ON A RAMANUJAN'S IDENTITY

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1. Introduction. In [**4**], Ramanujan stated the following beautiful identity which was later proved by Bailey [**2**] and [**3**];

$$q\prod_{n=1}^{\infty}\frac{(1-q^{5n})^5}{(1-q^n)}=\sum_{n=0}^{\infty}\frac{q^{5n+1}}{(1-q^{5n+1})^2}-\frac{q^{5n+2}}{(1-q^{5n+2})^2}-\frac{q^{5n+3}}{(1-q^{5n+3})^2}+\frac{q^{5n+4}}{(1-q^{5n+4})^2}. \tag{1}$$

We denote by $S(n)$ the coefficient of q^n on the right hand side of (1). The object of this paper is to prove the following two theorems.

THEOREM 1. (a) *$S(n)$ is multiplicative.* (b) *Let p be a prime. We have the following*:
(i) *if $p=5$, then $S(5^n)=5^n$;*
(ii) *if $p\equiv\pm1(\text{mod }5)$, then $S(p^n)=(p^{n+1}-1)/(p-1)$;*
(iii) *if $p\equiv\pm2(\text{mod }5)$, then* $S(p^n)=\begin{cases}(p^{n+1}-1)/(p+1) & \text{if } n \text{ is odd}\\ (p^{n+1}+1)/(p+1) & \text{if } n \text{ is even.}\end{cases}$

THEOREM 2. *Let $R(n)$ denote the number of integral solutions of the Diophantine equation*

$$5(x_1^2+x_2^2+x_3^2+x_4^2-x_1x_2-x_2x_3-x_3x_4)+x_1+x_2+x_3+x_4=n.$$

Then

$$R(n)=S(n+1).$$

2. Proof of Theorem 1. For $1\le k\le 4$, define

$$\sigma_k(n)=\sum_{\substack{d\mid n\\ d\equiv k(\text{mod }5)}} n/d.$$

and note that

$$\sum_{n=0}^{\infty}\frac{q^{5n+k}}{(1-q^{5n+k})^2}=\sum_{n=1}^{\infty}\sigma_k(n)q^n.$$

As $S(n)$ denotes the coefficients of q^n on the right hand side of (1), we have

$$S(n)=\sigma_1(n)-\sigma_2(n)-\sigma_3(n)+\sigma_4(n). \tag{2}$$

Now, the part (a) of the Theorem 1 easily follows by observing that

$$S(n)=\sum_{d\mid n}(n/d)\chi(d),$$

where χ is the Dirichlet character defined by $\chi(2)=-1$. Part (b) of the Theorem 1 follows by using (2).

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3. Proof of Theorem 2. In this section we shall prove the following lemma from which Theorem 2 follows immediately.

LEMMA. *For* $|q|<1$,

$$\prod_{n=1}^{\infty}\frac{(1-q^{5n})^5}{(1-q^n)}=\sum_{n=0}^{\infty}R(n)q^n.$$

Proof. First we note that

$$\sum_{n\in\mathbb{Z}}z^{An}q^{Bn^2+Cn}=\prod_{n=1}^{\infty}(1-q^{2Bn})(1+z^Aq^{2Bn+C-B})(1+z^{-A}q^{2Bn-C-B}), \tag{3}$$

by Jacobi's triple product identity [**1**, p. 21]. Hence $\prod_{n=1}^{\infty}(1-q^n)^{-1}$ is the constant term (i.e. the term independent of z) in

$$\prod_{n=1}^{\infty}(1+zq^{n-2})(1+z^{-1}q^{n+1}).$$

Now consider

$$\begin{aligned}
&\sum_{y_1,y_2,y_3,y_4,y_5\in\mathbb{Z}}\{q^{5/2(\sum_{i=1}^{5}y_i^2)+\frac{1}{2}(y_1+y_2+3y_3+5y_4+7y_5)}z^{y_1-y_2-y_3-y_4-y_5}\}\\
&=\left\{\prod_{n=1}^{\infty}(1-q^{5n})(1+zq^{5n-2})(1+z^{-1}q^{5n-3})\right\}\left\{\prod_{n=1}^{\infty}(1-q^{5n})(1+z^{-1}q^{5n-2})(1+zq^{5n-3})\right\}\\
&\quad\times\left\{\prod_{n=1}^{\infty}(1-q^{5n})(1+z^{-1}q^{5n-1})(1+zq^{5n-4})\right\}\left\{\prod_{n=1}^{\infty}(1-q^{5n})(1+z^{-1}q^{5n})(1+zq^{5n-5})\right\}\\
&\quad\times\left\{\prod_{n=1}^{\infty}(1-q^{5n})(1+z^{-1}q^{5n+1})(1+zq^{5n-6})\right\}\\
&=\prod_{n=1}^{\infty}(1-q^{5n})^5(1+zq^{n-2})(1+z^{-1}q^{n+1}).
\end{aligned} \tag{4}$$

By comparing the constant terms on both sides of (3) we have $y_1=y_2+y_3+y_4+y_5$. Hence by making the change of variables

$$x_1=y_2+y_3+y_4+y_5,\qquad x_2=y_3+y_4+y_5,\qquad x_3=y_4+y_5,\qquad x_4=y_5$$

to get the polynomial in the form given in Theorem 2, the lemma follows.

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