}\lambda^2h^{-2}t^{-1}} d\lambda. \qquad \text{.........(24)}$$

The steady state may be obtained from this equation by making $t \to \infty$ when it follows that

$$V(x, y, \infty) = \frac{Hx}{\pi} \int_{-\frac{1}{2}l + y}^{\frac{1}{2}l + y} (x^2 + \lambda^2)^{-1} d\lambda$$

$$= \frac{H}{\pi} \left\{ \tan^{-1}\left(\frac{y + \frac{1}{2}l}{x}\right) - \tan^{-1}\left(\frac{y - \frac{1}{2}l}{x}\right) \right\}, \qquad \text{.........(25)}$$

where $-\frac{1}{2}\pi \leq \tan^{-1}(\lambda/x) \leq \frac{1}{2}\pi$.

4. Next we discuss a surface distribution consisting of a line source of given strength. The solution of this problem may be obtained from (24) by making $l \to 0$ and $H \to \infty$ so that the product Hl remains constant and equal to θ. In the limit (24) takes the form

$$V = \frac{x\theta}{\pi(x^2+y^2)} \exp\left[-(x^2+y^2)/(4h^2t)\right] \qquad \text{(26)}$$

and the steady state

$$V = \frac{x\theta}{\pi(x^2+y^2)}. \qquad \text{(27)}$$

5. Finally we consider the case of an arbitrary surface temperature distribution, $F(y)$. By the superposition of line sources of intensity $F(y_0)dy_0$ situated at $y = y_0$, we can obtain the temperature distribution due to $F(y)$. It can be easily shown that the temperature distribution is

$$V = \frac{x}{\pi}\int_{-\infty}^{\infty} \exp\left\{-\left[x^2+(y-y_0)^2\right]/(4h^2t)\right\} \frac{F(y_0)dy_0}{x^2+(y-y_0)^2}. \qquad \text{(28)}$$

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