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ON THE EXISTENCE OF NON-NORM-ATTAINING OPERATORS

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Abstract In this article, we provide necessary and sufficient conditions for the existence of non-norm-attaining operators in $\mathcal{L}(E,F)$. By using a theorem due to Pfitzner on James boundaries, we show that if there exists a relatively compact set K of $\mathcal{L}(E,F)$ (in the weak operator topology) such that 0 is an element of its closure (in the weak operator topology) but it is not in its norm-closed convex hull, then we can guarantee the existence of an operator that does not attain its norm. This allows us to provide the following generalisation of results due to Holub and Mujica. If E is a reflexive space, F is an arbitrary Banach space and the pair (E,F) has the (pointwise-)bounded compact approximation property, then the following are equivalent:

(i) $\mathcal{K}(E,F) = \mathcal{L}(E,F)$;

(ii) Every operator from E into F attains its norm;

(iii) $(\mathcal{L}(E,F),\tau_c)^* = (\mathcal{L}(E,F),\|\cdot\|)^*$,

where τ_c denotes the topology of compact convergence. We conclude the article by presenting a characterisation of the Schur property in terms of norm-attaining operators.

Keywords and phrases: James theorem, norm-attaining operators, compact approximation property

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1. Introduction

The famous James theorem states that a Banach space E is reflexive if and only if every bounded linear functional on E attains its norm. By using this characterisation, one can check that if every bounded linear operator from a Banach space E into a Banach space

F is norm-attaining, then E must be reflexive, whereas the range space F is not forced to be reflexive in general. Indeed, every bounded linear operator from a reflexive space into a Banach space that satisfies the Schur property is compact (by the Eberlein-Šmulian theorem) and any compact operator from a reflexive space into an arbitrary Banach space is norm-attaining. Therefore, it seems natural to wonder whether it is possible to guarantee the existence of a non-norm-attaining operator from the existence of a noncompact operator. This brings us back to the 1970s when J.R. Holub proved that this is, in fact, true under approximation property assumptions (see [16, Theorem 2]). Almost 30 years later, J. Mujica improved Holub's result by using the compact approximation property (see [25, Theorem 2.1]), which is a weaker assumption than the approximation property. However, both results require the reflexivity on both domain and range spaces, so the following question arises naturally:

Given a reflexive space E and an arbitrary Banach space F, under which assumptions may we guarantee the existence of non-norm-attaining operators in $\mathcal{L}(E,F)$?

Coming back to Holub and Mujica's results, we would like to highlight what they proved in the direction of the above question. For a background on necessary definitions and notations, we refer the reader to Section 2. In what follows, τ_c denotes the topology of compact convergence and $\|\cdot\|$ denotes the norm topology in $\mathcal{L}(E,F)$.

Theorem ([16, Theorem 2] and [25, Theorem 2.1]). *Let E and F be both reflexive spaces.*

(a) *If $\mathcal{L}(E,F)$ is nonreflexive, there is a non-norm-attaining operator $S \in \mathcal{L}(E,F)$.*

(b) *If E or F has the (compact) approximation property, then the following statements are equivalent:*
   (i) *There exists a non-norm-attaining operator $S \in \mathcal{L}(E,F)$;*
   (ii) $\mathcal{L}(E,F) \neq \mathcal{K}(E,F)$;
   (iii) $\mathcal{L}(E,F)$ *is nonreflexive;*
   (iv) $(\mathcal{L}(E,F),\|\cdot\|)^* \neq (\mathcal{L}(E,F),\tau_c)^*$.

The proof of the above result relies on the fact that if F is a reflexive space, then $\mathcal{L}(E,F)$ is the dual space of the projective tensor product $E\widehat{\otimes}_\pi F^*$. However, if the range space F is nonreflexive, then $\mathcal{L}(E,F)$ is always nonreflexive (see, for instance, [30]).

As a way of extending the above results to the case of nonreflexive range spaces, we borrow some of the techniques used by R.C. James (see [17, 18]). As a matter of fact, one of his results [18, Theorem 1] implies that a separable Banach space E is nonreflexive if and only if given $0<\theta<1$, there exists a sequence (x_n^*) in B_{E^*} such that $x_n^* \xrightarrow{w^*} 0$ and dist $(0, \operatorname{co}\{x_n^* : n \in \mathbb{N}\}) > \theta$, which in turn is equivalent to the existence of a relatively weak-star compact set $K \subseteq B_{E^*}$ such that $0 \in \overline{K}^{w^*}$ and $0 \notin \overline{\operatorname{co}}^{\|\cdot\|}(K)$. This motivates us to define the following property.

Definition 1.1. We say that a pair (E,F) of Banach spaces has the *James property* if there exists a relatively weak operator topology (WOT)-compact set $K \subseteq \mathcal{L}(E,F)$ such that $0 \in \overline{K}^{WOT}$ and $0 \notin \overline{\operatorname{co}}^{\|\cdot\|}(K)$.

We will prove that the James property is a sufficient condition to guarantee the existence of an operator that does not attain its norm, which is our first aim in the present article.

Theorem A. *Let E and F be Banach spaces. If the pair (E,F) has the James property, then there exists a non-norm-attaining operator in $\mathcal{L}(E,F)$.*

Next, we prove that $(\mathcal{L}(E,F),\|\cdot\|)^*$ does not coincide with $(\mathcal{L}(E,F),\tau_c)^*$ whenever a pair (E,F) satisfies the James property (see Proposition 3.1). From this, we can see that whenever the pair (E,F) has the James property, the Banach space $\mathcal{L}(E,F)$ cannot be reflexive due to [25, Lemma 2.3].

We observe, for a reflexive space E and an arbitrary Banach space F, that (1) the unit ball of $\mathcal{K}(E,F)$ is closed in the strong operator topology if and only if it is sequentially closed in this topology (see Lemma 3.4) and (2) $\mathcal{K}(E,F) = \mathcal{L}(E,F)$ implies that $(\mathcal{L}(E,F),\|\cdot\|)^* = (\mathcal{L}(E,F),\tau_c)^*$ by using the result [12, Theorem 1] due to M. Feder and P. Saphar. In addition, we consider the concept of the pointwise-bounded compact approximation property for a pair of Banach spaces in the way it is done in [3] (see Definition 2.1) and prove that $\mathcal{K}(E,F) = \mathcal{L}(E,F)$ when either (3) the norm-closed unit ball of $\mathcal{K}(E,F)$ is closed in the strong operator topology or (4) $(\mathcal{L}(E,F),\|\cdot\|)^* = (\mathcal{L}(E,F),\tau_c)^*$ under the just mentioned approximation property assumption (see Lemma 3.7). Combining (1)–(4) together with Theorem A, we get a generalisation of Holub and Mujica's results, where we no longer need reflexivity on the target space F and E and F might not have the bounded compact approximation property (CAP) (see Example 2.2).

Theorem B. *Let E be a reflexive space and F be an arbitrary Banach space. Consider the following conditions:*

(a) $\mathcal{K}(E,F) = \mathcal{L}(E,F)$.

(b) *Every operator from E into F attains its norm.*

(c) *The unit ball $B_{\mathcal{K}(E,F)}$ is closed in the strong operator topology.*

(d) $(\mathcal{L}(E,F),\tau_c)^* = (\mathcal{L}(E,F),\|\cdot\|)^*$.

Then, we always have (a) $\implies$ (b) $\implies$ (c) and (a) $\implies$ (d) $\implies$ (c). Additionally, if the pair (E,F) has the bounded compact approximation property, then (c) $\implies$ (a) and therefore all of the statements are equivalent.

The following diagram summarises most of the results included in this article. In what follows, E is supposed to be any reflexive space and F is any arbitrary Banach space. See Definition 2.1 in Section 2 for the definition of the pointwise-bounded compact approximation property (BCAP).

Finally, as an application of Theorem B, we connect the Schur property with the case where every operator attains its norm and obtain the following characterisation, which follows from Theorem 3.10.

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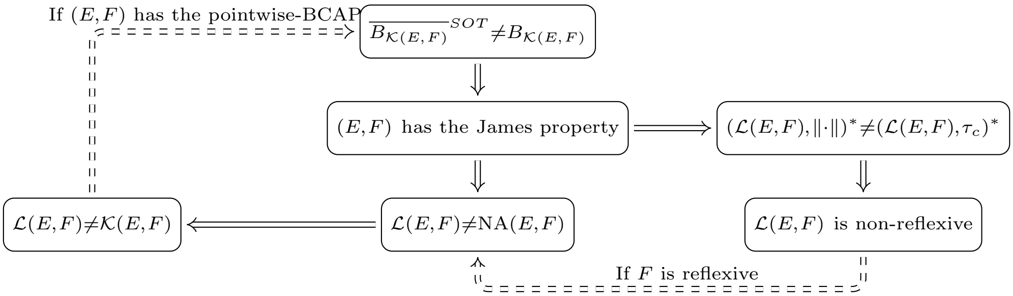

Theorem C. *Let F be a Banach space. The following statements are equivalent:*

(a) *F has the Schur property.*

(b) *$\mathcal{K}(E,F) = \mathcal{L}(E,F)$ for every reflexive space E.*

(c) *$\mathrm{NA}(E,F) = \mathcal{L}(E,F)$ for every reflexive space E.*

(d) *$\mathcal{K}(G,F) = \mathcal{L}(G,F)$ for every reflexive space G with basis.*

(e) *$\mathrm{NA}(G,F) = \mathcal{L}(G,F)$ for every reflexive space G with basis.*

2. Preliminaries

Throughout the article, E and F will be Banach spaces over a field $\mathbb{K}$, which can be either real or complex numbers. We denote by B_E and S_E the closed unit ball and the unit sphere of the Banach space E, respectively. For a subset K of E, $\mathrm{co}(K)$ (respectively, $\overline{\mathrm{co}}(K)$) denotes the convex hull (respectively, closed convex hull) of K. The space of all bounded linear operators from E into F is denoted by $\mathcal{L}(E,F)$. The symbol $\mathcal{K}(E,F)$ (respectively, $\mathcal{W}(E,F)$) stands for the space of all compact operators (respectively, weakly compact operators) from E into F, whereas the symbol $\mathcal{F}(E,F)$ is used to denote the space of all finite-rank operators. Finally, let us recall that an operator $T \in \mathcal{L}(E,F)$ *attains its norm* or is *norm-attaining* if there exists $x \in B_E$ such that $\|T(x)\| = \|T\|$. By $\mathrm{NA}(E,F)$ we mean the set of all norm-attaining operators from E into F. If $E = F$, then we simply write $\mathrm{NA}(E)$ instead of $\mathrm{NA}(E,E)$ and we do the same with the above classes of operators.

We will be using different topologies in $\mathcal{L}(E,F)$. We denote by τ_c the *topology of compact convergence*; that is, the topology of uniform convergence on compact subsets of E. The WOT is defined by the basic neighbourhoods

$$N(T;A,B,\varepsilon) := \left\{S \in \mathcal{L}(E,F) : |y^*(T-S)(x)| < \varepsilon, \text{ for every } y^* \in B, x \in A\right\},$$

where A and B are arbitrary finite sets in E and F^*, respectively, and $\varepsilon > 0$. Thus, in the WOT, a net (T_α) converges to T if and only if $(y^*(T_\alpha(x)))$ converges to $y^*(T(x))$ for every $x \in E$ and $y^* \in F^*$. Analogously, the *strong operator topology* (SOT) is defined by the basic neighbourhoods

$$N(T;A,\varepsilon) := \left\{S \in \mathcal{L}(E,F) : \|(T-S)(x)\| < \varepsilon, \text{ for every } x \in A\right\},$$

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where A is an arbitrary finite set in E and $\varepsilon > 0$. Thus, a net (T_α) converges in the SOT to T if and only if $(T_\alpha(x))$ converges in norm to $T(x)$ for every $x \in E$. We will deal with SOT and WOT closures of bounded sets in $\mathcal{L}(E,F)$. It is worth mentioning that for a bounded convex set in $\mathcal{L}(E,F)$, the WOT closure and the SOT closure coincide [10, Corollary VI.1.5]. Thus, the SOT and the WOT in some results in this article can be interchanged. For a more detailed exposition on topologies in $\mathcal{L}(E,F)$, we refer the reader to [8, 10].

Let us present now the necessary definitions on approximation properties we will need. A Banach space E is said to have the *approximation property* (AP) if the identity operator Id_E in $\mathcal{L}(E)$ belongs to $\overline{\mathcal{F}(E)}^{\tau_c}$. Given $\lambda \geqslant 1$, E is said to have the λ-*bounded approximation property* (λ-BAP) when Id_E belongs to $\lambda \overline{B_{\mathcal{F}(E)}}^{\tau_c}$. A Banach space is said to have the *bounded approximation property* (BAP) if it has the λ-BAP for some $\lambda \geqslant 1$. In the special case when $\lambda = 1$, we say that E has the *metric approximation property* (MAP). Also, recall that E is said to have the CAP if the identity operator Id_E in $\mathcal{L}(E)$ belongs to $\overline{\mathcal{K}(E)}^{\tau_c}$. The λ-*bounded compact approximation property* (λ-BCAP), BCAP and *metric compact approximation property* (MCAP) for a Banach space E can be defined in an analogous way. It is known that a reflexive space has the AP if and only if it has the MAP (see [14]). Analogously, every reflexive space with the CAP also has the MCAP (see [5, Proposition 1 and Remark 1]). We refer the reader to [22, 23] and [4] for background.

On the other hand, E. Bonde introduced in [3] the AP and λ-BAP for pairs of Banach spaces in the following way: a pair (E,F) of Banach spaces is said to have the *AP* if any operator $T \in \mathcal{L}(E,F)$ belongs to $\overline{\mathcal{F}(E,F)}^{\tau_c}$. If $\lambda \geqslant 1$ and every operator $T \in \mathcal{L}(E,F)$ belongs to $\lambda \|T\| \overline{B_{\mathcal{F}(E,F)}}^{\tau_c}$, then (E,F) is said to have the λ-*BAP* or simply the *BAP*.

It is clear that if E or F has the AP (respectively, BAP), then the pair (E,F) has the AP (respectively, BAP). It is observed in [3, Section 4] that there are pairs of Banach spaces (E,F) with the BAP such that E and F do not have the BAP. Similarly, we have the following.

Definition 2.1. The pair (E,F) of Banach spaces is said to have the *compact approximation property* (CAP) if every operator $T \in \mathcal{L}(E,F)$ belongs to $\overline{\mathcal{K}(E,F)}^{\tau_c}$. If $\lambda \geqslant 1$ and every operator $T \in \mathcal{L}(E,F)$ belongs to $\lambda \|T\| \overline{B_{\mathcal{K}(E,F)}}^{\tau_c}$, then (E,F) is said to have the λ-*BCAP* or simply the *BCAP*. In the case when $\lambda = 1$, we say that the pair (E,F) has the MCAP.

Moreover, as one of the anonymous referees and Miguel Martín suggested, we say that the pair (E,F) has the *pointwise-BCAP* if for every operator $T \in \mathcal{L}(E,F)$ there is a constant $\lambda_T \geqslant 1$ such that $T \in \lambda_T \overline{B_{\mathcal{K}(E,F)}}^{\tau_c}$.

Let us note that the BCAP implies the pointwise-BCAP and the pointwise-BCAP implies the CAP. In addition, for a Banach space E, the pair (E,E) has the BCAP if and only if it has the pointwise-BCAP (just take $\lambda := \lambda_I$ with I the identity on E from the definition of the pointwise-BCAP and note that $T = T \circ I \subseteq \overline{\{T \circ K : K \in \lambda_I B_{\mathcal{K}(E,E)}\}}^{\tau_c} \subseteq \lambda \|T\| \overline{B_{\mathcal{K}(E,E)}}^{\tau_c}$ for every $T \in \mathcal{L}(E,E)$). However, we do not know whether the pointwise-BCAP of a pair (E,F) is equivalent to the BCAP or the CAP for an arbitrary Banach

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space F even if E is assumed to be reflexive. The next example shows that a pair (E,F) might have the BCAP even if E and F do not have the CAP.

Example 2.2. In [3, Example 4.2], it is shown that whenever E is a subspace of ℓ_{p_1} and F is a subspace of ℓ_{p_2} with $1 \leqslant p_2 < 2 < p_1 < \infty$, the pair (E,F) has the BAP and therefore the BCAP. Nevertheless, for every $1 < p < \infty$ with $p \neq 2$ there is a subspace of ℓ_p failing the CAP (see [6] and [22, Theorem 1.g.4 and Remark 2 in pg. 111]). In particular, there are Banach spaces E and F such that (E,F) has the BCAP and E and F do not have the CAP. Therefore, assuming that a pair (E,F) of Banach spaces has the BCAP is more general than E or F having the CAP.

3. The Results

In this section, we shall prove Theorems A, B, C and their consequences. We start by proving Theorem A. To do so, let us recall that a set $B \subseteq B_{E^*}$ is called a *James boundary* of a Banach space E if for every $x \in S_E$ there exists $f \in B$ such that $f(x) = 1$. For a subset G of E^*, we shall denote by $w(E,G)$ the weak topology of X induced by G.

Proof of Theorem A. Let us assume by contradiction that every operator from E into F attains its norm. Then, the family

$$B := \left\{ x \otimes y^* : x \in S_E, y^* \in S_{F^*} \right\}$$

is a James boundary of $\mathcal{L}(E,F)$. Indeed, for an arbitrary operator $T \in \mathcal{L}(E,F) = \mathrm{NA}(E,F)$, take $x \in S_E$ to be such that $\|T(x)\| = \|T\|$ and then $y^* \in S_{F^*}$ to be such that $|y^*(T(x))| = \|T(x)\| = \|T\|$. Now, because (E,F) has the James property, there exists a relatively WOT-compact set $K \subseteq \mathcal{L}(E,F)$ such that $0 \in \overline{K}^{WOT}$ and $0 \notin \overline{\mathrm{co}}^{\|\cdot\|}(K)$. By the uniform boundedness principle, the set $\overline{K}^{WOT}$ is norm-bounded. Note that the WOT-topology is the topology of pointwise convergence on B, so it coincides with $w(\mathcal{L}(E,F),B)$. By hypothesis, $\overline{K}^{WOT}$ is WOT-compact or, equivalently, $w(\mathcal{L}(E,F),B)$-compact. By a theorem of Pfitzner (see [27] or [11, Theorem 3.121]), we have that $\overline{K}^{WOT}$ is weakly compact. Therefore, $0 \in \overline{K}^{WOT} = \overline{K}^{w}$, which in particular gives that $0 \in \overline{\mathrm{co}}^{w}(K) = \overline{\mathrm{co}}^{\|\cdot\|}(K)$. This contradiction yields a non-norm-attaining operator $T \in \mathcal{L}(E,F)$ as desired. □

Let us observe that if a pair (E,F) of Banach spaces has the James property, then the dual of $\mathcal{L}(E,F)$ endowed with the norm topology does not coincide with the dual of $\mathcal{L}(E,F)$ endowed with the topology τ_c of compact convergence. As a matter of fact, if K is a subset of E given as in Definition 1.1, then there exists $\varphi \in (\mathcal{L}(E,F), \|\cdot\|)^*$ such that $0 = \mathrm{Re}\,\varphi(0) > \sup\{\mathrm{Re}\,\varphi(T) : T \in \overline{\mathrm{co}}(K)\}$ thanks to the Hahn-Banach separation theorem. This implies that φ cannot be in $(\mathcal{L}(E,F), \tau_c)^*$ because $0 \in \overline{\mathrm{co}}^{WOT}(K) = \overline{\mathrm{co}}^{\tau_c}(K)$. Moreover, using [25, Lemma 2.3], we see that if $(\mathcal{L}(E,F), \|\cdot\|)^* \neq (\mathcal{L}(E,F), \tau_c)^*$, then the space $\mathcal{L}(E,F)$ cannot be reflexive. Summarising, we obtain the following result.

Proposition 3.1. *Let E and F be Banach spaces. If the pair (E,F) has the James property, then*

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(i) $(\mathcal{L}(E,F),\|\cdot\|)^* \neq (\mathcal{L}(E,F),\tau_c)^*$.

(ii) $\mathcal{L}(E,F)$ *is nonreflexive.*

One easy consequence of Theorem A is that if E is reflexive and a pair (E,F) has the James property, then $\mathcal{K}(E,F)$ cannot be equal to the whole space $\mathcal{L}(E,F)$. As a matter of fact, the following result gives us a rather general observation.

Proposition 3.2. *Let E be a reflexive space and F be an arbitrary Banach space. If $\mathcal{K}(E,F) = \mathcal{L}(E,F)$, then $(\mathcal{L}(E,F),\|\cdot\|)^* = (\mathcal{L}(E,F),\tau_c)^*$.*

Proof. Let $D : E\widehat{\otimes}_\pi F^* \longrightarrow (\mathcal{L}(E,F),\tau_c)^*$ be defined by $D(z)(T) := \sum_{n=1}^\infty y_n^*(T(x_n))$ for every $z \in E\widehat{\otimes}_\pi F^*$ with $z = \sum_{n=1}^\infty x_n \otimes y_n^*$ and $T \in \mathcal{L}(E,F)$. It is well known that D is a surjective map (see, for example, [8, Section 5.5, pg. 62]). Therefore, we have that $(\mathcal{L}(E,F),\tau_c)^* = (E\widehat{\otimes}_\pi F^*)/\ker D$. On the other hand, from the result [12, Theorem 1], we have that the map $V : E\widehat{\otimes}_\pi F^* \longrightarrow (\mathcal{K}(E,F),\|\cdot\|)^*$ defined by $V(z)(T) := \sum_{n=1}^\infty y_n^*(T(x_n))$ for $z = \sum_{n=1}^\infty x_n \otimes y_n^*$ and $T \in \mathcal{K}(E,F)$ satisfies the following: for every $\varphi \in (\mathcal{K}(E,F),\|\cdot\|)^*$, there exists $v \in E\widehat{\otimes}_\pi F^*$ such that $\varphi = V(v)$ and $\|\varphi\| = \|v\|$. In particular, we have that $(\mathcal{K}(E,F),\|\cdot\|)^* = (E\widehat{\otimes}_\pi F^*)/\ker V$. Thus, if $\mathcal{K}(E,F) = \mathcal{L}(E,F)$, then $D(z)(T) = V(z)(T)$ for every $z \in E\widehat{\otimes}_\pi F^*$ and every $T \in \mathcal{K}(E,F)$; hence, $\ker D = \ker V$ and $(\mathcal{L}(E,F),\|\cdot\|)^* = (\mathcal{L}(E,F),\tau_c)^*$. □

Let us now go towards the proof of Theorem B. We show the following result, which will help us to prove that if (E,F) does not satisfy the James property, then $\overline{B_{\mathcal{K}(E,F)}}^{SOT}$ coincides with $B_{\mathcal{K}(E,F)}$. Recall that the sequential closure of a set in a topological space is the family of all limit points of sequences on the set in consideration.

Lemma 3.3. *Let E and F be Banach spaces. Suppose that there exists a norm-closed convex set $C \subseteq \mathcal{L}(E,F)$ that is not sequentially closed in the strong operator topology. Then (E,F) has the James property.*

Proof. Suppose that $C \subseteq \mathcal{L}(E,F)$ is norm-closed but not SOT-sequentially closed. This implies that there exists a sequence of operators $(R_n) \subseteq C$ such that (R_n) converges in the SOT (and therefore in the WOT) to an operator $R \notin C$. We may (and we do) suppose that $R = 0$. Set $K := \{R_n : n \in \mathbb{N}\} \subseteq \mathcal{L}(E,F)$. Therefore, K is relatively WOT-compact, $0 \in \overline{K}^{WOT}$ but 0 cannot be in $\overline{\text{co}}(K)$ by hypothesis. Therefore, (E,F) has the James property. □

It is not difficult to check that for a bounded subset C of $\mathcal{L}(E,F)$, with E separable, the SOT-closure of C coincides with the SOT-sequential closure of C. Namely, if $D = \{x_n : n \in \mathbb{N}\}$ is a countable norm-dense subset of E and C is a bounded subset of $\mathcal{L}(E,F)$ with $T \in \overline{C}^{SOT}$, then for each $n \in \mathbb{N}$ we can pick $T_n \in C$ such that $\|T_n(x_m) - T(x_m)\| \leqslant \frac{1}{n}$ for every $m \leqslant n$. Because (T_n) is a uniformly bounded sequence of operators converging in norm on a dense set, it follows from a routine computation that $T_n(x)$ converges in norm to $T(x)$ for every $x \in E$, so T is in the SOT-sequential closure of C.

Furthermore, the following result shows that the unit ball of $\mathcal{K}(E,F)$ is SOT-closed if it is SOT-sequentially closed under the assumption that E has the separable

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complementation property. Recall that a Banach space E is said to have the *separable complementation property* if for every separable subspace Y in E there is a separable subspace Z with $Y \subseteq Z \subseteq E$ and Z is complemented in E. It is worth mentioning that D. Amir and J. Lindenstrauss proved in [2] that every weakly compactly generated Banach space (and therefore every reflexive space) has the separable complementation property.

Lemma 3.4. *Let E be a Banach space with the separable complementation property and F be an arbitrary Banach space. Then, the unit ball $B_{\mathcal{K}(E,F)}$ is SOT-closed if and only if it is SOT-sequentially closed.*

Proof. It is enough to check that if $B_{\mathcal{K}(E,F)}$ is not SOT-closed then it is not SOT-sequentially closed. Suppose that T is an operator that belongs to the SOT-closure of $B_{\mathcal{K}(E,F)}$ but not to $B_{\mathcal{K}(E,F)}$. Note that T is noncompact; hence, there exists a separable subspace E_0 of E such that $T|_{E_0}$ is noncompact. Choose a separable subspace Z of E such that $E_0 \subseteq Z \subseteq E$ and Z is complemented in E. Note that $T|_Z$ is noncompact and belongs to the SOT-closure of $B_{\mathcal{K}(Z,F)}$. Because Z is separable, we have that $T|_Z$ is indeed in the SOT-sequential closure of $B_{\mathcal{K}(Z,F)}$. Let (K_n) be a sequence in $B_{\mathcal{K}(Z,F)}$ converging to $T|_Z$ in the SOT. Letting P be a projection from E onto Z, it is immediate that $T|_Z \circ P$ is noncompact and $K_n \circ P$ is SOT-convergent to $T|_Z \circ P$. This proves that $B_{\mathcal{K}(E,F)}$ is not SOT-sequentially closed. □

It is worth mentioning, however, that the SOT-closure and SOT-sequential closure are different in general, as the following remark shows.

Remark 3.5. *In general, it is not true that the SOT-sequential closure of a bounded convex set C in $\mathcal{L}(E,F)$ coincides with the SOT-closure of C. An example is given by*

$$C := \Big\{ T \in B_{\mathcal{L}(\ell_2(\omega_1))} : \text{there is } \alpha < \omega_1 \text{ such that } (T(x))_\beta = 0 \text{ for every } \beta > \alpha,\ x \in \ell_2(\omega_1) \Big\}.$$

It is immediate that C is SOT-sequentially closed. Nevertheless, because the canonical projections $P_\alpha \in \mathcal{L}(\ell_2(\omega_1))$ with $\alpha < \omega_1$, defined by $(P_\alpha(x))_\beta = x_\beta$ if $\beta \leqslant \alpha$ and 0 otherwise are in C and satisfy that $\{P_\alpha\}_{\alpha<\omega_1}$ SOT-converges to the identity, which is not in C, it follows that C is not SOT-closed.

Notice that if E is reflexive, then it has the separable complementation property. By Lemma 3.4, $B_{\mathcal{K}(E,F)}$ is SOT-closed if and only if it is SOT-sequentially closed. Therefore, if we assume that $\overline{B_{\mathcal{K}(E,F)}}^{SOT} \neq B_{\mathcal{K}(E,F)}$, then (E,F) has the James property by Lemma 3.3. Therefore, we have the following result.

Proposition 3.6. *Let E and F be Banach spaces. If $\overline{B_{\mathcal{K}(E,F)}}^{SOT} \neq B_{\mathcal{K}(E,F)}$, then (E,F) has the James property.*

In order to prove Theorem B, we also need the following lemma. We thank Miguel Martín and one of the anonymous referees for suggesting the use of the pointwise-BCAP instead of the BCAP in the following lemma.

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Lemma 3.7. *Let E and F be Banach spaces. Suppose that the pair (E,F) has the pointwise-BCAP. Then $\mathcal{K}(E,F)=\mathcal{L}(E,F)$ if and only if the unit ball $B_{\mathcal{K}(E,F)}$ is SOT-closed.*

Proof. First, note that because the pair (E,F) has the pointwise-BCAP, we have $\mathcal{L}(E,F)=\bigcup_{\lambda>0}\lambda\overline{B_{\mathcal{K}(E,F)}}^{\tau_c}$. Because $\overline{B_{\mathcal{K}(E,F)}}^{\tau_c}\subseteq\overline{B_{\mathcal{K}(E,F)}}^{SOT}$, we have that $\mathcal{L}(E,F)=\bigcup_{\lambda>0}\lambda\overline{B_{\mathcal{K}(E,F)}}^{SOT}$. So, if we assume $B_{\mathcal{K}(E,F)}$ to be SOT-closed, then

$$\mathcal{L}(E,F)=\bigcup_{\lambda>0}\lambda\overline{B_{\mathcal{K}(E,F)}}^{SOT}=\bigcup_{\lambda>0}\lambda B_{\mathcal{K}(E,F)}=\mathcal{K}(E,F).$$

The other implication is immediate. □

Let us finally recall the following conditions and prove Theorem B as a consequence of Theorem A, Proposition 3.1, Proposition 3.2, Proposition 3.6 and Lemma 3.7.

(a) $\mathcal{K}(E,F)=\mathcal{L}(E,F)$.

(b) Every operator from E into F attains its norm.

(c) The unit ball $B_{\mathcal{K}(E,F)}$ is closed in the strong operator topology.

(d) $(\mathcal{L}(E,F),\tau_c)^*=(\mathcal{L}(E,F),\|\cdot\|)^*$.

Proof of Theorem B. Let E be reflexive and F be an arbitrary Banach space. It was noted in the Introduction that (a) $\Longrightarrow$ (b) holds. Moreover, (b) implies that (E,F) does not have the James property (by applying Theorem A), which in turn implies (c) (by applying Proposition 3.6). On the other hand, Proposition 3.2 shows (a) $\Longrightarrow$ (d). By Proposition 3.1, (d) implies that (E,F) does not have the James Property and, therefore, it implies (c) (by applying Proposition 3.6). Finally, if the pair (E,F) has the BCAP, then the implication (c) $\Longrightarrow$ (a) follows from Lemma 3.7 and all statements (a)–(d) are equivalent. □

M.I. Ostrovskii asked in [24, §12, pg. 65] whether there exist infinite-dimensional Banach spaces on which every operator attains its norm (this question is also asked in [20, Problem 8] and [15, Problem 217]). By Holub's theorem [16], if such an infinite-dimensional Banach space exists, it cannot have the AP. Theorem 3.8 is a generalisation of this fact. Let us recall that given a (norm-closed) operator ideal $\mathcal{A}$ and $\lambda\geqslant 1$, a Banach space E is said to have the λ-$\mathcal{A}$-*approximation property* (λ-$\mathcal{A}$-AP) if the identity operator Id_E belongs to $\overline{\{T\in\mathcal{A}(E,E):\|T\|\leqslant\lambda\}}^{\tau_c}$. We say that E has the bounded-$\mathcal{A}$-AP if it has the λ-$\mathcal{A}$-AP for some $\lambda\geqslant 1$. This general approximation property has been studied, for instance, in [13, 21, 26, 29].

Theorem 3.8. *If there is an infinite-dimensional Banach space E such that every operator on $\mathcal{L}(E)$ attains its norm, then E does not have the bounded $\mathcal{A}$-approximation property for any nontrivial ideal $\mathcal{A}$ (i.e., for any ideal $\mathcal{A}\neq\mathcal{L}(E)$).*

Proof. As highlighted in [24, §12, pg. 66], due to a result of N.J. Kalton, if such a Banach space E exists, then it must be separable. Therefore, the SOT-closure of the set $B:=\{T\in\mathcal{A}(E,E):\|T\|\leqslant 1\}$ in $\mathcal{L}(E)$ coincides with its SOT-sequential closure. Thus, if

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every operator on $\mathcal{L}(E)$ attains its norm, then B is SOT-closed by Lemma 3.3. Suppose that E has the bounded $\mathcal{A}$-approximation property. Then, because $\overline{B}^{\tau_c} \subseteq \overline{B}^{SOT} = B$, we have that B contains a multiple of the identity and therefore $\mathcal{A}$ contains the identity on E, so $\mathcal{A} = \mathcal{L}(E)$. □

We finally present the proof of Theorem C as a direct consequence of Theorem B and Proposition 3.9. Recall that a Banach space E has the *Schur property* if every weakly convergent sequence is norm convergent. It is known that a Banach space F has the Schur property if and only if every weakly compact operator from E into F is compact for any Banach space E (see, for example, [28, Section 3.2, pg. 61]). Also, it is proved in [9, Theorem 1] that a Banach space F has the Schur property if and only if the weak Grothendieck compactness principle holds in F; that is, every weakly compact subset of F is contained in the closed convex hull of a weakly null sequence. Then W.B. Johnson et al. gave an alternative proof in [19, Theorem 1.1] for this result by using the Davis-Figiel-Johnson-Pełczyński factorisation theorem [7]. Moreover, it was observed in [19, Theorem 3.3] that a Banach space F has the Schur property if and only if $\mathcal{W}_\infty(E,F) \subseteq \mathcal{W}(E,F)$ for every Banach space E (see the precise definition of these sets just after Proposition 3.9).

The following result will be used as an important tool in the proof of Theorem 3.10.

Proposition 3.9. *Let F be a Banach space. If F fails to have the Schur property, then there exists a reflexive space with basis E such that $\mathcal{K}(E,F) \neq \mathcal{L}(E,F)$.*

Proof. Take $(x_n) \subseteq S_F$ to be a weakly null sequence in F, which is not norm null. Because the absolute closed convex hull of $\{x_n : n \in \mathbb{N}\}$ is weakly compact, the operator $T \in \mathcal{L}(\ell_1, F)$ given by $T(e_n) := x_n$ for each $n \in \mathbb{N}$ defines a weakly compact operator (which is not compact). By the Davis-Figiel-Johnson-Pełczyński factorisation theorem [7], there exists a reflexive space E_0 such that $T = S \circ R$, where $R \in \mathcal{L}(\ell_1, E_0)$ and $S \in \mathcal{L}(E_0, F)$. In particular, note that S cannot be a compact operator. Now, pick a weakly null sequence $(v_n) \subseteq E_0$ so that $S(v_n)$ does not admit a convergent subsequence. Because (v_n) is weakly null, consider a subsequence that is a basic sequence of E_0 (see [1, Proposition 1.5.4]) and denote it again by (v_n). Let $E := \overline{\text{span}}\{v_n\}_{n\in\mathbb{N}}$. Then, E is a closed reflexive space with basis and $S(v_n)$ does not admit a convergent subsequence. Therefore, we conclude that $\mathcal{K}(E,F) \neq \mathcal{L}(E,F)$. □

Compared to the previously known results in [19], Theorem 3.10 not only provides a new characterisation of the Schur property in terms of norm-attaining operators but also shows that we can restrict the possible candidates for a domain space as in the below items (f)–(i) by considering only reflexive Banach spaces *with basis*. Recall that $T \in \mathcal{L}(E,F)$ is *completely continuous* if T sends weakly null sequences in E to norm null sequences in F. We denote by $\mathcal{V}(E,F)$ the space of all completely continuous operators from E into F. Let us denote by $\mathcal{W}_\infty(E,F)$ the space of all weakly ∞-compact operators from E into F, which are introduced in [31]. A subset C of a Banach space E is called *relatively weakly ∞-compact* if there exists a weakly null sequence (x_n) in E such that $C \subseteq \{\sum_{n=1}^{\infty} a_n x_n : (a_n) \in B_{\ell_1}\}$ and an operator $T \in \mathcal{L}(E,F)$ is said to be *weakly ∞-compact* if $T(B_E)$ is a relatively weakly ∞-compact subset of F.

It is immediate to notice that Theorem C follows from Theorem 3.10.

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Theorem 3.10. *Let F be an arbitrary Banach space. The following are equivalent:*

(a) *F has the Schur property.*

(b) $\mathcal{K}(E,F)=\mathcal{L}(E,F)$ *for every reflexive space E.*

(c) $\mathcal{W}_\infty(E,F)=\mathcal{L}(E,F)$ *for every reflexive space E.*

(d) $\mathcal{V}(E,F)=\mathcal{L}(E,F)$ *for every reflexive space E.*

(e) $\mathrm{NA}(E,F)=\mathcal{L}(E,F)$ *for every reflexive space E.*

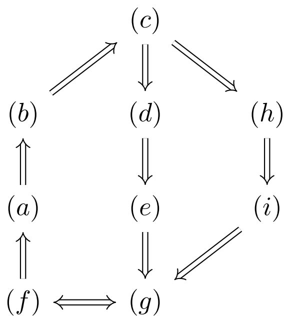
(f) $\mathcal{K}(G,F)=\mathcal{L}(G,F)$ *for every reflexive space G with basis.*

(g) $\mathrm{NA}(G,F)=\mathcal{L}(G,F)$ *for every reflexive space G with basis.*

(h) $\mathcal{W}_\infty(G,F)=\mathcal{L}(G,F)$ *for every reflexive space G with basis.*

(i) $\mathcal{V}(G,F)=\mathcal{L}(G,F)$ *for every reflexive space G with basis.*

Proof. The following diagram holds.



Indeed, by definition we have that $\mathcal{K}(E,F)\subseteq\mathcal{W}_\infty(E,F)\subseteq\mathcal{W}(E,F)$ for any Banach space E and F, and it is also known that $\mathcal{K}(E,F)\subseteq\mathcal{W}_\infty(E,F)\subseteq\mathcal{V}(E,F)$ (see [19, Proposition 3.1]). Moreover, if T is an element of $\mathcal{V}(E,F)$ with E reflexive, then $T\in\mathrm{NA}(E,F)$ thanks to the weak sequential compactness of B_E. Thus, it is immediate that (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) $\Longrightarrow$ (d) $\Longrightarrow$ (e) $\Longrightarrow$ (g) and (c) $\Longrightarrow$ (h) $\Longrightarrow$ (i) $\Longrightarrow$ (g) hold. Because a reflexive Banach space with basis has the MAP, (f) $\Longleftrightarrow$ (g) follows from Theorem B. Finally, (f) $\Longrightarrow$ (a) is already obtained by Proposition 3.9. □

Questions and comments. Let us conclude the article by recalling some open problems. In [16], Holub conjectured that if E and F are both reflexive, then $\mathcal{L}(E,F)$ is reflexive if and only if $\mathcal{L}(E,F)=\mathcal{K}(E,F)=NA(E,F)$. We do not know whether, in general, when E and F are both reflexive spaces all of the implications in the diagram of the Introduction are indeed equivalences. A similar open question posed by Miguel Martín and one of the anonymous referees asks whether Theorem A is in general an equivalence; that is, whether a pair (E,F) has the James property whenever there exists a non-norm-attaining operator in $\mathcal{L}(E,F)$. Note that James proved that this implication holds when E is separable and $F=\mathbb{R}$ (recall the paragraph preceding Definition 1.1).

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