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SOME PROPERTIES OF INFLATED BINOMIAL DISTRIBUTION

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1. **Introduction.** The role of binomial distribution in the probability theory and its applications is well known. Recently M. P. Singh [4] has been discussing the so-called inflated binomial distribution. This distribution was introduced for situations which are described by simple binomial except for zero cell which is inflated, that is, there more observations than can be expected on the basis of simple binomial. The investigation of this inflated binomial distribution seems to be useful.

The aim of this note is to give the distribution of sums of random variables in the case of inflated binomial distribution. The result which was obtained in [1] and [2] is a particular case of our results.

The random variable X is said to have the inflated binomial distribution with parameters λ, p, N, if

$$(1)\qquad p(x; \lambda, p, N) = P[X = x] = \begin{cases} 1-\lambda+\lambda q^N & \text{for} \quad x = 0, \\ \lambda \binom{N}{x} p^x q^{N-x} & \text{for} \quad x = 1, 2, \ldots, N, \end{cases}$$

where $0<\lambda\leq 1$, $0<p<1$, $p+q=1$.

2. **Distribution of the sums.** It is well known that the sum of the independent and identically distributed random variables having the binomial probability function is also binomially distributed. In the case of the independent random variables having the inflated binomial distribution the case is different.

THEOREM 1. *If $X_1, X_2, \ldots, X_m$ are the independent random variables having the same inflated binomial distribution* (1), *and if $Z=X_1+X_2+\cdots+X_m$, then*

$$(2)\quad P[Z = z] = \begin{cases} (1-\lambda+\lambda q^N)^m & \text{for} \quad z = 0, \\ \sum_{j=1}^{m} \lambda^j (1-\lambda)^{m-j} \binom{m}{j} \binom{Nj}{z} p^z q^{Nj-z} & \text{for} \quad z = 1, 2, \ldots, Nm. \end{cases}$$

Proof. It is easy to verify that characteristic function of (2) is given by

$$\varphi_Z(t) = [1-\lambda+\lambda(pe^{it}+q)^N]^m.$$

Using the inversion formula for characteristic functions, we obtain for $z=0$

$$P[Z = 0] = [1-\lambda(1-q^N)]^m,$$

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and for $z=1, 2, \ldots, Nm$

$$P[Z = z] = \sum_{j=1}^{m} \lambda^j(1-\lambda)^{m-j}\binom{m}{j}\binom{Nj}{z}p^z q^{Nj-z}.$$

H. J. Malik [2] and J. C. Ahuja [1] have investigated distribution of the sum of truncated binomial variates. One can observe that the probability function of truncated at the point 0 inflated binomial distribution

$$P[X = x] = \frac{\lambda\binom{N}{x}p^x q^{N-x}}{1-(1-\lambda+\lambda q^N)} = \frac{\binom{N}{x}p^x q^{N-x}}{1-q^N} \quad \text{for} \quad x = 1, 2, \ldots, N$$

is the same as the probability function of truncated simple binomial distribution. Hence, the distribution of their sums is as given in [1].

Now, we are going to discuss the distribution of the sum of the sums of truncated inflated binomial variates.

THEOREM 2. *Let $Z_1, Z_2, \ldots, Z_n$ be n independent and identically distributed random variables having probability function*

$$p(z; \lambda, p, N, m) = \sum_{j=1}^{m} \lambda^j(1-\lambda)^{m-j}\binom{m}{j}\binom{Nj}{z}p^z q^{Nj-z} \Big/ (1-[1-\lambda(1-q^N)]^m)$$

for $z=1, 2, \ldots, Nm$, where $0<\lambda\leq 1$, $0<p<1$, $p+q=1$ with N as a positive integer number. If $Y=Z_1+Z_2+\cdots+Z_n$, then the probability function of the random variable Y is given by

$$\text{(3)} \quad P[Y = y] = \{1-[1-\lambda(1-q^N)]^m\}^{-n}\sum_{r=1}^{n}\sum_{s=1}^{mr}(-1)^{n-r}\lambda^s(1-\lambda)^{mr-s}$$
$$\times [1-\lambda(1-q^N)]^{m(n-r)}\binom{n}{r}\binom{mr}{s}\binom{Ns}{y}p^y q^{Ns-y}$$
$$\text{for} \quad y = n, n+1, \ldots, Nmn$$

and 0 otherwise.

Proof. Since, the characteristic function of the random variables Z_i, $i=1, 2, \ldots, n$ is given by

$$\varphi_{Z_i}(t) = \sum_{z=1}^{Nm} e^{itz}\sum_{j=1}^{m}\lambda^j(1-\lambda)^{m-j}\binom{m}{j}\binom{Nj}{z}p^z q^{Nj-z} \Big/ (1-[1-\lambda(1-q^N)]^m)$$
$$= \{[1-\lambda+\lambda(pe^{it}+q)^N]^m-[1-\lambda(1-q^N)]^m\}/(1-[1-\lambda(1-q^N)]^m)$$

so, by means of the formula $\varphi_Y(t)=[\varphi_Z(t)]^n$, we have

$$\text{(4)} \quad \varphi_Y(t) = \{1-[1-\lambda(1-q^N)]^m\}^{-n}\cdot\{[1-\lambda+\lambda(pe^{it}+q)^N]^m-[1-\lambda(1-q^N)]^m\}^n.$$

By (4) and the inversion formula for characteristic functions, we obtain

$$P[Y=y]=\{1-[1-\lambda(1-q^N)]^m\}^{-n}\cdot\lim_{k\to\infty}\frac{1}{2k}\int_{-k}^{k}e^{-ity}\sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}$$

$$\times[1-\lambda+\lambda(pe^{it}+q)^N]^{mr}[1-\lambda(1-q^N)]^{m(n-r)}\,dt$$

$$=\{1-[1-\lambda(1-q^N)]^m\}^{-n}\sum_{r=1}^{n}\sum_{s=1}^{mr}(-1)^{n-r}\lambda^s(1-\lambda)^{mr-s}$$

$$\times[1-\lambda(1-q^N)]^{m(n-r)}\binom{n}{r}\binom{mr}{s}\binom{Ns}{y}p^yq^{Ns-y}$$

for $y=n, n+1, \ldots, Nmn$ and 0 otherwise.

It may be easily seen that the distribution function of Y is obtained as

$$F(y)=1-\sum_{x=y+1}^{Nmn}\{1-[1-\lambda(1-q^N)]^m\}^{-n}\sum_{r=1}^{n}\sum_{s=1}^{mr}(-1)^{n-r}\lambda^s(1-\lambda)^{mr-s}\tag{5}$$

$$\times[1-\lambda(1-q^N)]^{m(n-r)}\binom{n}{r}\binom{mr}{s}\binom{Ns}{x}p^xq^{Ns-x}$$

$$=1-\{1-[1-\lambda(1-q^N)]^m\}^{-n}\sum_{r=1}^{n}\sum_{s=1}^{mr}(-1)^{n-r}\lambda^s(1-\lambda)^{mr-s}$$

$$\times[1-\lambda(1-q^N)]^{m(n-r)}\binom{n}{r}\binom{mr}{s}T_p(y+1, Ns-y),$$

where $T_p(y+1, Ns-y)$ is the incomplete beta function tabulated by K. Pearson [4].

For $m=1$ and $\lambda=1$ the distribution (3) is identically the same as (6) in [1], and the distribution function (5) is the same as (7) there.

References

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