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ON LOOPS WHICH HAVE DIHEDRAL 2-GROUPS AS INNER MAPPING GROUPS

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In this paper we consider the situation that a group G has a subgroup H which is a dihedral 2-group and with connected transversals A and B in G. We show that G is then solvable and moreover, if G is generated by the set $A \cup B$, then H is subnormal in G. We apply these results to loop theory and it follows that if the inner mapping group of a loop Q is a dihedral 2-group then Q is centrally nilpotent.

1. INTRODUCTION

Quasigroups, loops and corresponding multiplication groups were first studied by Albert [**1, 2**] and Bruck [**3**]. In 1946 Bruck showed that if a finite loop Q is centrally nilpotent then $M(Q)$ (the multiplication group of Q) is solvable. On the other hand, if $M(Q)$ is nilpotent then Q is centrally nilpotent. When we study multiplication groups of loops, a central role is played by one special subgroup. This subgroup $I(Q)$ of $M(Q)$ is called the inner mapping group of the loop and it is the analogue for loops of the inner automorphism group of a group. Therefore it is reasonable to expect that $I(Q)$ reflects at least some properties of the loop Q. The relation between the properties of Q, $I(Q)$ and $M(Q)$ has been studied very thoroughly by Bruck [**4**, Section VIII] in the case that Q is a commutative Moufang loop. The general case (without any identities restricting the structure of the loop) was investigated by Kepka, Niemenmaa and Rosenberger [**7, 11, 12, 13, 14**]. They managed to show that if Q is a loop such that the inner mapping group is cyclic, then Q is an Abelian group [**7**, Theorem 2.4]. It also turned out that if Q is a finite loop such that the inner mapping group $I(Q)$ is Abelian then Q is centrally nilpotent [**13**, Corollary 6.4]. If the inner mapping group is not Abelian then, of course, things get more complicated. From [**10**, Theorem 3.4] it follows that if $I(Q)$ is non-abelian of order six then $M(Q)$ is solvable. However, in this case Q is not necessarily centrally nilpotent. The purpose of this paper is to investigate the situation that Q is a loop and $I(Q)$ is a dihedral 2-group. We show that then $M(Q)$ is solvable and Q is a centrally nilpotent loop.

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Now the link between multiplication groups of loops and certain transversal conditions was given by Kepka and Niemenmaa [**11**, Theorem 4.1]. It has turned out that many properties of loops can be reduced to the properties of so called connected transversals in the multiplication group. Therefore also in this paper we first list some basic properties of connected transversals in Section 2. In Section 3 we prove that if a finite group G has a subgroup H such that H is a dihedral 2-group and there exist H-connected transversals A and B in G, then G is solvable. In the proof we use Thompson's [**15**] result on a special class of non-solvable groups and we are led to investigate the existence of connected transversals in the groups $PSL(2,q)$. By using a neat little argument on centralisers we are able to show that also in the case of infinite G the above result is true. In Section 4 we assume that in addition G is generated by $A \cup B$ and we can show that then H is subnormal in G. As an application of this result we immediately have our main theorem about the central nilpotency of loops.

Our notation in group theory is standard as in [**6**]. Basic facts about loops, their multiplication groups and central nilpotency can be found in [**3**, **4**, **9**, **11**, **12**]. There are also other approaches to these problems. The relation between finite simple Moufang loops and finite simple groups is investigated in [**8**] and [**5**, **16**] contain interesting material about Coxeter groups and p-groups.

2. Preliminaries

The basic tool for our study is the concept of connected transversals. If G is a group, $H \leqslant G$ and A and B are two left transversals to H in G such that the commutator group $[A,B]$ is a subgroup of H then we say that A and B are H-connected transversals in G. By H_G we denote the core of H in G (the largest normal subgroup of G contained in H). The relation between multiplication groups of loops and connected transversals is given by

Theorem 2.1. *A group G is isomorphic to the multiplication group of a loop if and only if there exist a subgroup H satisfying $H_G = 1$ and H-connected transversals A and B such that $G = \langle A, B\rangle$.*

For the proof of the theorem, see [**11**, Theorem 4.1]. In the following lemmas we assume that A and B are H-connected transversals in G.

Lemma 2.2. *Now A^g and B^g are left and right transversals to H in G for every $g \in G$. If $H_G = 1$, then $1 \in A \cap B$.*

In this paper we often have the situation that H is a Sylow 2-subgroup of G and $H_G = 1$. By Lemma 2.2, this means that the transversals A and B can not contain any 2-elements.

Lemma 2.3. *If $C \subseteq A \cup B$ and $K = \langle H, C\rangle$, then $C \subseteq K_G$.*

Lemma 2.4. *If $H_G = 1$, then $N_G(H) = H \times Z(G)$.*

Lemma 2.5. *Let H be a cyclic subgroup of a group G. Then $G' \leqslant H$ if and only if there exists a pair A, B of H-connected transversals in G such that $G = \langle A, B\rangle$.*

Lemma 2.6. *If H is cyclic then G is solvable and $G^{(3)} = 1$.*

For the proofs, see [**11**, Lemma 2.1, Lemma 2.2, Proposition 2.7] and [**7**, Theorem 2.2 and Corollary 2.3].

In this paper we also need a knowledge of the structure of dihedral 2-groups and of the structure of the projective special linear groups PSL(2,q). All this material can easily be found in [**6**].

3. Solvability

In this section we investigate the case where $H \leqslant G$ is a dihedral 2-group. We start by considering the situation in projective special linear groups.

Lemma 3.1. *Let $G = PSL(2,q)$ where $q = 2^n \pm 1 \geqslant 17$ is a prime number. If H is a maximal subgroup of G and H is a dihedral 2-group of order $q \pm 1$ then there exist no H-connected transversals in G.*

Proof: Assume first that $q = 2^n - 1$ and $|H| = q + 1$. If A and B are H-connected transversals in G, then $|A| = |B| = q(q-1)/2$. Now $a^{-1}b^{-1}ab \in H$ and thus $a^b \in aH$ for every $a \in A$ and $b \in B$. If $a^b = a^c$ then $1 \neq bc^{-1} \in C_G(a)$. Thus we place $q(q-1)/2 - (q+1)$ elements in the set $D = C_G(a) - \{1\}$. By Lemma 2.2, a is not a 2 - element in G, hence the set D has at most $q - 1$ elements. Thus there exists $c \in D$ such that $c = b_1{b_2}^{-1} = b_3{b_4}^{-1}$. Now $b_1 = cb_2$ and $b_3 = cb_4$. If $a \in A$, then $[a, cb_i] = [a, b_i][a, c]^{b_i} \in H$, hence $[a, c] \in H^{{b_i}^{-1}} (i = 2, 4)$. But then $[a, c] \in H^{{b_2}^{-1}} \cap H^{{b_4}^{-1}} = F$. If 8 divides $|F|$ then F has a cyclic subgroup K whose order is divisible by four. Since H and its conjugates are maximal in G we conclude that K is normal in G. Thus $|F|$ is at most four, since G is simple. Now $[a, c] \in F$ for every $a \in A$ and this means that we put $q(q-1)/2$ commutators in the four places in F. It follows that there exists $h \in F$ such that $h = [a_1, c] = \ldots = [a_t, c]$ where $t \geqslant q(q-1)/8$. If $[a_1, c] = [a_2, c]$ then $a_1{a_2}^{-1} \in C_G(c)$, thus we place $t - 1$ elements in the set $C_G(c) - 1$ which has at most $q - 1$ elements. Now $q \geqslant 17$ and then certainly $a_1{a_j}^{-1} = a_1{a_i}^{-1}$ for some $a_j \neq a_i$ and this leads to a contradiction.

In the case that $q = 2^n + 1$, $|H| = q - 1$ and $|A| = |B| = q(q+1)/2$ we can proceed in a similar way. ∎

Now we prove

Theorem 3.2. *Let G be a finite group and $H \leqslant G$ a dihedral 2-group. If there exist H-connected transversals in G then G is solvable.*

PROOF: Let G be a minimal counterexample. If $H_G > 1$, then H/H_G is cyclic or dihedral, hence G/H_G is solvable by Lemma 2.6 or by induction. Thus we may assume that $H_G = 1$.

If H is not a maximal subgroup of G then G has a proper subgroup K such that $1 < H < K < G$. By Lemma 2.3, $K_G > 1$ and since HK_G/K_G is cyclic or dihedral we conclude that G/K_G is solvable. By induction, K is solvable and then, of course, G is solvable. Thus we may assume that H is a maximal subgroup of G (and a Sylow 2-subgroup of G).

If G is simple then following Thompson [15, p.461–462] we know that $G \cong PSL(2,q)$, where $q = |H| \pm 1 \geqslant 17$ is a prime. One should observe here that the groups PSL(2,7) and PSL(2,9) can also be found in Thompson's classification but they do not have maximal subgroups which are 2-groups. By Lemma 3.1, G does not have H-connected transversals. Thus we may also assume that G is not simple.

Following again Thompson [15] we conclude that there exists a normal subgroup N of G such that $G = NH$ and $[G:N] = 2$, $N \cap H$ is a dihedral 2-group and a Sylow 2-subgroup of N and $N \cong PSL(2,q)$, where $q = |N \cap H| \pm 1 \geqslant 7$is a prime number or $q = 9$.

Now we denote by A and B the H-connected transversals in G. If $A \cup B \subseteq N$ then A and B are $N \cap H$-connected tranversals in N. By induction, N is solvable. But this is not possible and therefore there exists $a \in A - N$ (or $B - N$). Now we divide the rest of the proof into three parts.

(1) Let $q \geqslant 17$ be a prime. If $a \in A - N$, then $a^2 \in N$ and $|a| = 2q$ or $|a|$ divides $q+1$ or $q-1$. Clearly, $C_G(a)$ can have at most $2q$ elements. Now we employ the same method and notation as in the proof of Lemma 3.1. We end up with the situation that $t \geqslant q(q-1)/8$ and we place $t-1$ elements of the form $a_1 a_i{}^{-1}$ in the group $C_G(c)$ (which has at most $2q$ elements). If $|C_G(c)| = 2q$ then $C_G(c)$ contains an involution u and we thus place $t-1$ elements in the set $C_G(c) - \{1,u\}$. If $q \geqslant 17$ then $q(q-1)/8 - 1 > 2q - 2$ and we are done.

(2) Let $N \cong PSL(2,7)$. Now $|H| = 16$ and then clearly $|A| = |B| = 21$. Now $a \in A - N$ implies that $a^2 \in N$, hence $|a^2| = 2,3,4$ or 7. By Lemma 2.2, $|a| = 4$ or 8 is not possible. If $|a| = 6$, then $|C_G(a)| = 6$ and we can employ the method from the proof of Lemma 3.1. We just have to observe that here $a^{-1}b^{-1}ab \in N \cap H$ and $|N \cap H| = 8$. After this, routine calculations as in Lemma 3.1 lead us to a contradiction. If $|a| = 14$, then a^7 is an involution and thus it belongs to a conjugate H_1 of H in G. Since $a^7 \in {H_1}^{a^k}$ (k=1,...,13) we have that $C_G(a^7) \geqslant \langle a, t_k \rangle$ where the elements t_k are involutions and $\langle t_k \rangle = Z\left({H_1}^{a^k}\right) < N$. Thus $T = C_N(a^7) \geqslant \langle a^2, t_k \rangle$, hence 14 divides T. From the information about the maximal subgroups of PSL(2,7) we conclude that

$T = N$, hence $C_N(a^7) = N$. Now $G = N\langle a^7\rangle$ and thus $a^7 \in Z(G)$. By induction, $G/Z(G)$ is solvable and then also G is solvable, a contradiction.

(3) Let $N \cong PSL(2,9)$. Now $|H| = 16$ and $|A| = |B| = 45$. If $a \in A - N$, then $a^2 \in N$ and $|a^2| = 3$ or 5 or $|a^2|$ divides 4. By Lemma 2.2, $|a| = 4$ or 8 is not possible. If $|a| = 6$, then $|C_G(a)| = 6$ or 18. We first assume that $|C_G(a)| = 6$. Once again we use the method and notation from Lemma 3.1. (Here one should observe that $a^{-1}b^{-1}ab \in N \cap H$ and $|N \cap H| = 8$.) Since $(45 - 8)/5 > 7$ it follows that the commutators $[a, c]$ belong to a subgroup which has at most two elements. Thus $t - 1 \geqslant 22$ and we put at least 22 elements in the set $C_G(c) - \{1\}$. This set has at most 17 elements and we have our contradiction. Let $|C_G(a)| = 18$. Now a^3 is an involution and $C_N(a^3)$ contains a group of order 9 and an elementary Abelian group of order four. (Compare with the situation of the element a^7 in PSL(2,7).) But this means that $C_N(a^3) = N$ and we conclude that $a^3 \in Z(G)$. By induction $G/Z(G)$ is solvable and therefore G is solvable, a contradiction. If $|a| = 10$, then we again use calculations similar to Lemma 3.1. Now $|C_G(c)|$ is at most 10 and we easily get the final contradiction. The proof is complete. □

Next we prove that our theorem also holds when G is infinite.

Theorem 3.3. *Let $H < G$, where H is a dihedral 2-group and assume that there exists a pair A, B of H-connected transversals. Then G is solvable.*

Proof: If $H_G > 1$, then H/H_G is cyclic or dihedral. In the former case G/H_G is solvable by Lemma 2.6 and we are done. In the latter case we can investigate the groups G/H_G and H/H_G and thus we can assume that H_G is trivial.

First assume that $G = \langle A, B\rangle$. Let a be a fixed element of A and h a fixed element of H and write $F(a,h) = \{b \in B : a^{-1}b^{-1}ab = h\}$. If b and c are elements of $F(a,h)$ then $bc^{-1} \in C_G(a)$ and $b \in C_G(a)c$. Thus $F(a,h) \subseteq C_G(a)b_h$ where b_h is a fixed element from $F(a,h)$ and $B = \bigcup F(a,h)$, where h goes through all the elements of H. Now $G = BH \subseteq C_G(a)\{b_h : h \in H\}H$ and thus $[G : C_G(a)] \leqslant |H|^2$. (We are indebted to Tomas Kepka who pointed out this way of using centralisers.) Since H is a finite subgroup of $\langle A, B\rangle$ it follows that $[G : C_G(H)]$ is finite, whence $[G : N_G(H)]$ is finite. Now $N_G(H) = H \times Z(G)$ and thus $[G : Z(G)]$ is finite. Clearly $HZ(G)/Z(G)$ is a dihedral 2-group and from Theorem 3.2 it follows that $G/Z(G)$ is solvable, hence G is solvable.

Then let $K = \langle A, B\rangle$ be a proper subgroup of G. Now A and B are $K \cap H$-connected transversals in K. Thus K is a solvable group by the first part of the proof. Since $[G : K]$ is finite we have a normal subgroup $N \leqslant K$ such that $[G : N]$ is finite. Again by Theorem 3.2, G/N is solvable and therefore G is solvable. The proof is complete. □

By using Theorems 2.1 and 3.3 we easily get the following

COROLLARY 3.4. *If Q is a loop such that the inner mapping group $I(Q)$ is a dihedral 2-group then $M(Q)$ is solvable.*

4. Central nilpotency of loops

In this section we deal with the central nilpotency of a loop. For this concept and related results we advise the reader to consult [**3, 9, 12, 13**]. However, we first prove a purely group theoretical result which binds together transversals, commutators and subnormality.

THEOREM 4.1. *Let G be a group and $H \leqslant G$ be a dihedral 2-subgroup with connected transversals A and B. If G is generated by $A \cup B$ then H is subnormal in G.*

PROOF: We first let G be finite and a counterexample of minimal order. If $H_G > 1$, then H/H_G is either cyclic or dihedral. If H/H_G is cyclic, then we can use Lemma 2.5. In any case H/H_G is subnormal in G/H_G and we are done. Thus we may assume that $H_G = 1$.

Then assume that H is a maximal subgroup of G. By Theorem 3.2, G is solvable. Let N be a minimal normal subgroup of G (thus N is an elementary Abelian p-group and we can assume that the prime $p \neq 2$). Now $G = NH$ and $N \cap H = 1$.

If $a \in A, b \in B$ then $a = nh$ and $b = mk$ where $m,n \in N$ and $h,k \in H$. Now $[a,b] = h^{-1}n^{-1}k^{-1}m^{-1}nhmk \in H$, hence $kn^{-1}k^{-1}m^{-1}nhmh^{-1} \in H \cap N = 1$ whence $[a,b] = [h,k] \in H'$. In the rest of the proof we denote by S the characteristic cyclic maximal subgroup of H. If $1 \neq a \in (A \cup B) - N$ then $a = nh$ where $n \in N$ and either $h \in H - S$ is an involution or $h \in S$. Now $\langle a \rangle$ contains an element $1 \neq d \in N$ and from $dnh = nhd$ we see that $dh = hd$. If $h \in S$ then $\langle h \rangle$ is normal in $\langle d, H \rangle = G$, a contradiction, since $H_G = 1$. Thus $h \in H - S$ is an involution.

Now assume that $A \neq B$. Thus we have $a \in A$, $b \in B$ such that $aH = bH$ and $a \neq b$. If $a = nh_1$ and $b = nh_2$ where $h_1 \neq h_2$ are involutions then $a^{-1}b = {h_1}^{-1}h_2 \in H$. In fact, $a^{-1}b = s \in S$ and thus $b = as$. Moreover, $[a,b] = a^{-1}s^{-1}a^{-1}aas \in H' \leqslant S$, hence $a^{-1}s^{-1}a \in S$ and $s \in S^{a^{-1}} \cap S = F$. Now F is normal in $\langle H^{a^{-1}}, H \rangle = G$, hence $F = 1$. Thus we may assume that $a = n$ and $b = nh$, where $h \in H - S$ is an involution. Now $[a,B] = 1$ and this means that $B \subseteq N\langle h \rangle$. If $A = N$, then $A \cup B$ does not generate G, hence we have $a \in A$ such that $a = mk$ (here $m \in N$ and $k \in H - S$ is an involution). Obviously $k \neq h$ and then it follows that $m \in B$, hence $A \subseteq N\langle k \rangle$. Now it is clear that if $a = b$ then $a \in N$ which means that $a \in Z(G)$. Then $G = H\langle a \rangle$ and H is normal in G. Thus if $aH = bH \neq H$, then either $a = n \in N$ and $b = nh$ or $b = m \in N$ and $a = mk$ (here $h \neq k$ are involutions from $H - S$). Now n commutes with h and m commutes with k. Also nm has to commute with, say, k and from $nmk = kmn$ it follows that $nk = kn$. Now n commutes with $hk \in S$ and then $\langle hk \rangle$ is normal in $\langle n, H \rangle = G$, which is a contradiction with $H_G = 1$.

Thus we may assume that $A = B$ and $G = \langle A \rangle$. Now by Lemma 2.2, we know that $Ha = Hb^{-1}$ for some $a, b \in A$ and then $ab \in H$ (also $ba \in H$). Of course, $a = b$ is not possible. If $a \in N$, then $A \subseteq C_G(a)$, hence $a \in Z(G)$, $G = H\langle a \rangle$ and H is normal in G. Thus we may assume that $a = nh$, $b = mk$ where $n, m \in N$ and h, k are involutions from $H - S$. Now $nhmk \in H$, hence $nhmh \in H \cap N = 1$ and $m^h = n^{-1}$. Similarly, $n^k = m^{-1}$ and we see that $n^h = n^k = m^{-1}$ which means that $hk \in S$ and $n \in C_G(hk)$. This is possible only if $h = k$. But then $mk = (nh)^{-1}$ and $b = a^{-1}$.

Thus we know that $A = A^{-1}$. Now assume that $a, b \in A$ and $ab \notin H$ (thus $b \neq a^{-1}$). We know now that a^{-1} and b^{-1} are from A. Thus $a^{-1}b^{-1}ab \in H$ and $b^{-1}aba^{-1} \in H$, hence $a^{-1}b^{-1}abb^{-1}aba^{-1} \in H$. Since $m = a^2 \in N$ (now $|a| = 2p$) we get $a^{-1}m^b a^{-1} \in H$, thus $a^{-2}m^b \in H^a$. Now $a^{-2} \in N$, thus $a^{-2}m^b \in H^a \cap N = 1$, hence $m^b = a^2$, that is, $b \in C_G(a^2)$. But then $A \subseteq C_G(a^2)$, hence $a^2 \in Z(G)$ and $G = \langle a^2 \rangle H$ and H is normal in G.

Thus H is not a maximal subgroup of G. Let $G > K > H$ and H be a maximal subgroup of K. Then $K_G > 1$, by Lemma 2.3. Now HK_G/K_G is subnormal in G/K_G, thus $HK_G = K$ is subnormal in G. If E is a subgroup of G, $E \neq K$, H is maximal in E, then $H = K \cap E$ is subnormal in G and we are ready. Thus there exists a unique subgroup K of G such that H is a maximal subgroup of K. Now H is not normal in K and $N_G(H) = H$. Thus H is a Sylow 2-subgroup of G. If $H < T$ and T is a maximal subgroup of G then T is subnormal in G and, in fact, T is normal in G. Now H is a Sylow 2-subgroup of T and by the Frattini lemma $G = TN_G(H) = TH = T$, a contradiction. This completes the proof for finite G.

If G is infinite then we proceed as in the proof of Theorem 3.3 and we see that $G/Z(G)$ is finite. Thus $HZ(G)$ is subnormal in G and then clearly H is subnormal in G. $\square$

Now we can prove our main theorem.

Theorem 4.2. *If Q is a loop such that $I(Q)$ is a dihedral 2-group, then Q is centrally nilpotent.*

Proof: We first observe the obvious fact that the factors of a dihedral group are either cyclic or dihedral. Then we proceed as in Section 4 of [12] and by applying Theorem 2.1, Lemma 2.5 and Theorem 4.1, we are done. $\square$

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