# MATRIX TRANSFORMATIONS IN AN INCOMPLETE SPACE

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Let $X = (X, p)$ be a seminormed complex linear space with zero $\theta$. Natural definitions of convergent sequence, Cauchy sequence, absolutely convergent series, etc., can be given in terms of the seminorm $p$. Let us write $C = C(X)$ for the set of all convergent sequences $\bar{x} = (x_k)$, $x_k \in X$; $\mathfrak{C}$ for the set of Cauchy sequences; and $L_\infty$ for the set of all bounded sequences. One has $C \subseteq \mathfrak{C} \subseteq L_\infty$ and $L_\infty$ becomes a seminormed space with the natural addition and scalar multiplication for sequences $\bar{x}$, with $\bar{\theta} = (\theta, \theta, \ldots)$ and seminorm $\bar{p}(\bar{x}) = \sup_n p(x_n)$. The spaces $C$ and $\mathfrak{C}$ may be seminormed with the $L_\infty$ seminorm.

In the usual way one says that $X$ is complete if and only if $C = \mathfrak{C}$, or what amounts to the same thing, if and only if every absolutely convergent series is convergent. When $X$ is complete, so are $C$ and $L_\infty$ under the given seminorm. Most of the Toeplitz theory on transformations of sequences seems to have been carried out for the case in which $X$ is the complex plane, but many of the important results are still valid in any complete seminormed space $X$.

Let us write $(C, C)$ for the set of all conservative infinite matrices $A = (a_{nk})$ of complex numbers, i.e., $A$ is in $(C, C)$ if and only if $A_n(x) = \sum_k a_{nk} x_k$ converges for each $n$ to an element of $X$, and also $(A_n) \in C$, whenever $\bar{x} \in C$. The set of Toeplitz matrices, which leave the limit of $\bar{x}$ invariant, is denoted by $(C, C)_P$. In general, for sequence spaces $E$ and $F$ we denote by $(E, F)$ the set of all $A$ such that $\sum a_{nk} x_k$ converges for each $n$ to an element of $X$ and $(A_n) \in F$ whenever $\bar{x} \in E$.

By $\phi$ we denote the space of finite sequences of complex numbers, i.e., sequences which have only a finite number of non-zero coordinates, and $R$ denotes the set of row-finite infinite matrices, i.e., matrices whose rows are in $\phi$.

When $X$ is complete, the usual conditions for $A$ to be in $(C, C)_P$ (or $(C, C)$) are valid. For example, $A \in (C, C)_P$ if and only if

$$||A|| = \sup_n \sum |a_{nk}| < \infty, \quad a_{nk} \to 0 \ (n \to \infty, k \text{ fixed}), \quad \sum a_{nk} \to 1 \ (n \to \infty).$$

(See, for example, **(3; 4; 5)**, where, more generally, matrices of operators are used rather than matrices of complex numbers.)

The purpose of this note is to consider regularity conditions and related matters in an incomplete space. Without the assumption of completeness, the results of **(3; 4; 5)** mentioned above are false, as the following example shows.

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Let $(X, p)$ be incomplete, so that $C \subset \mathfrak{C}$, with strict inclusion. Then the usual conditions for $A$ to be conservative are not sufficient in $X$. Let us define the conservative lower semi-matrix $A$ by $a_{nk} = 2^{-k}$, $0 \leqslant k \leqslant n$; $a_{nk} = 0$, $k > n$. Now there is a sequence $\bar{y} \in \mathfrak{C} - C$, whence there are positive integers $m_1 < m_2 < m_3 < \ldots$ such that

$$p(y_{m_{k+1}} - y_{m_k}) < 1/k2^k \qquad (k = 1, 2, \ldots).$$

Set $x_0 = y_{m_1}$, $x_k = (y_{m_{k+1}} - y_{m_k})2^k$ $(k \geqslant 1)$, so that $x_k \to \theta$ and

$$\sum_{k=0}^{n} a_{nk} x_k = y_{m_{n+1}}.$$

By the condition on $\bar{y}$ the subsequence $(y_{m_{n+1}})$ is divergent, whence $(A_n(\bar{x}))$ is divergent, even though $A$ is conservative and $\bar{x}$ converges.

In what follows we suppose that $(X, p)$ is incomplete. In the next lemma we give the convergence factor theorem which enables us to say when a matrix $A$ applies to a convergent sequence in $X$.

LEMMA 1. *$\sum a_k x_k$ converges whenever $\bar{x} = (x_k) \in C$ if and only if $a \in \phi$.*

*Proof.* The sufficiency is trivial. For the necessity let us suppose that $a \notin \phi$. Then there is a sequence of positive integers $m_1 < m_2 < \ldots$ such that $a_{m_k} \neq 0$. Also, there is a sequence $\bar{y} \in \mathfrak{C} - C$, whence there exist positive integers $n_1 < n_2 < \ldots$ such that

$$p(y_{n_{k+1}} - y_{n_k}) < |a_{m_k}|/k \qquad (k = 1, 2, \ldots).$$

Define $x_{m_1} = y_{n_1}/a_{m_1}$, $x_{m_k} = (y_{n_k} - y_{n_{k-1}})/a_{m_k}$ $(k \geqslant 2)$ and $x_n = \theta$ $(n \neq m_k)$, so that $x_n \to \theta$. Then $\sum a_k x_k$ diverges, since its sequence of partial sums has the divergent subsequence with general term

$$a_1 x_1 + a_2 x_2 + \ldots + a_{m_p} x_{m_p} = y_{n_p}.$$

The necessity now follows.

Before we prove the first main result (Theorem 1) we introduce the notation $u = (u_k)$ for the sequence of column limits of the conservative matrix $A$, i.e., $u_k = \lim_n a_{nk}$ $(k = 0, 1, \ldots)$. We may note that when $X$ is complete the assertion that $A$ is conservative implies that $u \in l_1$. It will be seen shortly that row-finiteness of $A$ and finiteness of $u$ are essential for $A$ to be conservative in an incomplete space.

For convenience we write $T$ for $(C, C)$ and $T_P$ for $(C, C)_P$ when $C$ is the space of convergent sequences of complex numbers, reserving $(C, C)$ and $(C, C)_P$ for the case of the incomplete space $X$.

THEOREM 1. *$A \in (C, C)$ if and only if $A \in R \cap T$ and $u \in \phi$.*

*Proof.* The sufficiency is essentially trivial and the proof is omitted.

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Next consider the necessity. If, for each $n$, $\sum a_{nk} x_k$ is to converge to an element of $X$ whenever $\bar{x} \in C$ it is necessary by Lemma 1 that

$$a^{(n)} = (a_{n0}, a_{n1}, \ldots) \in \phi,$$

i.e., $A \in R$. Thus $a_{nm} = 0$ $(m > f(n))$, so let us define, for each fixed $k \leqslant f(n)$, $x_m = \theta$ $(0 \leqslant m < k)$, $x_m = y/p(y)$ $(m \geqslant k)$, where $p(y) > 0$. Then $(A_n(\bar{x}))$ converges and so is a Cauchy sequence. By the "absolute homogeneity" of the seminorm $p$ we see that

$$\left(\sum_{m=k}^{f(n)} a_{n,m}\right)$$

is a Cauchy sequence for each fixed $k$, and hence is convergent. It remains to show that

$$||A|| = \sup_n \sum |a_{nk}| < \infty.$$

Here the Banach-Steinhaus theorem, so useful in the complete case, is not available; we do not know that $C$ is of the second category; a priori it may be of the first category. However, the original argument used by Toeplitz **(6)** can easily be adapted to show that $||A|| < \infty$, and this completes the proof that $A \in R \cap T$.

Since $A \in R$, we have $a_{nk} = 0$ $(k > f(n))$. Consider two cases: (i) $(f(n))$ bounded, (ii) $(f(n))$ unbounded. In (i) we have $a_{nk} = 0$ $(k > \max f(n))$, whence $u \in \phi$. Now suppose that (ii) holds but that $u \notin \phi$. We are then assuming that $(A_n(\bar{x})) \in C$ whenever $\bar{x} \in C$, which implies that $(A_n(\bar{x})) \in C$ whenever $\bar{x}$ converges to $\theta$ and since $A \in R \cap T$ it follows that the sequence

$$\left(\sum_{m=0}^{f(n)} u_m x_m\right)_{n=1,2,\ldots} \tag{1}$$

converges whenever $(x_m)$ converges to $\theta$. Using the proof of Lemma 1, with $u$ in place of $a$, we construct $\bar{x}$ in terms of the Cauchy sequence $\bar{y}$ and then extract from (1) a subsequence of $\bar{y}$, which will converge since (1) converges. This contradicts the fact that $\bar{y} \in \mathfrak{C} - C$ and so completes the proof of Theorem 1.

An immediate corollary to Theorem 1 is that $A \in (C, C)_P$ if and only if $A \in R \cap T_P$.

Perhaps it is worth mentioning at this stage that the restriction that $A$ should be row-finite is not especially heavy. Even in the case when $X$ is complete there are many useful row-finite matrices in $(C, C)_P$, e.g., the Riesz and Cesàro methods. However, the Abel method loses its significance when $X$ is incomplete.

Next we turn to mappings of $\mathfrak{C}$ into itself and prove

THEOREM 2. *$A \in (\mathfrak{C}, \mathfrak{C})$ if and only if $A \in R \cap T$.*

*Proof.* First consider the necessity. By Lemma 1 we must have $A \in R$.

Now suppose that $(A_n(\bar{x})) \in \mathfrak{C}$ whenever $\bar{x} \in \mathfrak{C}$. Using the proof of Theorem 1 we find that the convergence of the sequence

$$\left(\sum_{m=k}^{f(n)} a_{n,m}\right)$$

for each fixed $k$ is necessary. We shall show that $||A|| < \infty$ by supposing the contrary and exhibiting a Cauchy sequence (in fact a null sequence) such that $(A_n(\bar{x}))$ has an unbounded subsequence. This argument is just a slight modification of Toeplitz' original argument mentioned in Theorem 1, but we might perhaps indicate it for completeness.

If $||A|| = \sup_n \sum |a_{rk}| = \infty$, we determine $g(k) = f(n_k)$ such that the sequence $(g(k))$ increases to infinity and such that

$$\sum_0^{g(1)} |a_{n1,k}| > 2^2, \qquad \sum_0^{g(i)} |a_{ni,k}| > (2M_{g(i-1)} + 2^i)\cdot 2^i \quad (i \geqslant 2),$$

where, since $a_{nk} \to u_k$ $(n \to \infty)$,

$$M_s = \sup_n \sum_{k=0}^{s} |a_{nk}| < \infty,$$

for each fixed $s$. Now take a fixed $y \in X$ such that $p(y) > 0$ and define $\bar{x} = (x_k)$ by

$$x_k = \frac{y}{2}\operatorname{sgn} a_{n1,k} \qquad (0 \leqslant k \leqslant g(1)),$$

$$x_k = \frac{y}{2^i}\operatorname{sgn} a_{ni,k} \qquad (g(i-1) < k \leqslant g(i);\, i \geqslant 2).$$

Then $\bar{p}(\bar{x}) < p(y)$ and for $i \geqslant 1$,

$$\begin{aligned} p\left(\sum_0^{g(i)} a_{ni,k}\, x_k\right) &\geqslant \sum_{g(i-1)+1}^{g(i)} \frac{p(y)}{2^i} |a_{ni,k}| - \sum_0^{g(i-1)} p(y)|a_{ni,k}| \\ &\geqslant \frac{p(y)}{2^i} \sum_0^{g(i)} |a_{ni,k}| - 2p(y)M_{g(i-1)} \\ &\geqslant p(y)\cdot 2^i \to \infty \quad (i \to \infty), \end{aligned}$$

whence $||A|| < \infty$ is necessary.

Now consider the sufficiency. We have

$$A_n(\bar{x}) = \sum_0^{\infty} a_{nk}\, x_k,$$

where $\bar{x} \in \mathfrak{C}$ and $A \in R \cap T$. The sufficiency follows on writing

$$A_m(\bar{x}) - A_n(\bar{x}) = \sum_0^{\infty} (a_{mk} - a_{nk})(x_k - x_n) + (A_m - A_n)x_n,$$

where

$$A_n = \sum_0^\infty a_{nk}.$$

THEOREM 3. *$A \in (L_\infty, L_\infty)$ if and only if $A \in R$ and $||A|| < \infty$.*

*Proof.* The sufficiency is trivial and the necessity follows from the proof of Theorem 2.

THEOREM 4. *$A \in (L_\infty, C)$ if and only if $A \in R \cap T^*$ and $u \in \phi$, where $T^*$ is the set of all $A = (a_{nk})$ such that $\lim_n a_{nk} = u_k$ and $\sum|a_{nk}|$ converges uniformly in $n$.*

*Proof.* The sufficiency is immediate and since $(L_\infty, C) \subseteq (C, C)$ the necessity of $A \in R \cap T$ and $u \in \phi$ also follows. To show that $A \in T^*$ it is enough to show that $A \in (L_\infty, C)$ with $a_{nk} \to 0$ implies $\sum|a_{nk}| \to 0$. For, in the general case we replace $(a_{nk})$ by $(a_{nk} - u_k)$ and then $\sum|a_{nk} - u_k| \to 0$ implies $\sum|a_{nk}| \to \sum|u_k|$, whence by Dini's theorem $\sum|a_{nk}|$ is uniformly convergent.† It is now easy to adapt the argument of (**1**, Theorem 3) to show that $A \in T^*$ is necessary for $A \in (L_\infty, C)$. For, on the assumption that

$$\limsup_n \sum|a_{nk}| > 0$$

one can construct a bounded sequence $\bar{x}$ such that $(A_n(\bar{x}))$ has a subsequence which is not a Cauchy sequence.

THEOREM 5. *$A \in (L_\infty, \mathfrak{C})$ if and only if $A \in R \cap T^*$.*

*Proof.* The sufficiency is straightforward. From the inclusion

$$(L_\infty, \mathfrak{C}) \subseteq (\mathfrak{C}, \mathfrak{C})$$

it follows that $A \in R \cap T$ is necessary. Now $A \in (L_\infty, \mathfrak{C})$ implies that $b_{nk} = a_{nk} - u_k \to 0$ and since $u \in l_1$ it follows that $B \in (L_\infty, \mathfrak{C})$. On the assumption that $\sum|b_{nk}| \not\to 0$ we can construct, as in Theorem 4, a bounded sequence $\bar{x}$ such that $A_n(\bar{x})$ has a non-Cauchy subsequence. Hence, as in Theorem 4, we deduce that $A \in T^*$.

In the next theorem we shall denote by $N$ the set of all conull‡ matrices $A$, i.e., the set of all $A \in T$ such that

$$\lim_n \sum a_{nk} = \sum u_k.$$

THEOREM 6. *$A \in (\mathfrak{C}, C)$ if and only if $A \in R \cap N$ and $u \in \phi$.*

*Proof.* If $u_k = 0$ for $k > M$ and $\bar{x} \in \mathfrak{C}$, we write

$$(2) \qquad A_n(\bar{x}) = x_n \sum_{k=0}^{f(n)} (a_{nk} - u_k) + \sum_{k=0}^{f(n)} (a_{nk} - u_k)(x_k - x_n) + \sum_{k=0}^{M} u_k x_k.$$

†This is the essence of the argument given in (**1**, Theorem 3).
‡See Wilansky (**7**).

When $A \in N$, it follows that $(A_n(\bar{x}))$ converges to $\sum_0^M u_k x_k$, whence the sufficiency.

Since $(\mathfrak{C}, C) \subseteq (C, C)$, it follows from Theorem 1 that $A \in R \cap T$ and $u \in \phi$ are necessary. Hence from (2) the convergence of $(A_n(\bar{x}))$ for all $\bar{x} \in \mathfrak{C}$ implies that

$$\left(x_n \sum_0^{f(n)} (a_{nk} - u_k)\right)$$

converges.

Now write

$$a_n = \sum_0^{f(n)} (a_{nk} - u_k),$$

so that $(a_n x_n) \in C$ whenever $\bar{x} \in \mathfrak{C}$. Clearly this implies that $\lim a_n$ exists. Suppose that $\lim a_n = 2l \neq 0$. Then there is an $n_0$ such that $|a_n| > l$ for $n \geqslant n_0$. Write $b_n = a_n^{-1}$ for $n \geqslant n_0$ and take a sequence $\bar{y} \in \mathfrak{C} - C$. Now define, for $n \geqslant n_0$, $x_n = b_n y_n$ and set $x = \theta$ otherwise. Then $\bar{x} \in \mathfrak{C}$, since

$$p(x_n - x_m) \leqslant \frac{1}{l} p(y_n - y_m) + |b_n - b_m| \bar{p}(\bar{y}) \qquad (n, m \geqslant n_0),$$

but $(a_n x_n) = (y_n)$ is not convergent. Consequently it is necessary that $a_n \to 0$, which completes the proof.

The following theorem completes the solution of the problem of finding all the necessary and sufficient conditions for $A \in (E, F)$, where each of $E$, $F$ may be any of $C$, $\mathfrak{C}$, or $L_\infty$.

THEOREM 7. $(C, L_\infty) = (\mathfrak{C}, L_\infty) = (L_\infty, L_\infty)$ *and* $(C, \mathfrak{C}) = (\mathfrak{C}, \mathfrak{C})$.

*Proof.* Obviously $(L_\infty, L_\infty) \subseteq (\mathfrak{C}, L_\infty) \subseteq (C, L_\infty)$, and from the proof of Theorem 2 it follows that $A \in R$ and $||A|| < \infty$ are necessary for $A \in (C, L_\infty)$. Hence, by Theorem 3, $(C, L_\infty) \subseteq (L_\infty, L_\infty)$, which proves the first part of the theorem.

The equality of the classes $(C, \mathfrak{C})$ and $(\mathfrak{C}, \mathfrak{C})$ follows from the inclusion $(\mathfrak{C}, \mathfrak{C}) \subseteq (C, \mathfrak{C})$ and the proof of Theorem 2.

For the next two theorems, which are of classical Steinhaus type,† we employ the terminology that a sequence $\bar{x}$ is summable $A$ if $A_n(\bar{x})$ exists for each $n$ and $(A_n(\bar{x})) \in C$.

THEOREM 8. *If $N'$ denotes the set $N$ of conull matrices satisfying $u \in \phi$, then $(C, C) - N'$ and $(L_\infty, C)$ are disjoint.*

For the proof we need only observe that $A \in (L_\infty, C)$ implies that

$$\lim_n \sum a_{nk} = \sum u_k.$$

†The original theorem of Steinhaus, well known in summability theory, is that given any Toeplitz matrix $A$ there is always a bounded sequence which is not summable $A$.

As a corollary to Theorem 8 we have that $(C, C)_P$ and $(L_\infty, C)$ are disjoint, from which follows the theorem of Steinhaus type: Given any $A$ in $(C, C)_P$ there is always a bounded sequence which is not summable $A$. Using Theorem 6 we can immediately improve this last result by the following theorem.

THEOREM 9. *In an incomplete seminormed space $X$, if $A$ is a given Toeplitz matrix, i.e., $A \in (C, C)_P$, then there is always a Cauchy sequence which is not summable $A$.*

Our last result involves a generalization of the notion of "solid (or normal) sequence space" which is familiar from the work of Köthe and Toeplitz (**2**, Definition 4), the aim being to show that the summability field of a Toeplitz matrix is not solid; this result will in fact be a special case of Theorem 10.

Let $X$ be a seminormed space, not necessarily complete. A sequence space $\lambda$ over $X$ may be defined as a linear manifold in $X^\infty$ such that $\lambda$ contains the space $\phi(X)$ of finite sequences, i.e., almost all components of $\bar{x} \in \phi(X)$ are equal to $\theta$. Writing $p(\bar{x}) = (p(x_n))$ with $\bar{x} = (x_n)$ and defining $p(\bar{x}) \leqslant p(\bar{y})$ to mean $p(x_n) \leqslant p(y_n)$ for every $n$, we say that $\lambda$ is solid if $p(\bar{x}) \leqslant p(\bar{y})$ and $\bar{y} \in \lambda$ together imply $\bar{x} \in \lambda$. Thus $L_\infty$ and $C_\theta$, the space of sequences convergent to $\theta$, are solid. Unfortunately, $\phi(X)$ is not generally solid; it is solid if and only if $p$ is a norm.

Now let $A$ be a given coregular matrix, i.e., $A \in (C, C)$ and

$$\lim \sum a_{nk} - \sum u_k \neq 0,$$

and let $(A)$ be the set of $A$-summable sequences. Then we have

THEOREM 10. *$(A)$ is not solid.*

*Proof*. We have only to show that there is at least one $\bar{y} \in (A)$ and $\bar{x} \notin (A)$ such that $p(\bar{x}) \leqslant p(\bar{y})$. However, it is easy to prove slightly more. For, if we exclude the trivial case in which $p \equiv 0$, then we can show that for every $\bar{y} \in S$ there is an $\bar{x}$, depending on $\bar{y}$, with $\bar{x} \notin (A)$, such that $p(\bar{x}) \leqslant p(\bar{y})$. Here $S$ denotes the non-empty set of all $\bar{y} \in C$ such that $p(\lim y_n) > 0$. Of course when $p$ is a norm, $S = C - C_\theta$.

To prove our assertion we take any $\bar{y} \in S$ with $\lim y_n = y$, say. Then there is an $n_0$ such that $2p(y_n) > p(y)$ for $n \geqslant n_0$. Define

$$x_n = \begin{cases} \dfrac{p(y)z_n}{2\bar{p}(\bar{z})} & (n \geqslant n_0), \\ \theta & (n < n_0), \end{cases}$$

where $\bar{z} \in L_\infty - (A)$; $\bar{z}$ exists by the Steinhaus theorem. It is now clear that $\bar{x} \in L_\infty - (A)$ and $p(\bar{x}) \leqslant p(\bar{y})$.

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