EXTENSION OF COMPLETELY BOUNDED A-B BIMODULE MAPS†

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0. Abstract. In this paper, we present an "order" characterization of completely bounded bimodule maps for bimodules over unital operator algebras. We use this result to prove a bimodule generalization of Wittstock's generalized Hahn-Banach theorem. Our proofs simplify and unify some of Wittstock's arguments.

1. Introduction. We are concerned with operator spaces that are modules over operator algebras and module maps between operator modules. Recall that an operator space is a norm closed subspace of a C*-algebra. Although we will make no use of the fact here, we point out that by Ruan's theorem [**8**] such spaces may be characterized abstractly as matrically normed Banach spaces satisfying the so-called L^∞-condition. Similarly, an operator algebra is a (norm closed) subalgebra of a C*-algebra. All our operator algebras will be unital. By [**2**], unital operator algebras may be characterized abstractly as operator spaces in which multiplication is completely contractive. To say that an operator space X is a *(left) operator module* over an operator algebra A is simply to say that X is a unital (left) A module in the usual sense for which multiplication is completely contractive as a bilinear map. Using the results of [**8**] and [**2**], it is possible to show that if X is a left operator module over an operator algebra A, then it is possible to imbed A and X in a C*-algebra completely isometrically in such a way that the module multiplication is transferred to the multiplication in the C*-algebra (see [**1**]). Thus, when speaking about operator modules that are given as subspaces of a C*-algebra, we always assume the multiplication and operator space structures are inherited from the C*-algebra. Right operator modules are defined similarly, as are operator bimodules.

In [**11**, Theorem 3.1 and 4.1], Wittstock proved two theorems about extending module maps from submodules to larger modules when the operator algebras concerned are C*-algebras. The hypotheses in these two theorems are the same except that one, Theorem 3.1, discusses bimodules over C*-algebras where the algebra is the same on both sides, while in the other, Theorem 4.1, only left modules are considered. Although both proofs use the notion of "sublinear set valued functionals" (see Definition 3.1 below), they are quite different in detail. Both theorems are module-theoretic generalizations of his earlier extension of the Hahn-Banach theorem [**12**]. The purpose of this note is first to provide an "order theoretic characterization of bimodule maps from an operator bimodule into a C*-algebra (see Theorem 2.1, below) and then to use this characterization to prove a bimodule generalization of Wittstock's generalized Hahn-Banach Theorem (Theorem 3.4, below). Here the bimodules considered can have different algebras on one side and on the other. Moreover, the "order" characterization of bimodule maps makes the proof of the module extension theorem more perspicuous, a more immediate corollary of Wittstock's Hahn-Banach theorem, than do the proofs in [**11**].

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We note in passing that Suen **[10]** has given another proof of Wittstock's Theorem 3.1 that is different from ours. We note, too, that Ruan showed us how to derive our Theorem 3.4 from Wittstock's Theorem 3.1 using Putnam's technique for generalizing Fuglede's Theorem to the setting of operators intertwining two normal operators (see **[5]**). Thus, while the A-B version of the extension theorem, our Theorem 3.4, is logically equivalent to the A-A version, Wittstock's Theorem 3.1, our proof does not simplify if one restricts to A-A bimodules.

2. Order and bimodule maps. In the following theorem we give an "order" theoretic characterization of completely bounded bimodule maps from an operator bimodule into a C*-algebra. Although our application of it in this note uses the result only under the additional hypotheses that the bimodule is one over C*-algebras, the greater generality requires no additional effort to prove and may be useful elsewhere.

THEOREM 2.1. *Suppose that A and B are unital operator subalgebras of an unital C*-algebra $\mathcal{D}$, and suppose that X is an A-B operator bimodule. Then a real linear map $\phi : X \to \mathcal{D}$ is a completely bounded A-B bimodule map if and only if there exists a nonnegative constant c such that ϕ satisfies*

$$\begin{pmatrix} 0 & \phi_n(axb) \\ \phi_n(axb)^* & 0 \end{pmatrix} \leq c\,\|x\|_n \begin{pmatrix} aa^* & 0 \\ 0 & b^*b \end{pmatrix} \tag{2.1}$$

for every $n \in \mathbf{N}$, $a \in M_n(A)$, $x \in M_n(X)$ and $b \in M_n(B)$. Moreover, the cb-norm $\|\phi\|_{cb}$ is the infimum over all the constants c satisfying (2.1).

Proof. Recall that for any C*-algebra $\mathcal{A}$ with unit, an element

$$\begin{pmatrix} a & v \\ v^* & b \end{pmatrix}$$

in $M_2(\mathcal{A})$ is nonnegative if and only if $a \geq 0$, $b \geq 0$, and for each $\epsilon > 0$, $\|(A+\epsilon)^{-1/2} v (b+\epsilon)^{-1/2}\| \leq 1$ (see **[4]**).

Suppose first that $\phi : X \to \mathcal{D}$ is a completely bounded A-B bimodule map. Then for each $n \in \mathbf{N}$, $a \in M_n(A)$, $x \in M_n(X)$, and $b \in M_n(X)$,

$$\begin{pmatrix} 0 & \phi_n(axb) \\ \phi_n(axb)^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ b^* & 0 \end{pmatrix} \begin{pmatrix} 0 & \phi_n(x)^* \\ \phi_n(x) & 0 \end{pmatrix} \begin{pmatrix} 0 & b \\ a^* & 0 \end{pmatrix}. \tag{2.2}$$

We claim that for any $x \in M_n(X)$,

$$\begin{pmatrix} 0 & \phi_n(x)^* \\ \phi_n(x) & 0 \end{pmatrix} \leq \|\phi\|_{cb}\, \|x\|_n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.3}$$

Indeed,

$$\|\phi\|_{cb}\,\|x\|_n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & \phi_n(x)^* \\ \phi_n(x) & 0 \end{pmatrix} = \begin{pmatrix} \|\phi\|_{cb}\,\|x\|_n & -\phi_n(x)^* \\ -\phi_n(x) & \|\phi\|_{cb}\,\|x\|_n \end{pmatrix} \geq 0,$$

because $\|\phi\|_{cb}\,\|x\|_n \geq 0$ and for each $\epsilon > 0$

$$\begin{aligned} \|(\|\phi\|_{cb}\,\|x\|_n + \epsilon)^{-1/2}(-\phi_n(x)^*)(\|\phi\|_{cb}\,\|x\|_n + \epsilon)^{-1/2}\| &= \|(\|\phi\|_{cb}\|x\| + \epsilon)^{-1}\phi_n(x)^*\|_n \\ &\leq (\|\phi\|_{cb}\,\|x\|_n + \epsilon)^{-1}\,\|\phi_n(x)^*\|_n \leq 1. \end{aligned}$$

From (2.2) and (2.3) we have

$$\begin{pmatrix} 0 & \phi_n(axb) \\ \phi_n(axb)^* & 0 \end{pmatrix} \le \|\phi\|_{cb} \|x\|_n \begin{pmatrix} 0 & a \\ b^* & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & a \\ b^* & 0 \end{pmatrix}^*$$

$$= \|\phi\|_{cb} \|x\|_n \begin{pmatrix} aa^* & 0 \\ 0 & a^*b \end{pmatrix}$$

for all $a \in M_n(A)$, $x \in M_n(X)$, $b \in M_n(B)$.

Conversely, suppose that $\phi : X \to \mathfrak{D}$ satisfies (2.1). Letting $a = 1 \in M_n(A)$, and $b = 1 \in M_n(B)$, we have

$$\|\phi_n(x)\|_n = \left\| \begin{pmatrix} 0 & \phi_n(x) \\ \phi_n(x)^* & 0 \end{pmatrix} \right\| \le c\, \|x\|_n \left\| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\| = c\, \|x\|_n,$$

for all $x \in M_n(X)$. It follows that ϕ is a completely bounded map and $\|\phi\|_{cb} = \inf c$. Therefore in (2.1) we may replace c with $\|\phi\|_{cb}$ and write

$$\begin{pmatrix} 0 & \phi_n(axb) \\ \phi_n(axb)^* & 0 \end{pmatrix} \le \|\phi\|_{cb} \|x\|_n \begin{pmatrix} aa^* & 0 \\ 0 & b^*b \end{pmatrix} \tag{2.4}$$

for all $a \in M_n(A)$, $x \in M_n(X)$, and $b \in M_n(B)$. Now, let $b = 1 \in M_n(B)$. Then

$$\begin{pmatrix} 0 & \phi_n(ax) \\ \phi_n(ax)^* & 0 \end{pmatrix} \le \|\phi\|_{cb} \|x\|_n \begin{pmatrix} aa^* & 0 \\ 0 & 1 \end{pmatrix}$$

for all $a \in M_n(A)$, and $x \in M_n(X)$. In particular, we see that for any $a \in A$, and $x \in X$, we have

$$\begin{pmatrix} 0 & \phi_2\left(\begin{pmatrix} a & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\right) \\ \phi_2\left(\begin{pmatrix} a & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\right)^* & 0 \end{pmatrix}$$

$$\le \|\phi\|_{cb} \left\| \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \right\| \begin{pmatrix} \begin{pmatrix} a & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a^* & -1 \\ 0 & 0 \end{pmatrix} & 0 \\ 0 & 1 \end{pmatrix}.$$

That is

$$\begin{pmatrix} 0 & 0 & \phi(ax) & 0 \\ 0 & 0 & \phi(-x) & 0 \\ \phi(ax)^* & \phi(-x)^* & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \le \|\phi\|_{cb} \|x\| \begin{pmatrix} aa^* & -a & 0 & 0 \\ -a^* & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore

$$\begin{pmatrix}1 & a & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & \phi(ax) & 0\\ 0 & 0 & \phi(-x) & 0\\ \phi(ax)^* & \phi(-x)^* & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ a^* & 0\\ 0 & 1\\ 0 & 0\end{pmatrix}$$

$$\leq \|\phi\|_{cb}\,\|x\|\begin{pmatrix}1 & a & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}\begin{pmatrix}aa^* & -a & 0 & 0\\ -a^* & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ a^* & 0\\ 0 & 1\\ 0 & 0\end{pmatrix}.$$

That is,

$$\begin{pmatrix}0 & \phi(ax)-a\phi(x)\\ \phi(ax)^*-\phi(x)^*a^* & 0\end{pmatrix}\leq \|\phi\|_{cb}\,\|x\|\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}.$$

Thus for any $a \in A$, and $x \in X$, we have

$$\begin{pmatrix}0 & -(\phi(ax)-a\phi(x))\\ -(\phi(ax)^*-\phi(x)^*a^*) & \|\phi\|_{cb}\,\|x\|\end{pmatrix}\geq 0.$$

Therefore for any $\epsilon > 0$, we have

$$\|(0+\epsilon)^{-1/2}(\phi(ax)-a\phi(x))(\|\phi\|_{cb}\,\|x\|+\epsilon)^{-1/2}\|\leq 1.$$

It follows that

$$\|\phi(ax)-a\phi(x)\|\leq \epsilon^{1/2}(\|\phi\|_{cb}\,\|x\|+\epsilon)^{1/2}.$$

Letting $\epsilon \to 0$, we have $\phi(ax) = a\phi(x)$ for all $a \in A$, and $x \in X$, proving that ϕ is a left A-module map.

Similarly, if we let $a = 1$ in (2.4), we get

$$\begin{pmatrix}0 & \phi_n(xb)\\ \phi_n(xb)^* & 0\end{pmatrix}\leq \|\phi\|_{cb}\,\|x\|\begin{pmatrix}1 & 0\\ 0 & b^*b\end{pmatrix}.$$

Therefore,

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & \phi_n(xb)\\ \phi_n(xb)^* & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\leq \|\phi\|_{cb}\,\|x\|\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & b^*b\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

That is

$$\begin{pmatrix}0 & \phi_n(xb)^*\\ \phi_n(xb) & 0\end{pmatrix}\leq \|\phi\|_{cb}\,\|x\|_n\begin{pmatrix}b^*b & 0\\ 0 & 1\end{pmatrix}$$

for all $x \in M_n(X)$, and $b \in M_n(B)$. So with $n = 2$, we have

$$\begin{pmatrix}0 & \phi_2\left(\begin{pmatrix}x & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}b & -1\\ 0 & 0\end{pmatrix}\right)^*\\ \phi_2\left(\begin{pmatrix}x & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}b & -1\\ 0 & 0\end{pmatrix}\right) & 0\end{pmatrix}$$

$$\leq \|\phi\|_{cb}\left\|\begin{pmatrix}x & 0\\ 0 & 0\end{pmatrix}\right\|\begin{pmatrix}\begin{pmatrix}b^* & 0\\ -1 & 0\end{pmatrix}\begin{pmatrix}b & -1\\ 0 & 0\end{pmatrix} & 0\\ 0 & 1\end{pmatrix}$$

for all $x \in X$, and $b \in B$. That is

$$\begin{pmatrix} 0 & 0 & \phi(xb)^* & 0 \\ 0 & 0 & \phi(-x)^* & 0 \\ \phi(xb) & \phi(-x) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \leq \|\phi\|_{cb} \|x\| \begin{pmatrix} b^*b & -b^* & 0 & 0 \\ -b & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} 1 & b^* & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \phi(xb)^* & 0 \\ 0 & 0 & \phi(-x)^* & 0 \\ \phi(xb) & \phi(-x) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\leq \|\phi\|_{cb} \|x\| \begin{pmatrix} 1 & b^* & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} b^*b & -b^* & 0 & 0 \\ -b & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

That is,

$$\begin{pmatrix} 0 & \phi(xb)^* - b^*\phi(x) \\ \phi(xb) - \phi(x)b & 0 \end{pmatrix} \leq \|\phi\|_{cb} \|x\| \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

for any $a \in A$, and $x \in X$. Thus for any $\epsilon > 0$, we have

$$\|(0+\epsilon)^{-1/2}(\phi(xb)^* - b^*\phi(x)^*)(\|\phi\|_{cb} \|X\| + \epsilon)^{-1/2}\| \leq 1.$$

Equivalently,

$$\|\phi(xb)^* - b^*\phi(x)^*\| \leq \epsilon^{1/2}(\|\phi\|_{cb} \|x\| + \epsilon)^{1/2}.$$

Letting $\epsilon \to 0$, we have $\phi(xb)^* = b^*\phi(x)^*$, or, $\phi(xb) = \phi(x)b$ for all $x \in X$, and $b \in B$, proving that ϕ is a right B-module map as well. This completes the proof.

Since a left A operator module (respectively, right B operator module) may be regarded as an A-$\mathbf{C}$ operator bimodule (respectively, $\mathbf{C}$-B operator bimodule), similar arguments apply to a module map. Therefore letting $b = 1$ (respectively, $a = 1$) in Theorem 2.1, we get an "order" characterization of completely bounded one sided modules maps. Observe that the following corollary was noted in [**11**].

COROLLARY 2.2. *Suppose that A is a unital operator subalgebra of an unital C^*-algebra $\mathcal{D}$, and that X is an A-A operator bimodule. If $\phi : X \to \mathcal{D}$ is a completely bounded A-A bimodule map, then*

$$\operatorname{Re} \phi_n(axa^*) \leq \|\phi\|_{cb} \|x\|_n \, aa^*$$

for all $a \in M_n(A)$, $x \in M_n(X)$, where $\operatorname{Re} \phi_n(axa^*) = (1/2)(\phi_n(axa^*) + \phi_n(axa^*)^*)$.

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Proof. By Theorem 2.1, we have

$$\begin{pmatrix} 0 & \phi_n(axa^*) \\ \phi_n(axa^*)^* & 0 \end{pmatrix} \le c\ \|x\|_n \begin{pmatrix} aa^* & 0 \\ 0 & a^{**}a^* \end{pmatrix}$$

for all $a \in M_n(A)$, and $x \in M_n(X)$. By multiplying by $(1 \quad 1)$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ on left side and right side, respectively, of this inequality, we get the desired result.

3. A Hahn-Banach theorem for bimodule maps.

DEFINITION 3.1. (see [**11**]). Given a unital C*-algebra $\mathcal{D}$, let $\mathcal{D}_h$ denote the hermitian part of $\mathcal{D}$. Let K, L be two subsets of $\mathcal{D}_h$. We write $K \le L$ if for every $l \in L$ there exists a $k \in K$ with $k \le l$.

Let E be a K-vector space ($K = \mathbf{R}$ or $\mathbf{C}$). A set valued functional $\theta : E \to \mathcal{D}_h$ is called *sublinear* if it has the following properties:

(i) $(X) \ne \emptyset$, for all $X \in E$,

(ii) $\theta(x_1 + x_2) \le \theta(x_1) + \theta(x_2)$, for all $x_1, x_2 \in E$,

(iii) $0 \le \theta(0)$,

(iv) $\theta(\lambda x) \le \lambda\theta(x)$, for all $x \in E$ and $\lambda \in \mathbf{R}_+$.

A family $\theta = (\theta_n)_{n \in \mathbf{N}}$ of set valued sublinear functionals $\theta_n : E \otimes_{\mathbf{R}} (M_n)_h \to M_n(\mathcal{D})_h$ is called a *matrical sublinear functional* if in addition

(v) $\theta_m(\gamma^* x \gamma) \le \gamma^* \theta_n(x)\gamma$, for all $x \in E \otimes_{\mathbf{R}} (M_n)_h$, and all $n \times m$ matrices γ.

DEFINITION 3.2. (see [**3**], [**6**]). A C*-algebra $\mathcal{D}$ is called *injective* if it has Arveson's extension property; i.e., it is an injective object in the category of operator systems and completely positive maps.

Recall Wittstock's generalization of Hahn-Banach Theorem [**11**] or [**12**].

LEMMA 3.3. *Let E be a K-vector space ($K = \mathbf{R}$ or $\mathbf{C}$), $\mathcal{D}$ a unital C*-algebra, and $\theta : E \to \mathcal{D}_h$ a matrical sublinear functioinal. Then there exists a K-linear map $\phi : E \to \mathcal{D}_h \otimes K$ such that* $\operatorname{Re} \phi_n(x) \le \theta_n(x)$, for all $x \in E \otimes_{\mathbf{R}} (M_n)_h$.

Notice that if E is a $\mathbf{C}$-vector space, then $E \otimes_{\mathbf{R}} (M_n)_h$ is the underlying real vector space of $E \otimes_{\mathbf{C}} M_n$ (see [**11**]). The following theorem combines and generalizes Theorem 3.1 and 4.1 in [**11**].

THEOREM 3.4. *Let $\mathcal{D}$ be a unital injective C*-algebra, A and B unital C*-subalgebras of $\mathcal{D}$. Let Y be an A-B operator bimodule, X be an A-B operator subbimodule of Y, and let $\phi : X \to \mathcal{D}$ be a completely bounded A-B bimodule map. Then there exists a completely bounded A-B bimodule map $\tilde{\phi} : Y \to \mathcal{D}$ that extends ϕ with the same cb-norm. In other words, $\mathcal{D}$ is an injective operator A-B bimodule.*

Proof. Consider the family of real linear maps $\Phi_n : M_n(X) \to M_{2n}(\mathcal{D})$, where

$$\Phi_n(x) = \begin{pmatrix} 0 & \phi_n(x) \\ \phi_n(x)^* & 0 \end{pmatrix}$$

for all $x \in M_n(X)$.

Then $\|\Phi_n(x)\|_{cb} \le \|\phi\|_{cb}\ \|x\|_n$ for all $x \in M_n(X)$. For each $x \in M_n(Y)$, let $\theta_n(x)$ be the set of elements of $M_{2n}(\mathscr{D})$ of the form

$$\|\phi\|_{cb}\,\|z\|\begin{pmatrix}\gamma^*aa^*\gamma & 0\\ 0 & \gamma^*b^*b\gamma\end{pmatrix}+\Phi_n(y)$$

corresponding to all possible expressions of x in the form $x=\gamma^*azb\gamma+y$, where $z\in M_m(Y)$, $y\in M_n(X)$, $a\in M_m(A)$, $b\in M_m(B)$, and $\gamma\in M_{m,n}$.

For each $n\in\mathbf{N}$, let P_n be a unitary matrix in M_{2n} such that $P_n(e_k)=e_{2k-1}$, $P_n(e_{n+k})=e_{2k}$, $1\le k\le n$, where $e_1,e_2,\ldots,e_{2n}$ are the usual unit basis of $\mathbf{C}^{2n}$. Then $x\to P_nxP_n^*$ is the canonical shuffle $*$-isomorphism from $M_2(M_n(\mathscr{D}))$ onto $M_n(M_2(\mathscr{D}))$ (see [**6**]).

We claim that the family $(P_n\theta_nP_n^*)$ of set valued functionals

$$P_n\theta_nP_n^*:Y\bigotimes_{\mathbf{R}}(M_n)_h\to M_n(M_2(\mathscr{D})_h)$$

is a matrical sublinear functional $P\theta P^*:Y\to M_2(\mathscr{D})_h$. In fact,

(i) for each $x\in M_n(Y)$,

$$\|\phi\|_{cb}\,\|x\|_n\begin{pmatrix}1&0\\0&1\end{pmatrix}\in\theta_n(x).$$

Therefore $\theta_n(x)\neq\emptyset$, and hence $P_n\theta_n(x)P_n^*\neq\emptyset$.

(ii) Let $x_i=\gamma_i^*a_iz_ib_i\gamma_i+y_i$, $i=1,\ 2$, be any two expressions of x_1 and x_2; here $z_i\in M_m(Y)$, $y_i\in M_n(X)$, $a_i\in M_m(A)$, $b_i\in M_m(B)$, $\gamma_i\in M_{m,n}$, $i=1,2$. We may choose the same m for both x_1 and x_2 by adding zeros if necessary. If $\min(\|z_1\|,\|z_2\|)>0$, let $\lambda_i=\|z_i\|^{-1}\min(\|z_1\|,\|z_2\|)$, $i=1,2$. In this case set

$$\gamma=\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix},\qquad a=\begin{pmatrix}\lambda_1^{-1/2}a_1 & 0\\ 0 & \lambda_2^{-1/2}a_2\end{pmatrix},\qquad b=\begin{pmatrix}\lambda_1^{-1/2}b_1 & 0\\ 0 & \lambda_2^{-1/2}b_2\end{pmatrix},\qquad z=\begin{pmatrix}\lambda_1z_1 & 0\\ 0 & \lambda_2z_2\end{pmatrix},$$

and $y=y_1+y_2$. Then $x_1+x_2=\gamma^*azb\gamma+y$, and $\|z\|=\min(\|z_1\|,\|z_2\|)$. Moreover,

$$\|\phi\|_{cb}\,\|z\|\begin{pmatrix}\gamma^*aa^*\gamma & 0\\ 0 & \gamma^*b^*b\gamma\end{pmatrix}\le\sum_1^2\|\phi\|\,\|z_i\|\begin{pmatrix}\gamma_i^*a_ia_i^*\gamma_i & 0\\ 0 & \gamma_i^*b_i^*b_i\gamma_i\end{pmatrix}. \tag{3.5}$$

If $\min(\|z_1\|,\|z_2\|)=0$, let $\mathrm{sign}(x)=0$, when $x=0$, $\mathrm{sign}(x)=1$, when $x>0$, $\mathrm{sign}(x)=-1$, when $x<0$. In this case set

$$\gamma=\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix},\qquad a=\begin{pmatrix}\mathrm{sign}(\|z_1\|)a_1 & 0\\ 0 & \mathrm{sign}(\|z_2\|)a_2\end{pmatrix},$$

$$b=\begin{pmatrix}\mathrm{sign}(\|z_1\|b_1 & 0\\ 0 & \mathrm{sign}(\|z_2\|)b_2\end{pmatrix},\qquad z=\begin{pmatrix}z_1 & 0\\ 0 & z_2\end{pmatrix},$$

and $y=y_1+y_2$. Then $x_1+x_2=\gamma^*azb\gamma+y$, $\|z\|=\max(\|Z_1\|,\|z_2\|)$, and (3.5) still holds in this case. Therefore it follows that $\theta_n(x_1+x_2)\le\theta_n(x_1)+\theta_n(x_2)$. Consequently, $P_n\theta_n(x_1+x_2)P_n^*\le P_n\theta_n(x_1)P_n^*+P_n\theta_n(x_2)P_n^*$.

(iii) Let 0 be written as $\gamma^*azb\gamma+y$, where $z\in M_m(Y)$, $y\in M_n(X)$, $a\in M_m(A)$,

$b \in M_m(B)$, $\gamma \in M_{m,n}$. For each $\epsilon > 0$, let $b_\epsilon = (\gamma^* b^* b\gamma + \epsilon)^{-1/2} \in M_n(B)$, $a_\epsilon = (\gamma^* aa^*\gamma + \epsilon)^{-1/2} \in M_n(A)$. Then $y_\epsilon = a_\epsilon y b_\epsilon \in M_n(X)$, and

$$\begin{aligned}
\|y_\epsilon\|_n &= \|a_\epsilon y b_\epsilon\|_n = \|a_\epsilon \gamma^* azb\gamma b_\epsilon\|_n \\
&\le \|a_\epsilon \gamma^* a\| \, \|z\|_m \, \|b\gamma b_\epsilon\| \\
&= \|(\gamma^* aa^*\gamma + \epsilon)^{-1/2}\gamma^* aa^*\gamma(\gamma^* aa^*\gamma + \epsilon)^{-1/2}\|^{1/2} \times \|z\|_m \\
&\quad \times \|(\gamma^* b^* b\gamma + \epsilon)^{-1/2}\gamma^* b^* b\gamma(\gamma^* b^* b\gamma + \epsilon)^{-1/2}\|^{1/2} \\
&= \|(\gamma^* aa^*\gamma + \epsilon)^{-1}\gamma^* aa^*\gamma\|^{1/2} \, \|z\|_m \, \|(^* b^* b\gamma + \epsilon)^{-1}\gamma^* b^* b\gamma\|^{1/2} \\
&\le \|z\|_m.
\end{aligned}$$

Therefore

$$\|a_\epsilon \phi_n(y) b_\epsilon\|_n = \|\phi_n(y_\epsilon)\|_n \le \|\phi\|_{cb} \, \|z\|_m,$$

and

$$\|\phi\|_{cb} \, \|z\|_m \begin{pmatrix} \gamma^* aa^*\gamma & 0 \\ 0 & \gamma^* b^* b\gamma \end{pmatrix} + \Phi_n(y) = \begin{pmatrix} \|\phi\|_{cb} \, \|z\|_m \, \gamma^* aa^*\gamma & \phi_n(y) \\ \phi_n(y)^* & \|\phi\|_{cb} \, \|z\|_m \, \gamma^* b^* b\gamma \end{pmatrix}$$

is an arbitrary element of $\theta_n(0)$. Since $\|\phi\|_{cb} \, \|z\|_m \, \gamma^* aa^*\gamma \ge 0$, $\|\phi\|_{cb} \, \|z\|_m \, \gamma^* b^* b\gamma \ge 0$, and for each $\epsilon > 0$, we have

$$\begin{aligned}
&\|(\|\phi\|_{cb} \, \|z\|_m \, \gamma^* aa^*\gamma + \epsilon)^{-1/2}\phi_n(y)(\|\phi\|_{cb} \, \|z\|_m \, \gamma^* b^* b\gamma + \epsilon)^{-1/2}\| \\
&\quad = \|(\|\phi\|_{cb} \, \|z\|_m)^{-1/2} a_{\epsilon'} \phi_n(y) b_{\epsilon'} (\|\phi\|_{cb} \, \|z\|_m)^{-1/2}\| \\
&\quad = (\|\phi\|_{cb} \, \|z\|_m)^{-1} \, \|a_{\epsilon'} \phi_n(y) b_{\epsilon'}\| \le 1,
\end{aligned}$$

where $\epsilon' = (\|\phi\|_{cb} \, \|z\|_m)^{-1}\epsilon$, the typical element in $\theta_n(0)$ is nonnegative. It follows that $0 \le \theta_n(0)$. Hence $0 \le P_n\theta_n(0)P_n^*$.

(iv) For each $x = \gamma^* azb\gamma + y$, where $z \in M_m(Y)$, $y \in M_n(X)$, $a \in M_m(A)$, $b \in M_m(B)$, $\gamma \in M_{m,n}$, $x \in M_n(Y)$, and each $\lambda \in \mathbf{R}_+$, we have

$$\lambda\left(\|\phi\|_{cb} \, \|z\|_m \begin{pmatrix} \gamma^* aa^*\gamma & 0 \\ 0 & \gamma^* b^* b\gamma \end{pmatrix} + \Phi_n(y)\right) = \|\phi\|_{cb} \, \|\lambda z\|_m \begin{pmatrix} \gamma^* aa^*\gamma & 0 \\ 0 & \gamma^* b^* b\gamma \end{pmatrix} + \Phi_n(\lambda y)$$

belongs to $\theta_n(\lambda x)$, and $\lambda x = \gamma^* a(\lambda z)b\gamma + \lambda y$. Therefore $\theta_n(\lambda x) \le \lambda\theta_n(x)$. Hence $P_n\theta_n(\lambda x)P_n^* \le \lambda P_n\theta_n(x)P_n^*$.

(v) We need to prove that for any $x \in M_n(Y)$ and $\gamma \in M_{n,m}$ we have

$$P_m\theta_m(\gamma^* x\gamma)P_m^* \le \gamma^* P_n\theta_n(x)P_n^*\gamma.$$

It is equivalent to prove that

$$\theta_m(\gamma^* x\gamma) \le P_m^*\gamma^* P_n\theta_n(x)P_n^*\gamma P_m. \tag{3.6}$$

Notice that $P_n\theta_n(x)P_n^* \in M_n(M_2(\mathscr{D})_h)$. Write $\gamma = (\gamma_{i,j})$ and $P_n\theta_n(x)P_n^* = (A_{i,j})$, $A_{i,j} \in M_2(\mathscr{D})_h$. Then

$$\gamma^* P_n\theta_n(x)P_n^*\gamma = (\gamma \otimes 1_2)^*(A_{i,j})(\gamma \otimes 1_2),$$

where 1_2 is the unit of $M_2(\mathscr{D})$. Therefore (3.6) becomes

$$\begin{aligned}\theta_m(\gamma^*x\gamma) &\le P_m^*\gamma^*P_n\theta_n(x)P_n^*\gamma P_m\\ &= P_m^*(\gamma\otimes 1_2)^*P_n\theta_n(x)P_n^*(\gamma\otimes 1_2)P_m\\ &= (P_n^*(\gamma\otimes 1_2)P_m)^*\theta_n(x)(P_n^*(\gamma\otimes 1_2)P_m)\\ &= \begin{pmatrix}(\gamma_{i,j}) & 0\\ 0 & (\gamma_{i,j})\end{pmatrix}^*\theta_n(x)\begin{pmatrix}(\gamma_{i,j}) & 0\\ 0 & (\gamma_{i,j})\end{pmatrix}\\ &= \begin{pmatrix}\gamma^* & 0\\ 0 & \gamma^*\end{pmatrix}\theta_n(x)\begin{pmatrix}\gamma & 0\\ 0 & \gamma\end{pmatrix}.\end{aligned}$$

Therefore we need to prove that

$$\theta_m(\gamma^*x\gamma) \le \begin{pmatrix}\gamma^* & 0\\ 0 & \gamma^*\end{pmatrix}\theta_n(x)\begin{pmatrix}\gamma & 0\\ 0 & \gamma\end{pmatrix}.$$

In fact, for any $x\in M_n(Y)$, each element in $\theta_n(x)$ is of the form

$$\|\phi\|_{cb}\,\|z\|_k\begin{pmatrix}\mu^*aa^*\mu & 0\\ 0 & \mu^*b^*b\mu\end{pmatrix} + \Phi_n(y),$$

where $x = \mu^*azb\mu + y$, $z\in M_k(Y)$, $y\in M_n(X)$, $a\in M_k(A)$, $b\in M_k(B)$, $\mu\in M_{k,n}$, $n, m\in\mathbf{N}$. However

$$\begin{aligned}&\begin{pmatrix}\gamma^* & 0\\ 0 & \gamma^*\end{pmatrix}\left(\|\phi\|_{cb}\,\|z\|_k\begin{pmatrix}\mu^*aa^*\mu & 0\\ 0 & \mu^*b^*b\mu\end{pmatrix} + \Phi_n(y)\right)\begin{pmatrix}\gamma & 0\\ 0 & \gamma\end{pmatrix}\\ &\qquad = \|\phi\|_{cb}\,\|z\|_k\begin{pmatrix}(\mu\gamma)^*aa^*(\mu\gamma) & 0\\ 0 & (\mu\gamma)^*b^*b(\mu\gamma)\end{pmatrix} + \Phi_m(\gamma^*y\gamma)\in\theta_m(\gamma^*x\gamma),\end{aligned}$$

where $\gamma^*x\gamma = (\mu\gamma)^*azb(\mu\gamma) + (\gamma^*x\gamma)$. Thus (3.6) is fulfilled.

We may now apply the Lemma 3.3, since $Y\otimes_{\mathbf{R}}(M_n)_h = M_n(Y)$ as real space, and $M_2(\mathscr{D})$ is an injective C*-algebra (see **[10]**), to assert that there exists a real linear map $\Psi: Y\to M_2(\mathscr{D})_h$ such that $\Psi_n(x)\le P_n\theta_n(x)P_n^*$ for all $x\in M_n(Y)$. If we write

$$\Psi(x) = \begin{pmatrix}\Psi_{1,1}(x) & \Psi_{1,2}(x)\\ \Psi_{2,1}(x) & \Psi_{2,2}(x)\end{pmatrix},$$

then we have

$$\begin{pmatrix}(\Psi_{1,1})_n(x) & (\Psi_{1,2})_n(x)\\ (\Psi_{2,1})_n(x) & (\Psi_{2,2})_n(x)\end{pmatrix} = P_n^*\Psi_n(x)P_n \le \theta_n(x) \tag{3.7}$$

for all $x\in M_n(Y)$. It follows that for any $x\in M_n(Y)$, $a\in M_n(A)$, and $b\in M_n(B)$, we have

$$\begin{pmatrix}(\Psi_{1,1})_n(axb) & (\Psi_{1,2})_n(axb)\\ (\Psi_{2,1})_n(axb) & (\Psi_{2,2})_n(axb)\end{pmatrix} \le \|\phi\|_{cb}\,\|x\|_n\begin{pmatrix}aa^* & 0\\ 0 & b^*b\end{pmatrix}. \tag{3.8}$$

If we multiply this inequality on the left by $(1\ 0)$, and on the right by $\begin{pmatrix}1\\0\end{pmatrix}$, we obtain

$$(\Psi_{1,1})_n(axb) \le \|\phi\|_{cb}\,\|x\|_n aa^*. \tag{3.9}$$

Similarly, by multiplying (3.8) on the left by (0 1) and on the right by $\begin{pmatrix}0\\1\end{pmatrix}$, we get

$$(\Psi_{2,2})_n(axb) \le \|\phi\|_{cb}\ \|x\|_n\ b^*b, \tag{3.10}$$

for any $x \in M_n(Y)$, $a \in M_n(A)$, $b \in M_n(B)$. Now in (3.9), let $a = 1$, $b = k1$, $k \in \mathbf{N}$; and in (3.10), let $b = 1$, $a = k1$, $k \in \mathbf{N}$. Then

$$k\ \|(\Psi_{1,1})_n(x)\|_n \le \|\phi\|_{cb}\ \|x\|_n$$

and

$$k\ \|(\Psi_{2,2})_n(x)\|_n \le \|\phi\|_{cb}\ \|x\|_n$$

for all $x \in M_n(Y)$, $k \in \mathbf{N}$. This implies that $\Psi_{1,1} = \Psi_{2,2} = 0$. So (3.8) becomes

$$\begin{pmatrix} 0 & (\Psi_{1,2})_n(axb) \\ (\Psi_{2,1})_n(axb) & 0 \end{pmatrix} \le \|\phi\|_{cb}\ \|x\|_n \begin{pmatrix} aa^* & 0 \\ 0 & b^*b \end{pmatrix}.$$

Moreover, the inequality

$$\begin{pmatrix} \|\phi\|_{cb}\ \|x\|_n aa^* & -(\Psi_{1,2})_n(axb) \\ -a\Psi_{2,1})_n(axb) & \|\phi\|_{cb}\ \|x\|_n b^*b \end{pmatrix} \ge 0$$

implies that $(\Psi_{2,1})_n(axb) = (\Psi_{1,2})_n(axb)^*$. Applying Theorem 2.1, we conclude that $\Psi_{1,2}: Y \to \mathcal{D}$ is a completely bounded A-B bimodule map with $\|\Psi_{1,2}\|_{cb} \le \|\phi\|_{cb}$.

Now for any $x \in M_n(X)$, (3.7) implies that

$$\begin{pmatrix} 0 & (\Psi_{1,2})_n(x) \\ (\Psi_{1,2})_n(x)^* & 0 \end{pmatrix} \le \begin{pmatrix} 0 & \phi_n(x) \\ \phi_n(x)^* & 0 \end{pmatrix}.$$

It follows that for any $\epsilon > 0$, we have

$$\|(0+\epsilon)^{-1/2}(\phi_n(x) - (\Psi_{1,2})_n(x))(0+\epsilon)^{-1/2}\| \le 1.$$

Equivalently,

$$\|\phi_n(x) - (\Psi_{1,2})_n(x)\| \le \epsilon$$

for all $x \in M_n(X)$. Since $\epsilon > 0$ is arbitrary, we have $\phi_n(x) = (\Psi_{1,2})_n(x)$ for all $x \in M_n(X)$. Therefore $\Psi_{1,2}$ extends ϕ and has same cb-norm $\|\phi\|_{cb}$. This completes the proof.

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