INFLUENCE OF THE DYNAMICAL FIGURE OF THE MOON ON ITS ROTATIONAL-TRANSLATIONAL MOTION

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Abstract. The influence of the dynamical figure of the Moon on its rotation with respect to its mass centre (the physical libration) is determined by means of the theorem on the angular moment of a rigid body. In the expansion of the Moon's force function in spherical harmonics all the second and the third order harmonics are taken into consideration. For the determination of the Moon's physical libration components a linear system of differential equations of the second order with constant coefficients is constructed.

The integration displays the essential influence of the new terms in the force function expansion. For evaluation of the disturbed elements of the lunar orbit due to the nonsphericity of the Moon's dynamical figure the Lagrange's equations are solved. The disturbing function is taken in an expansion form in powers of the eccentricity of the lunar orbit and of the inclinations of the Moon's equator and its orbit with respect to the ecliptic. The commensurability of the Moon's mean motion and its angular velocity of rotation produces in the major semi-axis of the lunar orbit secular perturbations of the first order.

1. Introduction

The dynamical figure of the Moon implies the geometric figure of a homogeneous rigid body for which the expansion of the force function in spherical harmonics has the same coefficients C_{kj} and S_{kj} as the real Moon.

The expansion of the Moon's force function with respect to spherical harmonics has the form

$$U_m = \kappa \frac{m}{r} \left[1 + \sum_{k=2}^{\infty} \sum_{j=0}^{k} \left(\frac{b}{r} \right)^k (C_{kj} \cos j\lambda + S_{kj} \sin j\lambda)\, P_k^j(\sin\delta) \right],$$

where κ is the gravitational constant, m is the Moon's mass, b is its mean radius, $P_k^j(\sin\delta)$ represent the associated Legendre's polynomials, the coefficients C_{kj} and S_{kj} being calculated in the selenocentric equatorial coordinate system $Oxyz$. The axis Oz is directed along the Moon's rotational axis, the axis Ox coincides with its 'first radius', the plane Oxy is the lunar equator. Selenocentric equatorial coordinates of a point outside the Moon are r, λ, δ, the longitudes being counted from the axis Ox.

From the optical observations the Moon's dynamical figure is found to be a triaxial ellipsoid, its axis of the minimum moment of inertia A being associated with the lunar 'first radius', the one of the maximum moment of inertia C being the Moon's rotational axis. The optical observations make it possible to find the ratios of the moments of inertia of the Moon, α and β, and, therefore, to evaluate the coefficients C_{20} and C_{22} in the series (1). In Goudas (1964) some coefficients C_{kj} for $k=2, 3, 4$ were calculated on the basis of Moon's optical observations assuming that the near and the far sides of the Moon are symmetrical. From the calculations it follows that under such an assumption the coefficient C_{40} has to be not less than C_{20}.

The examination of the Moon's gravity field with the aid of Lunar Artificial Sat-

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ellites (LAS) provides more exact characteristics of the dynamical figure of the Moon. Lately, on the basis of data received from observations of different LAS a number of investigations were carried out for deriving the coefficients C_{kj}, S_{kj} (Akim, 1966; Michael *et al.*, 1970; Lorell, 1970; Michael and Blackshear, 1972, among others). The numerical values of C_{kj} S_{kj} with the same indices in these papers differ from each other appreciably except for C_{20} and C_{22}. Nevertheless the values of C_{k0}, C_{k1} and C_{22} are nearly of the same order for $k>2$. These data allow us to conclude that the dynamical figure of the real Moon differs essentially from the triaxial ellipsoid, for which the values of C_{k0} and C_{k1} would tend to zero with the increase of k.

In previous papers concerning the influence of nonsphericity of the Moon's gravity field on its rotational and translational motions the Moon's dynamical figure was determined by the values of only two coefficients, C_{20} and C_{22}. Now the Moon's dynamical figure is characterized by the first 12 coefficients C_{kj} and S_{kj} ($k=2, 3$; $j=0, \ldots, k$), the values of which are taken from Michael *et al.* (1970):

$$
\begin{array}{ll}
C_{20}=-2.0707\times10^{-4} & \\
C_{21}=-0.4425\times10^{-6} & S_{21}=-0.4573\times10^{-5}\\
C_{22}=0.2242\times10^{-4} & S_{22}=0.2119\times10^{-6}\\
C_{30}=-0.6303\times10^{-5} & \\
C_{31}=0.2437\times10^{-4} & S_{31}=0.2301\times10^{-5}\\
C_{32}=0.5016\times10^{-5} & S_{32}=0.2031\times10^{-5}\\
C_{33}=0.1657\times10^{-5} & S_{33}=-0.6798\times10^{-6}.
\end{array}
$$

The coefficients C_{21}, S_{21} and S_{22} determine the positions of the principal axes of inertia with respect to the coordinate system $Oxyz$, while C_{3j} and S_{3j} characterize the deviation of the Moon's dynamical figure from the triaxial ellipsoid.

2. Physical Libration of the Moon

The problem of the lunar physical libration in the gravity field of the point-Earth is considered. The force function of the mutual attraction of the Moon and the Earth has the form $U=U_m m_0$, where m_0 is the Earth's mass and by r, λ, δ in U_m the Earth's coordinates are implied. The lunar kinetic energy depends both on moments and products of inertia. The orbital motion of the Moon is performed in accordance with Brown's theory.

For the variables ξ, η, τ (Hayn, 1923) the system of the linear differential equations with the constant coefficients is deduced as

$$
\begin{aligned}
\ddot{\eta}+a_{11}\dot{\xi}+a_{12}\eta&=\Sigma_1,\\
\ddot{\xi}-a_{21}\dot{\eta}+a_{22}\xi&=\Sigma_2,\\
\ddot{\tau}+a_{32}\tau&=\Sigma_3.
\end{aligned}
\tag{2}
$$

The equations of this system are similar to ones derived by Hayn and Koziel (Koziel, 1948) except that a_{22} and a_{32} depend now on C_{31} and C_{33} in addition to

C_{20} and C_{22}. Furthermore, the sums of the trigonometric terms in Σ_i have as a factor each of C_{kj}, S_{kj} $(k=2,3; j=0,\ldots,k)$ and two constant terms in Σ_1 and Σ_2 are connected with kinetic energy, namely with the products of inertia defined by C_{21} and S_{21}.

In Hayn's paper in the equation for τ a resonance phenomenon takes place for the value of the mechanical ellipticity of the Moon, $f=0.662$. Now, existence of this resonance for such a value of f which is called critical, depends on magnitudes of C_{31} and C_{33}.

For the usual variables τ, $\sin I\sigma$, ϱ the main terms in the solution of system (2) are as follows (except for those with coefficients C_{20} and C_{22}):

	τ	$\sin I\sigma$	ϱ
C_{21}	$+8''.51\cos\omega$	$-360''.53\cos(g+\omega)$	$+360''.65\sin(g+\omega)$
S_{21}	$-43''.92\sin\omega$	$+39''.41\sin\omega$	$+40''.61\cos\omega$
S_{21}		$+5824''.51\sin(g+\omega)$	$+5824''.51\cos(g+\omega)$
S_{22}	$-971''.82$	$-14''.12$	
C_{32}	$-8''.82\cos\omega$	$+67''.78\cos(g+\omega)$	$-67''.80\sin(g+\omega)$
S_{33}	$+104''.29$		

where g is the lunar mean anomaly and ω is the argument of the perigee of the lunar orbit.

However, according to the optical observations the maximum amplitude of the lunar physical libration must not exceed 100″–120″ (Weimer, 1968). This contradiction seems to be due to very large values of C_{21}, S_{21} and S_{22} accepted here. It is worth noting that these values of C_{21}, S_{21} and S_{22} are the least among those in the studies cited above on the determination of the lunar gravity field.

The terms of the solution dependent on C_{20} and C_{22} are in good agreement with solutions by Hayn and Koziel except for the terms in $\sin I\sigma$ and ϱ with the argument $2g'+2\omega'$ (doubled longitude of the Sun counted from the ascending node of the lunar orbit on the ecliptic). According to Hayn the amplitude of these terms is equal to 3″, but here it is less than $0''.1$. A similar discrepancy with Hayn's results was noted by Habibullin (1966).

3. Influence of Nonsphericity of the Moon's Dynamical Figure on Its Translational Motion

Now we deal with the problem of the motion of the mass centre of the rigid Moon disturbed by nonsphericity of its dynamical figure in the gravitational field of the point-Earth. The Moon's rotation with respect to its mass centre is performed in accordance with Cassini's laws, the Moon's physical libration being neglected.

Introduce the ecliptic coordinate system $O_1x_1y_1z_1$ with the origin in the Earth's mass centre the axes of which are parallel to those of $Ox_1y_1z_1$. The dimension and the position of the lunar orbit with respect to $O_1x_1y_1z_1$, as well as the position of the Moon's mass centre on the orbit are defined by six osculating elements as follows:

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a and e are the major semi-axis and the eccentricity of the lunar orbit, i represents the inclination of the orbit with respect to the ecliptic, Ω means the longitude of the ascending node of the lunar orbit in ecliptic, ω is the argument of the lunar orbit perihelion and M_0 is the Moon's mean anomaly at the initial epoch. For the determination of the perturbation of these elements the system of Lagrange's equations is constructed, the disturbing function R being of the form

$$R=\kappa\frac{m+m_0}{r}\sum_{k=2}^{\infty}\sum_{j=0}^{k}\left(\frac{b}{r}\right)^k(C_{kj}\cos j\lambda+S_{kj}\sin j\lambda)\,P_k^j(\sin\delta),\tag{3}$$

where r, λ and δ mean the selenocentric equatorial coordinates of the Earth which have to be expressed in terms of the osculating elements of the lunar orbit.

If one considers the problem of calculating the perturbations of the osculating elements of the Earth's selenocentric orbit disturbed by the nonsphericity of the Moon's gravity field, then the disturbing function R_1 would be of the same form (3) but r, λ and δ should be expressed in terms of the osculating elements of the Earth's selenocentric orbit a', e', i', Ω', ω' and M_0'. In this case it is convenient to take R_1 in the form of an expansion in powers of e', i' and ϑ (Brumberg *et al.*, 1971) since those are the small quantities. It is evident that between the osculating elements of the lunar geocentric orbit and those of the Earth's selenocentric orbit there exists a simple relation,

$$a=a',\qquad e=e',\qquad i=i',\qquad \Omega=\Omega',\qquad \omega=\omega'+180^\circ,\qquad M_0=M_0'.$$

Therefore, R can be presented as an expansion in powers of e, i, ϑ which is similar to the one mentioned above,

$$R=\kappa\frac{m+m_0}{r}\sum_{k=2}^{\infty}\sum_{j=0}^{k}\sum_{l=-k}^{+k}\sum_{s=0}^{k}\sum_{q=-\infty}^{+\infty}\left(\frac{b}{a}\right)^k\times$$
$$\times(-1)^h A_{kjl}(\vartheta)\,F_{kls}(i)\,X_{k-2s+q}^{-k-1,k-2s}(e)\,(C_{kj}\cos\Delta+S_{kj}\sin\Delta),\tag{4}$$

where

$$h=E\left(\frac{k-j}{2}\right)+E\left(\frac{k-l}{2}\right)+\max\{0,j-l\}+k-l-j,$$
$$\Delta=(k-2s+q)\,M+(k-2s)\,\omega+l(\Omega-\psi)-j\varphi-\nabla_{kj}90^\circ,$$

M is the Moon's mean anomaly,

$$\nabla_{kj}=\begin{cases}0, & k-j=2p,\\ 1, & k-j=2p+1,\end{cases}$$

where p is integer, A_{kjl}, F_{kls} are hypergeometric functions (Brumberg *et al.*, 1971), $X_{k-2s+q}^{-k-1,k-2s}$ being Hansen's coefficients (Brumberg, 1967).

Due to the smallness of C_{kj} and S_{kj}, it is preferable to find out the solution of the Lagrange's equations for the osculating elements with the disturbing function in the form (4) by the method of successive approximations:

$$a=a_0+\delta_1 a+\cdots+\delta_\nu a+\cdots,$$
$$e=e_0+\delta_1 e+\cdots+\delta_\nu e+\cdots,$$
$$\cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots,$$

where $a_0, e_0, \ldots$ are the values of the osculating elements of the lunar orbit for the certain epoch, and $\delta_\nu a, \delta_\nu e, \ldots$ are the perturbations of the νth order.

In order to obtain the first-order perturbations one should substitute into R, instead of $a, e, \ldots$ their undisturbed values corresponding to some epoch, and instead of φ, ψ and ϑ, their values related to the undisturbed rotation of the spherical body as follows:

$$\varphi=n(t-t_0)+\varphi_0, \qquad \psi=\psi_0, \qquad \vartheta=\vartheta_0,$$

where t_0 is the initial epoch, and φ_0, ψ_0 and ϑ_0 are the values of the angles for $t=t_0$. Then R becomes

$$R=\sum H^{\cos}_{\sin}[(k-2s+q)\,M-j\varphi+P], \tag{5}$$

where H are constant coefficients which depend in linear way on C_{kj} and S_{kj}. H vanishes simultaneously with all C_{kj} and S_{kj}, $M=n(t-t_0)+M_0$ and P are linear combinations of ω_0, Ω_0, ψ_0. Since the Moon's mean motion equals to its mean angular velocity of rotation with respect to the proper mass centre the major semi-axis of the lunar orbit and the Moon's mean motion have the secular perturbations of the first order. Let us recall that here the two-body problem is considered, with the disturbing function being due to nonsphericity of the Moon's dynamical figure. For this case the theorem can be stated on the secular first order perturbations of the major semi-axis of the planetary orbit which is analogous to that by Laplace–Lagrange for the three-body problem.

THEOREM 1. Let the dynamical figure of the planet have no axial symmetry. If the mean planetary motion n_1 with respect to the central body is not commensurable with its mean angular velocity of rotation n_2 about its proper mass centre, then the major semi-axis and the mean planetary motion have no secular perturbations of the first order.

Indeed, in this case the disturbing function is of the form (5), where $M=n_1(t-t_0)+$ $+M_0$ and $\varphi=n_2(t-t_0)+\varphi_0$. Since n_1 and n_2 are not commensurable, $\partial R/\partial M_0$ has a constant part only for $k-2s+q=0$ and $j=0$. But in this case $\partial R/\partial M_0\equiv 0$ and, therefore, the major semi-axis of the planetary orbit has no secular perturbations of the first order.

THEOREM 2. Let the dynamical figure of the planet have the axial symmetry and the planet performs its rotation about this axis. Then the major semi-axis of the planetary orbit has no secular perturbations of the first order even in the case of

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commensurability between the mean planetary motion n_1 and its mean angular velocity of rotation n_2.

The axial symmetry of the dynamical figure of a planet assumes that its dynamical figure is defined by the values of C_{k0}. All the other coefficients of (1) are identically equal to zero. In this case R reduces to

$$R = \sum H_{\sin}^{\cos} [(k-2s+q)\, M + P],$$

where H depends lineary on C_{k0} and vanishes simultaneously with all C_{k0}. In $\partial R/\partial M_0$ the terms corresponding to $k-2s+q=0$ do not depend on time, but $\partial R/\partial M_0 \equiv 0$ whatever values n_1 and n_2 have. Thus, the major semi-axis of the planetary orbit has no secular perturbations of the first order even in the case of commensurability of n_1 and n_2.

Since, for the Moon, $n_1 = n_2$, the secular perturbations of the lunar orbital elements are produced due to the terms in R, and its derivatives with indices k, j, s, q satisfying the relation $k-2s+q-j=0$. After differentiating R with respect to the osculating elements the Eulerian angles φ, ψ, ϑ are expressed in terms of the orbital elements according to Cassini's laws. The first-order perturbations of the major semi-axis determined by S_{22} and S_{33} are as follows (per Julian century):

$$(1/a_0)\, \delta_1 a = +0''089 - 0''009.$$

The periodic perturbations of the osculating elements of the lunar orbit are determined by the terms of the perturbation function R and its derivatives with indices k, j, s, q satisfying the condition $k-2s+q-j \neq 0$. As was expected the periodic perturbations appeared to be small. The largest of them are the following:

$$\delta_1 \tau = +0''02 \sin M,$$
$$\delta_1 M_0 = -0''02 \sin M,$$

the amplitude of these perturbations being determined by C_{22}. Since in Section 2 the conclusion was drawn that C_{21}, S_{21}, S_{22}, accepted here exceed their real values, the perturbations due to the corresponding terms in R are too large.

Let us note that the secular perturbations of Ω and π which are determined by C_{20} and C_{22} are in good agreement with the results obtained earlier (Eckert, 1965).

4. Conclusion

The work carried out demonstrates the essential influence of the formerly neglected harmonics of the third order ($k=3$) of the Moon's force function expansion (1) on its rotational and translational motions. The terms of the disturbing function with coefficients C_{21}, S_{21} and S_{22} appeared to influence essentially the Moon's rotational-translational motion. The discrepancy found between the results of the theory of the Moon's libration proposed here and the results of observations allows us to conclude that the values of the coefficients C_{21}, S_{21} and S_{22} are exaggerated, though the values of these coefficients are the least among those in other papers on the determination of the Moon's gravity field. In Lidov and Neishtadt (1973) the same conclusion is made.

The evaluation of the perturbations of the rotational-translational motion corresponding to some other values of C_{kj} and S_{kj} can be readily performed.

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