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Cuspidal modules of finite general linear group

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There are certain irreducible complex representations of the general linear group $\mathrm{GL}_n(q)$ over a finite field $\mathbb{F}_q$, called cuspidal or discrete series representations, which form the building blocks for all modules ([**1**]). The aim of this thesis is to give an elementary and explicit description of cuspidal representations for $\mathrm{GL}_n(q)$. Lusztig in his book [**2**] gives construction of some cuspidal modules of $\mathrm{GL}_n(q)$, but not all. In this thesis, using Lusztig's work as a foundation, we show that modules for all irreducible cuspidal characters can be obtained as eigenspaces of a certain endomorphism given by Lusztig. Furthermore, we show that Lusztig's formula for the eigenvalue associated with the distinguished cuspidal module generalizes in the obvious way for all cuspidal modules. However, whereas Lusztig's work is independent of Green's construction of the characters of $\mathrm{GL}_n(q)$, we need to use a knowledge of the characters to compute the eigenvalue.

The content of this thesis is roughly as follows. Chapter 1 contains some preliminary material on split BN-pairs and Harish–Chandra theory.

Chapter 2 is devoted to proving one of our main results. We consider two sugroups A and A' of $\mathrm{GL}_n(q)$ containing the unipotent subgroup U, either of which may be identified with the group of all affine transformations of an $(n-1)$-dimensional affine space. Since A and A' together generate $\mathrm{GL}_n(q)$, a representation of $\mathrm{GL}_n(q)$ is completely determined by its restrictions to these two subgroups. Now the restrictions of an irreducible cuspidal representation to A and to A' are both isomorphic to irreducible induced representations. Specifically, they are induced from a 1-dimensional representation of U that is in general position (meaning that it is nontrivial on all the fundamental root subgroups). We investigate the restrictions of these induced representations to the group $A \cap A'$, with a view to determining how they are identified in the cuspidal representation. We show that the restrictions are multiplicty-free, and describe the irreducible constituents. There remains a problem of finding one scalar for each of these constituents to determine how the representations of A and A' are identified.

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In Chapter 3, we use calculations with idempotents in the group algebra to investigate endomorphism algebras, and to compare the actions of A and A' on a cuspidal module. The first main result shows that the restriction of a cuspidal module to $A \cap A'$ is multiplicity-free. We work with two bases of the cuspidal module: with respect to one basis A acts via monomial transformations, with respect to the other A' acts via monomial transformations. The strategy is to obtain information on how elements of A act with respect to the second basis or how elements of A' act with respect to the first basis. The key technical result here gives the action of a particular element of A on a particular basis vector, describing it in terms of $q-1$ unknown constants. With further calculations it is shown that one of these constants is an eigenvalue of a certain endomorphism of the Gel'fand–Graev module, and the corresponding eigenspace can be identified with the cuspidal module.

The aim of Chapter 4 is to compute the $q-1$ constants mentioned above. For this we need to make use of Green's work on the irreducible characters of $\mathrm{GL}_n(q)$. We begin Chapter 4 with some preliminary material on conjugacy classes and cuspidal characters of $\mathrm{GL}_n(q)$. An element of $\mathrm{GL}_n(q)$ whose characteristic polynomial is equal to its minimum polynomial is conjugate in $\mathrm{GL}_n(q)$ to its companion matrix. It is readily shown that companion matrices corresponding to polynomials with a given constant term all lie in the same left coset of U. We show that this left coset contains exactly $q^{1/2(n-1)(n-2)}$ elements of each of these conjugacy classes, and no other elements. This enables us to compute the $q-1$ constants mentioned above.

Chapter 5 is based on the work of Lusztig [**2**], who gave a construction of modules for certain irreducible cuspidal characters of $\mathrm{GL}_n(q)$. The modules Lusztig constructs are all given as eigenspaces of one specific endomorphism of a certain module obtained from the homology of a simplicial complex on which $\mathrm{GL}_n(q)$ acts naturally. Lusztig gives a formula for the relevant eigenvalues. We show that modules for all the irreducible cuspidal characters can be obtained in exactly the same way: they are all eigenvalues of the endomorphism given by Lusztig, and his formula for the eigenvalues generalizes in the most obvious way.

References

[**1**] J.A. Green, 'The characters of the finite general linear groups', *Trans. Amer. Math. Soc.* **80** (1955), 402–447.

[**2**] G. Lusztig, *The discrete series of* GL_n *over a finite field*, Annals of Mathematics Studies **81** (Princeton University Press, Princeton, N.J., 1974).

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