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# A CHARACTERIZATION OF WEIGHTED BERGMAN-ORLICZ SPACES ON THE UNIT BALL IN $\mathbb{C}^n$

YASUO MATSUGU and JUN MIYAZAWA

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**Abstract**

Let $B$ denote the unit ball in $\mathbb{C}^n$, and $\nu$ the normalized Lebesgue measure on $B$. For $\alpha > -1$, define $d\nu_\alpha(z) = c_\alpha(1-|z|^2)^\alpha d\nu(z)$, $z \in B$. Here $c_\alpha$ is a positive constant such that $\nu_\alpha(B) = 1$. Let $H(B)$ denote the space of all holomorphic functions in $B$. For a twice differentiable, nondecreasing, nonnegative strongly convex function $\varphi$ on the real line $\mathbb{R}$, define the Bergman-Orlicz space $A_\varphi(\nu_\alpha)$ by

$$A_\varphi(\nu_\alpha) = \left\{ f \in H(B) : \int_B \varphi(\log|f|)\, d\nu_\alpha < \infty \right\}.$$

In this paper we prove that a function $f \in H(B)$ is in $A_\varphi(\nu_\alpha)$ if and only if

$$\int_B \varphi''\big(\log|f(z)|\big) \frac{|\mathscr{R} f(z)|^2}{|z|^2|f(z)|^2} \big(1-|z|^2\big)^2\, d\nu_\alpha(z) < \infty,$$

where $\mathscr{R} f(z) = \sum_{j=1}^n z_j \,\partial f(z)/\partial z_j$ is the radial derivative of $f$.

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## 1. Introduction

Let $n \geq 1$ be a fixed integer. Let $H(B)$ denote the space of all holomorphic functions in the unit ball $B \equiv B_n$ of the complex $n$-dimensional Euclidean space $\mathbb{C}^n$. Let $\nu$ denote the normalized Lebesgue measure on $B$. For each $\alpha \in (-1, \infty)$, we set $c_\alpha = \Gamma(n+\alpha+1)/\{\Gamma(n+1)\Gamma(\alpha+1)\}$ and $d\nu_\alpha(z) = c_\alpha(1-|z|^2)^\alpha d\nu(z)$, $z \in B$. Note that $\nu_\alpha(B) = 1$. Let $\mathscr{S}T^2(\mathbb{R})$ denote the class of those nondecreasing convex functions $\varphi : [-\infty, \infty) \to [0, \infty)$ which are twice differentiable in $(-\infty, \infty)$ and satisfy the

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growth condition $\lim_{t\to\infty}\varphi(t)/t=\infty$. For each $\alpha\in(-1,\infty)$ and $\varphi\in\mathscr{S}T^2(\mathbb{R})$, we define the *weighted Bergman-Orlicz space* $A_\varphi(\nu_\alpha)$ by

$$A_\varphi(\nu_\alpha)=\left\{f\in H(B):\|f\|_{A_\varphi(\nu_\alpha)}\equiv\int_B\varphi\big(\log|f|\big)\,d\nu_\alpha<\infty\right\}.$$

The *Hardy-Orlicz space* $H_\varphi(B)$ is as usual defined by

$$H_\varphi(B)=\left\{f\in H(B):\|f\|_{H_\varphi(B)}\equiv\sup_{0\le r<1}\int_S\varphi\big(\log|f_r|\big)\,d\sigma<\infty\right\},$$

where $\sigma$ is the normalized Lebesgue measure on the unit sphere $S\equiv\partial B$ and $f_r(z)=f(rz)$ for $0\le r<1$, $z\in\mathbb{C}^n$ with $rz\in B$. In 1985, Beatrous and Burbea [1] gave the following characterization of the Bergman spaces $A^p(\nu_\alpha)\equiv H(B)\cap L^p(\nu_\alpha)$, $0<p<\infty$.

THEOREM 1.1 (Beatrous and Burbea). *Let* $f\in H(B)\backslash\{0\}$, $\alpha\in(-1,\infty)$ *and let* $0<p<\infty$. *Then* $f\in A^p(\nu_\alpha)$ *if and only if*

$$\int_B|f(z)|^p\frac{|\mathscr{R}f(z)|^2}{|z|^2|f(z)|^2}\big(1-|z|^2\big)^2\,d\nu_\alpha(z)<\infty,$$

*where* $\mathscr{R}f(z)=\sum_{j=1}^n z_j\,\partial f(z)/\partial z_j$ *is the radial derivative of* $f$.

This characterization of the weighted Bergman spaces is of the same type as that of the Hardy spaces by Yamashita [8] and Stoll [6]. The purpose of the present paper is to give the characterization of the weighted Bergman-Orlicz spaces $A_\varphi(\nu_\alpha)$, $\varphi\in\mathscr{S}T^2(\mathbb{R})$, $-1<\alpha<\infty$, which is of the Beatrous-Burbea's type. Our main result (Section 4, Theorem 4.1) contains, as the limiting case $\alpha=-1$, a characterization of the Hardy-Orlicz spaces $H_\varphi(B)$, $\varphi\in\mathscr{S}T^2(\mathbb{R})$.

THEOREM 1.2. *Let* $\varphi\in\mathscr{S}T^2(\mathbb{R})$ *and* $f\in H(B)\backslash\{0\}$. *Then* $f\in H_\varphi(B)$ *if and only if*

$$\int_B\varphi''\big(\log|f(z)|\big)\frac{|\mathscr{R}f(z)|^2}{|z|^2|f(z)|^2}\big(1-|z|^2\big)\,d\nu(z)<\infty.$$

This characterization is a little bit different from that of by Ouyang and Riihentaus [2].

THEOREM 1.3 (Ouyang and Riihentaus). *Let* $\varphi\in\mathscr{S}T^2(\mathbb{R})$ *and* $f\in H(B)\backslash\{0\}$. *Then* $f\in H_\varphi(B)$ *if and only if*

$$\int_B\varphi''\big(\log|f(z)|\big)\frac{|\nabla f(z)|^2}{|f(z)|^2}\big(1-|z|^2\big)\,d\nu(z)<\infty,$$

*where* $|\nabla f(z)|^2=\sum_{j=1}^n|\partial f(z)/\partial z_j|^2$.

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We note that the results of Stoll [5] and Ouyang-Riihentaus [2] hold for more general domains in $\mathbb{C}$ and $\mathbb{C}^n$ than for $\mathbb{D}$ and $B$, respectively.

## 2. Notation

Let $\mathcal{M}$ denote the group of biholomorphic maps of $B$ onto itself. For each $a \in B$, let $\varphi_a \in \mathcal{M}$ be the involution described in [3, page 25]. Let $\lambda$ be the measure on $B$ defined by

$$d\lambda(z) = \frac{1}{(1-|z|^2)^{n+1}}\, d\nu(z), \quad z \in B.$$

Then $\lambda$ is the invariant volume measure induced by the Bergman metric on $B$. Thus

$$\int_B f\, d\lambda = \int_B (f \circ \psi)\, d\lambda$$

for each $f \in L^1(\lambda)$ and all $\psi \in \mathcal{M}$ ([3, Theorem 2.2.6]). For $f \in C^2(B)$ and $a \in B$, define

$$\tilde{\Delta} f(a) = \frac{1}{n+1}\Delta(f \circ \varphi_a)(0),$$

where $\Delta \equiv 4\sum_{j=1}^n \partial^2/\partial z_j \partial \bar{z}_j$ is the ordinary Laplacian. Then as in [3, Theorem 4.1.3],

$$\tilde{\Delta} f(a) = \frac{4}{n+1}(1-|a|^2)\sum_{i,j=1}^n (\delta_{ij} - a_i \bar{a}_j)\frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}(a).$$

The operator $\tilde{\Delta}$ is invariant under $\mathcal{M}$, that is, $\tilde{\Delta}(f \circ \psi) = (\tilde{\Delta} f) \circ \psi$ for all $\psi \in \mathcal{M}$ ([3, Theorem 4.1.2]). Let $\tilde{\nabla}$ denote the gradient with respect to the Bergman metric on $B$ ([7, page 27]). Then as in [7, page 30], for $f \in H(B)$

$$|\tilde{\nabla} f(a)|^2 = \frac{2}{n+1}(1-|a|^2)\left[\sum_{j=1}^n \left|\frac{\partial f}{\partial z_j}(a)\right|^2 - \left|\sum_{j=1}^n a_j \frac{\partial f}{\partial z_j}(a)\right|^2\right], \quad a \in B.$$

An upper semicontinuous function $u : B \to [-\infty, \infty)$, $u \not\equiv -\infty$, is said to be *$\mathcal{M}$-subharmonic* if for each $a \in B$

$$u(a) \le \int_S u(\varphi_a(r\zeta))\, d\sigma(\zeta), \quad 0 < r < 1.$$

A continuous function $u$ defined in $B$ is said to be *$\mathcal{M}$-harmonic* if equality holds in the above inequality. A function $u$ in $B$ is said to be *$\mathcal{M}$-superharmonic* if $-u$ is $\mathcal{M}$-subharmonic.

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As in [7, Section 6.2], the invariant Green's function on $B$ is given by $G(z, a) = g(\varphi_a(z))$ for $(z, a) \in B \times B$, where

$$g(z) = \frac{n+1}{2n} \int_{|z|}^{1} (1-t^2)^{n-1} t^{-2n+1}\, dt.$$

Note that $g$ is $\mathscr{M}$-harmonic in $B\backslash\{0\}$, and $\mathscr{M}$-superharmonic in $B$. Let $f$ be an $\mathscr{M}$-subharmonic function in $B$. The *Riesz measure* of $f$ is the non-negative regular Borel measure $\mu_f$ in $B$ which satisfies

$$\int_B \psi\, d\mu_f = \int_B f\, \tilde{\Delta}\psi\, d\lambda$$

for all $\psi \in C_c^2(B)$. Here $C_c^2(B)$ is the class of twice continuously differentiable functions in $B$ with compact support. If $f$ is in $C^2(B)$, then by Green's identity [7, Proposition 3.1] $d\mu_f = \tilde{\Delta} f\, d\lambda$.

In the case $n = 1$, $B_1 \equiv \mathbb{D}$ is the unit disc and $S_1 \equiv \mathbb{T}$ is the unit circle in the complex plane $\mathbb{C}$. Moreover, $g(z) = \log(1/|z|)$ and $(\tilde{\Delta} f)(z) = \frac{1}{2}(1-|z|^2)^2(\Delta f)(z)$ for $f \in C^2(\mathbb{D})$ and $z \in \mathbb{D}$.

## 3. Preliminaries

According to [1, page 41], we introduce positive functions $\{K_\alpha : -1 \le \alpha < \infty\}$ defined in the interval $(0, 1)$ as follows. For $t \in (0, 1)$,

$$K_\alpha(t) = 2nc_\alpha \int_t^1 \rho^{2n-1}(1-\rho^2)^\alpha \log\frac{\rho}{t}\, d\rho \quad \text{if } \alpha > -1$$

and $K_{-1}(t) = \log(1/t)$. The following lemma is easily verified. (See, for example, [1, Proposition 2.3].)

LEMMA 3.1. *The following two inequalities hold.*

$$0 < 1 - t^2 < 2K_{-1}(t) \quad (0 < t < 1), \qquad K_{-1}(t) < 1 - t^2 \quad (1/2 < t < 1).$$

*For each $\alpha \in (-1, \infty)$, there exist two positive numbers $c_{\alpha 1}$ and $c_{\alpha 2}$ such that*

$$c_{\alpha 1}(1-t^2)^{\alpha+2} \le K_\alpha(t) \le c_{\alpha 2}(1-t^2)^{\alpha+2} \quad (0 < t < 1).$$

For $f \in H(B)$ we denote the zero set of $f$ by $Z(f) \equiv \{z \in B : f(z) = 0\}$. A simple computation shows the following lemma.

LEMMA 3.2. *Suppose $\varphi \in \mathscr{S}T^2(\mathbb{R})$ and $f \in H(\mathbb{D})\backslash\{0\}$. Put $v = \varphi(\log|f|)$ in $\mathbb{D}$. Then*

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(1) $\Delta v = \varphi''(\log|f|)|f'|^2/|f|^2$ *in* $\mathbb{D}\setminus Z(f)$.
(2) $(\tilde{\Delta} v)(z) = \frac{1}{2}(1-|z|^2)^2(\Delta v)(z) = \frac{1}{2}(1-|z|^2)^2\varphi''(\log|f(z)|)|f'(z)|^2/|f(z)|^2$ *for* $z \in \mathbb{D}\setminus Z(f)$.

LEMMA 3.3. *Suppose* $\varphi \in \mathscr{S}T^2(\mathbb{R})$ *and* $f \in H(\mathbb{D})\setminus\{0\}$. *Put* $v = \varphi(\log|f|)$ *in* $\mathbb{D}$. *Then the Riesz measure* $\mu_v$ *is given by*

$$\begin{aligned} d\mu_v(z) &= \frac{1}{2}\varphi''(\log|f(z)|)\frac{|f'(z)|^2}{|f(z)|^2}\,d\nu(z) \\ &= \frac{1}{2}\Delta(v|\mathbb{D}\setminus Z(f))(z)\,d\nu(z) = \frac{1}{2}\tilde{\Delta}(v|\mathbb{D}\setminus Z(f))(z)\,d\lambda(z) \end{aligned}$$

*for* $z \in \mathbb{D}$. *Here we use the convention that the right hand sides of these equations are defined to be* 0 *in* $Z(f)$.

PROOF. The first equation follows from [5, (3.1), pages 1035–1037] and the two remaining equations follow then from Lemma 3.2. □

LEMMA 3.4. *Suppose* $\varphi \in \mathscr{S}T^2(\mathbb{R})$ *and* $f \in H_\varphi(\mathbb{D})\setminus\{0\}$. *Put* $v = \varphi(\log|f|)$ *in* $\mathbb{D}$. *Then* $v$ *has a harmonic majorant in* $\mathbb{D}$. *And the least harmonic majorant of* $v$ *is the Poisson integral* $P[v^*]$ *of* $v^* = \varphi(\log|f^*|)$. *Here* $f^*(\zeta) = \lim_{r\uparrow 1} f(r\zeta)$ *for almost every* $\zeta \in \mathbb{T}$ *and*

$$P[v^*](z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|1-z\bar{\zeta}|^2}v^*(\zeta)\,d\sigma(\zeta) \quad (z \in \mathbb{D}).$$

PROOF. First, one sees easily that $f \in N_*(\mathbb{D}) \subset N(\mathbb{D})$ (for the definition of the Smirnov class $N_*(\mathbb{D})$ see, for example, [4, page 85] or [3, 19.1.11, page 407]). Then by [3, 5.6.4. Theorem, page 85] $f^*$ is defined (almost everywhere on $\mathbb{T}$) and the least harmonic majorant of $\log|f|$ is $u = P[\log|f^*|d\sigma + d\gamma]$, where $\gamma$ is a singular measure on $\mathbb{T}$. By [4, Theorem 2, page 84] the boundary measure of $v = \varphi(\log|f|)$ is $\varphi(\log|f^*|)d\sigma$, hence the least harmonic majorant of $v$ is $P[\varphi(\log|f^*|)]$. □

LEMMA 3.5. *Suppose* $\varphi \in \mathscr{S}T^2(\mathbb{R})$ *and* $f \in H_\varphi(\mathbb{D})\setminus\{0\}$. *Then*

$$\|f\|_{H_\varphi(\mathbb{D})} = \varphi\big(\log|f(0)|\big) + \frac{1}{2}\int_{\mathbb{D}} \varphi''\big(\log|f(z)|\big)\frac{|f'(z)|^2}{|f(z)|^2}\log\frac{1}{|z|}\,d\nu(z).$$

PROOF. Since $\lim_{t\to\infty}\varphi(t)/t = \infty$, it holds that

$$\|f\|_{H_\varphi(\mathbb{D})} = \int_{\mathbb{T}} \varphi\big(\log|f^*|\big)\,d\sigma = P\big[\varphi\big(\log|f^*|\big)\big](0).$$

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With Lemma 3.3 and Lemma 3.4, the Riesz decomposition theorem ([7, Corollary 6.11]) implies that

$$\begin{aligned}\varphi(\log|f(z)|) &= P\big[\varphi(\log|f^*|)\big](z) - \int_{\mathbb{D}} G(z,w)\tilde{\Delta}\left(\{\varphi(\log|f|)\}\right)(w)\,d\lambda(w)\\ &= P\big[\varphi(\log|f^*|)\big](z) - \frac{1}{2}\int_{\mathbb{D}} G(z,w)\varphi''(\log|f(w)|)\frac{|f'(w)|^2}{|f(w)|^2}\,d\nu(w)\end{aligned}$$

for all $z \in \mathbb{D}$. Putting $z = 0$ in the above equations, we obtain the lemma. □

For $\varphi \in \mathscr{S}T^2(\mathbb{R})$ and $f \in H_\varphi(B)\backslash\{0\}$, we define

$$f_\varphi^\sharp(z) = \varphi''(\log|f(z)|)\frac{|(\mathscr{R}f)(z)|^2}{|f(z)|^2}|z|^{-2} \quad (z \in B\backslash[Z(f)\cup\{0\}]).$$

Let $A(B)$ denote the ball algebra: $A(B) \equiv C(\overline{B}) \cap H(B)$.

LEMMA 3.6. *Suppose* $\varphi \in \mathscr{S}T^2(\mathbb{R})$ *and* $f \in A(B)\backslash\{0\}$. *Then*

$$\|f\|_{H_\varphi(B)} = \varphi(\log|f(0)|) + \frac{1}{2n}\int_B f_\varphi^\sharp(z)|z|^{-2(n-1)}\log\frac{1}{|z|}\,d\nu(z).$$

PROOF. Almost every $\zeta \in S$, $f_\zeta \in A(\mathbb{D})\backslash\{0\} \subset H_\varphi(\mathbb{D})\backslash\{0\}$. Here $A(\mathbb{D}) \equiv C(\overline{\mathbb{D}}) \cap H(\mathbb{D})$ and $f_\zeta(t) = f(t\zeta)$ for $t \in \overline{\mathbb{D}}$. By Lemma 3.5,

$$\begin{aligned}\|f_\zeta\|_{H_\varphi(\mathbb{D})} - \varphi(\log|f(0)|) &= \frac{1}{2}\int_{\mathbb{D}} \varphi''(\log|f_\zeta(t)|)\frac{|(f_\zeta)'(t)|^2}{|f_\zeta(t)|^2}\log\frac{1}{|t|}\,d\nu_1(t)\\ &= \frac{1}{2}\int_{\mathbb{D}} \varphi''(\log|f(t\zeta)|)|t|^{-2}\frac{|(\mathscr{R}f)(t\zeta)|^2}{|f(t\zeta)|^2}\log\frac{1}{|t|}\,d\nu_1(t)\\ &= \frac{1}{2}\int_{\mathbb{D}} f_\varphi^\sharp(t\zeta)\log\frac{1}{|t|}\,d\nu_1(t).\end{aligned}$$

On the other hand, the assumption $f \in A(B)$ implies

$$\begin{aligned}\|f\|_{H_\varphi(B)} &= \int_S \varphi(\log|f|)\,d\sigma = \int_S d\sigma(\zeta)\frac{1}{2\pi}\int_0^{2\pi}\varphi\big(\log|f(e^{i\theta}\zeta)|\big)\,d\theta\\ &= \int_S d\sigma(\zeta)\frac{1}{2\pi}\int_0^{2\pi}\varphi\big(\log|f_\zeta(e^{i\theta})|\big)\,d\theta = \int_S \|f_\zeta\|_{H_\varphi(\mathbb{D})}\,d\sigma(\zeta).\end{aligned}$$

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We used here the formula in [3, 1.4.7. Proposition (1), page 15]. Hence we have

$$\begin{aligned}
\|f\|_{H_\varphi(B)} - \varphi(\log|f(0)|) &= \int_S \{\|f_\zeta\|_{H_\varphi(\mathbb{D})} - \varphi(\log|f(0)|)\}\, d\sigma(\zeta) \\
&= \int_S d\sigma(\zeta) \frac{1}{2} \int_{\mathbb{D}} f_\varphi^\sharp(t\zeta) \log\frac{1}{|t|}\, d\nu_1(t) \\
&= \frac{1}{2} \int_S d\sigma(\zeta) 2 \int_0^1 r\, dr \frac{1}{2\pi} \int_0^{2\pi} f_\varphi^\sharp(re^{i\theta}\zeta) \log\frac{1}{r}\, d\theta \\
&= \int_0^1 r \log\frac{1}{r}\, dr \int_S d\sigma(\zeta) \frac{1}{2\pi} \int_0^{2\pi} f_\varphi^\sharp(re^{i\theta}\zeta)\, d\theta \\
&= \frac{1}{2n} 2n \int_0^1 r \log\frac{1}{r}\, dr \int_S f_\varphi^\sharp(r\zeta)\, d\sigma(\zeta) \\
&= \frac{1}{2n} 2n \int_0^1 r^{2n-1}\, dr \int_S |r\zeta|^{-2(n-1)} \log\frac{1}{|r\zeta|} f_\varphi^\sharp(r\zeta)\, d\sigma(\zeta) \\
&= \frac{1}{2n} \int_B |z|^{-2(n-1)} \log\frac{1}{|z|} f_\varphi^\sharp(z)\, d\nu(z). \qquad \square
\end{aligned}$$

LEMMA 3.7. *Suppose* $\varphi \in \mathscr{S}T^2(\mathbb{R})$, $f \in H(B)\backslash\{0\}$ *and* $0 < r < 1$. *Then*

$$\|f_r\|_{H_\varphi(B)} = \varphi(\log|f(0)|) + \frac{1}{2n} \int_{rB} f_\varphi^\sharp(z)|z|^{-2(n-1)} \log\frac{r}{|z|}\, d\nu(z),$$

*where* $rB = \{z \in \mathbb{C}^n : |z| < r\}$.

PROOF. Since $f_r \in A(B)\backslash\{0\}$, Lemma 3.6 implies that

$$\|f_r\|_{H_\varphi(B)} - \varphi(\log|f(0)|) = \frac{1}{2n} \int_B (f_r)_\varphi^\sharp(z)|z|^{-2(n-1)} \log\frac{1}{|z|}\, d\nu(z).$$

By the change of variables $w = rz$, $z \in B$, we have

$$\begin{aligned}
\|f_r\|_{H_\varphi(B)} - \varphi(\log|f(0)|) &= \frac{1}{2n} \int_{rB} f_\varphi^\sharp(w) \frac{r^{2n}}{|w|^{2(n-1)}} \log\frac{r}{|w|} r^{-2n}\, d\nu(w) \\
&= \frac{1}{2n} \int_{rB} f_\varphi^\sharp(w)|w|^{-2(n-1)} \log\frac{r}{|w|}\, d\nu(w). \qquad \square
\end{aligned}$$

LEMMA 3.8. *Suppose* $-1 < \alpha < \infty$, $\varphi \in \mathscr{S}T^2(\mathbb{R})$ *and* $f \in H(B)\backslash\{0\}$. *Then*

$$\|f\|_{A_\varphi(\nu_\alpha)} = \varphi(\log|f(0)|) + \frac{1}{2n} \int_B f_\varphi^\sharp(z)|z|^{-2(n-1)} K_\alpha(|z|)\, d\nu(z).$$

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PROOF. Using Lemma 3.7, Fubini's theorem and the definition of the function $K_\alpha$, we have

$$\begin{aligned}
&\|f\|_{A_\varphi(\nu_\alpha)} - \varphi(\log|f(0)|)\\
&= \int_B \varphi(\log|f|)d\nu_\alpha - \varphi(\log|f(0)|)\\
&= c_\alpha 2n \int_0^1 r^{2n-1}(1-r^2)^\alpha\, dr \int_S \varphi(\log|f_r(\zeta)|)\, d\sigma(\zeta) - \varphi(\log|f(0)|)\\
&= 2nc_\alpha \int_0^1 r^{2n-1}(1-r^2)^\alpha \left\{\|f_r\|_{H_\varphi(B)} - \varphi(\log|f(0)|)\right\} dr\\
&= 2nc_\alpha \int_0^1 r^{2n-1}(1-r^2)^\alpha\, dr \frac{1}{2n}\int_{rB} f_\varphi^\sharp(z)|z|^{-2(n-1)} \log\frac{r}{|z|}\, d\nu(z)\\
&= c_\alpha \int_B f_\varphi^\sharp(z)|z|^{-2(n-1)}\, d\nu(z) \int_{|z|}^1 r^{2n-1}(1-r^2)^\alpha \log\frac{r}{|z|}\, dr\\
&= \frac{1}{2n}\int_B f_\varphi^\sharp(z)|z|^{-2(n-1)} K_\alpha(|z|)\, d\nu(z). \qquad \square
\end{aligned}$$

LEMMA 3.9. *Suppose* $\varphi \in \mathscr{S}T^2(\mathbb{R})$ *and* $f \in H(B)\backslash\{0\}$. *Then*

$$\|f\|_{H_\varphi(B)} = \varphi(\log|f(0)|) + \frac{1}{2n}\int_B f_\varphi^\sharp(z)|z|^{-2(n-1)} K_{-1}(|z|)\, d\nu(z).$$

PROOF. For any $r \in (0, 1)$, by Lemma 3.7

$$\|f_r\|_{H_\varphi(B)} = \varphi(\log|f(0)|) + \frac{1}{2n}\int_{rB} f_\varphi^\sharp(z)|z|^{-2(n-1)} \log\frac{r}{|z|}\, d\nu(z).$$

Using the subharmonicity of the function $\varphi(\log|f|)$ and the monotone convergence theorem, we have

$$\begin{aligned}
\|f\|_{H_\varphi(B)} - \varphi(\log|f(0)|) &= \lim_{r\uparrow 1} \|f_r\|_{H_\varphi(B)} - \varphi(\log|f(0)|)\\
&= \frac{1}{2n}\int_B f_\varphi^\sharp(z)|z|^{-2(n-1)} \log\frac{1}{|z|}\, d\nu(z)\\
&= \frac{1}{2n}\int_B f_\varphi^\sharp(z)|z|^{-2(n-1)} K_{-1}(|z|)\, d\nu(z). \qquad \square
\end{aligned}$$

For the sake of convenience we define $A_\varphi(\nu_{-1}) \equiv H_\varphi(B)$ for $\varphi \in \mathscr{S}T^2(\mathbb{R})$, and $\|f\|_{A_\varphi(\nu_{-1})} \equiv \|f\|_{H_\varphi(B)}$ for $f \in H(B)\backslash\{0\}$. Then we can unify Lemma 3.8 and Lemma 3.9 in the following form.

LEMMA 3.10. *Suppose* $-1 \le \alpha < \infty$, $\varphi \in \mathscr{S}T^2(\mathbb{R})$ *and* $f \in H(B)\backslash\{0\}$. *Then*

$$\|f\|_{A_\varphi(\nu_\alpha)} = \varphi(\log|f(0)|) + \frac{1}{2n}\int_B f_\varphi^\sharp(z)|z|^{-2(n-1)} K_\alpha(|z|)\, d\nu(z).$$

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This is a generalization of [1, Theorem 3.3].

For $\zeta \in S$ and $\beta \in (1, \infty)$ we define the *Korányi approach region* $D_\beta(\zeta)$ by

$$D_\beta(\zeta) \equiv \{z \in \mathbb{C}^n : |1 - \langle z, \zeta\rangle| < \beta(1 - |z|^2)/2\}.$$

Let $-1 \leq \alpha < \infty$, $\varphi \in \mathscr{S}T^2(\mathbb{R})$ and $1 < \beta < \infty$. For $f \in H(B)\backslash\{0\}$ and $\zeta \in S$, we define

$$L_\varphi(\zeta : f, \alpha, \beta) \equiv \int_{D_\beta(\zeta)} f_\varphi^\sharp(z)(1 - |z|^2)^{\alpha+2-n}\, d\nu(z)$$

and

$$\mathscr{L}_\varphi(f, \alpha, \beta) \equiv \int_S L_\varphi(\zeta : f, \alpha, \beta)\, d\sigma(\zeta).$$

For any $\beta \in (1, \infty)$ and $z \in B$, we define

$$Q_\beta(z) \equiv \{\zeta \in S : |1 - \langle z, \zeta\rangle| < \beta(1 - |z|^2)/2\}$$

and $\omega_\beta(z) \equiv \sigma(Q_\beta(z))$. We note that $\omega_\beta$ is a radial function in $B$:

$$\omega_\beta(Uz) = \omega_\beta(z) \quad (z \in B,\ U \in \mathscr{U}),$$

where $\mathscr{U}$ is the unitary group of $\mathbb{C}^n$. Hence there exists a function $F_\beta$ defined in the interval $[0, 1)$ such that $F_\beta(|z|) = \omega_\beta(z)$ $(z \in B)$. For any $\beta \in (1, \infty)$, we define $r_\beta \equiv \max\{0, (2 - \beta)/\beta\}$ and $G_\beta(r) \equiv F_\beta(r) r^{2(n-1)}(1 - r^2)^{-n}$ $(0 \leq r < 1)$.

LEMMA 3.11. *Let $\beta \in (1, \infty)$. Then $G_\beta(r) = 0$ if $0 \leq r \leq r_\beta$ and $G_\beta(r) > 0$ if $r_\beta < r < 1$. Moreover, $G_\beta$ is a continuous bounded function in the interval* $[0, 1)$.

PROOF. See [1, Proposition 4.2]. □

LEMMA 3.12. *Let $-1 \leq \alpha < \infty$, $1 < \beta < \infty$ and $r_\beta < r < 1$. Suppose $\varphi \in \mathscr{S}T^2(\mathbb{R})$ and $f \in H(B)\backslash\{0\}$. Then*

$$\left\{\inf_{r<t<1} G_\beta(t)\right\} \int_{B\backslash r\bar{B}} f_\varphi^\sharp(z)|z|^{-2(n-1)}(1 - |z|^2)^{\alpha+2}\, d\nu(z)$$
$$\leq \mathscr{L}_\varphi(f, \alpha, \beta) \leq \left\{\sup_{r_\beta<t<1} G_\beta(t)\right\} \int_{B\backslash r_\beta\bar{B}} f_\varphi^\sharp(z)|z|^{-2(n-1)}(1 - |z|^2)^{\alpha+2}\, d\nu(z).$$

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PROOF. (See [1, Theorem 4.3].) By the definition of $\mathscr{L}_\varphi$, Fubini's theorem and Lemma 3.11

$$\begin{aligned}
\mathscr{L}_\varphi(f, \alpha, \beta) &= \int_S L_\varphi(\zeta : f, \alpha, \beta)\, d\sigma(\zeta) \\
&= \int_S d\sigma(\zeta) \int_{D_\beta(\zeta)} f_\varphi^\sharp(z)(1-|z|^2)^{\alpha+2-n}\, d\nu(z) \\
&= \int_S d\sigma(\zeta) \int_B \chi_{D_\beta(\zeta)}(z) f_\varphi^\sharp(z)(1-|z|^2)^{\alpha+2-n}\, d\nu(z) \\
&= \int_B f_\varphi^\sharp(z)(1-|z|^2)^{\alpha+2-n}\, d\nu(z) \int_S \chi_{Q_\beta(z)}(\zeta)\, d\sigma(\zeta) \\
&= \int_B f_\varphi^\sharp(z)(1-|z|^2)^{\alpha+2-n} \sigma(Q_\beta(z))\, d\nu(z) \\
&= \int_B f_\varphi^\sharp(z)(1-|z|^2)^{\alpha+2-n} F_\beta(|z|)\, d\nu(z) \\
&= \int_B f_\varphi^\sharp(z)(1-|z|^2)^{\alpha+2-n} G_\beta(|z|)|z|^{-2(n-1)}(1-|z|^2)^n\, d\nu(z) \\
&= \int_B f_\varphi^\sharp(z)(1-|z|^2)^{\alpha+2}|z|^{-2(n-1)} G_\beta(|z|)\, d\nu(z) \\
&= \int_{B\setminus r_\beta \bar{B}} f_\varphi^\sharp(z)(1-|z|^2)^{\alpha+2}|z|^{-2(n-1)} G_\beta(|z|)\, d\nu(z)
\end{aligned}$$

Since $B\setminus r\bar{B} \subset B\setminus r_\beta \bar{B}$, the above equations prove the lemma. □

LEMMA 3.13. *Let* $-1 \le \alpha < \infty$, $0 < r < 1$ *and* $z \in rB$. *Then it holds that*

$$K_\alpha(|z|) \le K_\alpha(|z|/r) + \log(1/r).$$

PROOF. See [1, pages 48–49]. □

LEMMA 3.14. *Let* $-1 \le \alpha < \infty$ *and* $0 < r < 1$. *Suppose* $\varphi \in \mathscr{S}T^2(\mathbb{R})$ *and* $f \in H(B)\setminus\{0\}$. *Then*

(1) $$\int_{rB} f_\varphi^\sharp(z)|z|^{-2(n-1)}\, d\nu(z) < \infty$$

*and*

(2) $$\int_{rB} f_\varphi^\sharp(z)|z|^{-2(n-1)} K_\alpha(|z|)\, d\nu(z) < \infty.$$

PROOF. (See [1, Lemma 5.1].) Choose a number $r'$ so that $r < r' < 1$. Since

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$f_r \in A(B)\backslash\{0\}$, by Lemma 3.7, we have

$$(3)\quad \int_{rB} f_\varphi^\sharp(z)|z|^{-2(n-1)} K_{-1}\left(\frac{|z|}{r}\right) d\nu(z) = \int_{rB} f_\varphi^\sharp(z)|z|^{-2(n-1)} \log\frac{r}{|z|}\, d\nu(z)$$
$$= 2n\{\|f_r\|_{H_\varphi(B)} - \varphi(\log|f(0)|)\} < \infty$$

and also $\int_{r'B} f_\varphi^\sharp(z)|z|^{-2(n-1)} \log(r'/|z|)\, d\nu(z) < \infty$. Hence

$$\int_{rB} f_\varphi^\sharp(z)|z|^{-2(n-1)}\, d\nu(z) \le \int_{rB} f_\varphi^\sharp(z)|z|^{-2(n-1)} \frac{\log(r'/|z|)}{\log(r'/r)}\, d\nu(z)$$
$$\le \frac{1}{\log(r'/r)} \int_{r'B} f_\varphi^\sharp(z)|z|^{-2(n-1)} \log\frac{r'}{|z|} d\nu(z) < \infty.$$

This proves (1). In the case $-1 < \alpha$, by Lemma 3.8

$$\int_{rB} f_\varphi^\sharp(z)|z|^{-2(n-1)} K_\alpha\left(\frac{|z|}{r}\right) d\nu(z) = \int_B r^2 f_\varphi^\sharp(rz)|z|^{-2(n-1)} K_\alpha(|z|)\, d\nu(z)$$
$$= \int_B (f_r)_\varphi^\sharp(z)|z|^{-2(n-1)} K_\alpha(|z|)\, d\nu(z)$$
$$= 2n\{\|f_r\|_{A_\varphi(\nu_\alpha)} - \varphi(\log|f(0)|)\} < \infty.$$

By this, (3), (1) and Lemma 3.13 we obtain (2). □

## 4. Main result

THEOREM 4.1. *Let* $-1 \le \alpha < \infty$, $\varphi \in \mathscr{S}T^2(\mathbb{R})$ *and* $f \in H(B)\backslash\{0\}$. *Then the following statements are equivalent*:

(a) $f \in A_\varphi(\nu_\alpha)$ *if* $-1 < \alpha < \infty$. $f \in H_\varphi(B)$ *if* $\alpha = -1$.
(b) $\int_B f_\varphi^\sharp(z)(1-|z|^2)^{\alpha+2}\, d\nu(z) < \infty$.
(c) $\mathscr{L}_\varphi(f, \alpha, \beta) < \infty$ *for any* $\beta \in (1, \infty)$.
(d) $\mathscr{L}_\varphi(f, \alpha, \beta) < \infty$ *for some* $\beta \in (1, \infty)$.

PROOF. (a) implies (b). Using Lemma 3.1 and Lemma 3.10, we have

$$\int_B f_\varphi^\sharp(z)(1-|z|^2)^{\alpha+2}\, d\nu(z) \le \frac{1}{c_{\alpha 1}} \int_B f_\varphi^\sharp(z) K_\alpha(|z|)\, d\nu(z)$$
$$\le \frac{1}{c_{\alpha 1}} \int_B f_\varphi^\sharp(z)|z|^{-2(n-1)} K_\alpha(|z|)\, d\nu(z)$$
$$= \frac{2n}{c_{\alpha 1}}\{\|f\|_{A_\varphi(\nu_\alpha)} - \varphi(\log|f(0)|)\} < \infty,$$

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where $c_{\alpha 1} = 1/2$ if $\alpha = -1$.

(b) implies (a). By Lemma 3.10,

$$\begin{aligned}(4)\quad \|f\|_{A_\varphi(\nu_\alpha)} - \varphi(\log|f(0)|) &= \frac{1}{2n}\int_B f_\varphi^\sharp(z)|z|^{-2(n-1)}K_\alpha(|z|)\,d\nu(z)\\ &= \frac{1}{2n}\left(\int_{\frac{1}{2}B} + \int_{B\setminus\frac{1}{2}\bar{B}}\right) f_\varphi^\sharp(z)|z|^{-2(n-1)}K_\alpha(|z|)\,d\nu(z).\end{aligned}$$

By Lemma 3.14,

$$(5)\qquad \int_{\frac{1}{2}B} f_\varphi^\sharp(z)|z|^{-2(n-1)}K_\alpha(|z|)\,d\nu(z) < \infty.$$

By Lemma 3.1 and (b),

$$\begin{aligned}(6)\quad \int_{B\setminus\frac{1}{2}\bar{B}} f_\varphi^\sharp(z)|z|^{-2(n-1)}K_\alpha(|z|)\,d\nu(z) &\le 2^{2(n-1)}c_{\alpha 2}\int_{B\setminus\frac{1}{2}\bar{B}} f_\varphi^\sharp(z)(1-|z|^2)^{\alpha+2}\,d\nu(z)\\ &\le 2^{2(n-1)}c_{\alpha 2}\int_B f_\varphi^\sharp(z)(1-|z|^2)^{\alpha+2}\,d\nu(z) < \infty,\end{aligned}$$

where $c_{\alpha 2} = 1$ if $\alpha = -1$. By (4), (5) and (6), we have $\|f\|_{A_\varphi(\nu_\alpha)} < \infty$. This shows that $f \in A_\varphi(\nu_\alpha)$.

(c) implies (d). This is trivial.

(d) implies (b). Fix a number $r \in (r_\beta, 1)$. Using Lemma 3.1 and Lemma 3.12, we have

$$\begin{aligned}(7)\quad &\int_B f_\varphi^\sharp(z)(1-|z|^2)^{\alpha+2}\,d\nu(z)\\ &= \left(\int_{rB} + \int_{B\setminus r\bar{B}}\right) f_\varphi^\sharp(z)(1-|z|^2)^{\alpha+2}\,d\nu(z)\\ &\le \frac{1}{c_{\alpha 1}}\int_{rB} f_\varphi^\sharp(z)|z|^{-2(n-1)}K_\alpha(|z|)\,d\nu(z)\\ &\quad + \int_{B\setminus r\bar{B}} f_\varphi^\sharp(z)(1-|z|^2)^{\alpha+2}|z|^{-2(n-1)}\,d\nu(z)\\ &\le \frac{1}{c_{\alpha 1}}\int_{rB} f_\varphi^\sharp(z)|z|^{-2(n-1)}K_\alpha(|z|)\,d\nu(z) + \frac{1}{\inf_{r<t<1} G_\beta(t)}\mathscr{L}_\varphi(f,\alpha,\beta).\end{aligned}$$

Since $r_\beta < r < 1$, by Lemma 3.11,

$$(8)\qquad 0 < \frac{1}{\inf_{r<t<1} G_\beta(t)} < \infty.$$

By (7), Lemma 3.14, (8) and (d), we obtain (b).

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(b) implies (c). Fix any $\beta \in (1, \infty)$. By Lemma 3.12,

$$\mathcal{L}_\varphi(f, \alpha, \beta) \le \left\{ \sup_{0 \le t < 1} G_\beta(t) \right\} \int_{B \setminus r_\beta \bar{B}} f_\varphi^\sharp(z)(1 - |z|^2)^{\alpha+2} |z|^{-2(n-1)} \, d\nu(z).$$

By Lemma 3.11, $0 < \gamma_\beta \equiv \sup_{0 \le t < 1} G_\beta(t) < \infty$. Hence we have

$$\begin{aligned} \mathcal{L}_\varphi(f, \alpha, \beta) &\le \gamma_\beta r_\beta^{-2(n-1)} \int_{B \setminus r_\beta \bar{B}} f_\varphi^\sharp(z)(1 - |z|^2)^{\alpha+2} \, d\nu(z) \\ &\le \gamma_\beta r_\beta^{-2(n-1)} \int_B f_\varphi^\sharp(z)(1 - |z|^2)^{\alpha+2} \, d\nu(z) < \infty. \end{aligned}$$

This completes the proof. □

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Department of Mathematical Sciences
Faculty of Science
Shinshu University
390-8621 Matsumoto
Japan
e-mail: matsugu@math.shinshu-u.ac.jp

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