STAR-COMPLEXES, AND THE DEPENDENCE PROBLEMS FOR HYPERBOLIC COMPLEXES

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Introduction. Given a group presentation (or more generally† a 2-complex) one can associate with it an object which has variously been called the *co-initial graph, star-graph, star-complex,* and which has proved useful in several contexts [**2**], [**6**], [**7**], [**8**], [**9**], [**10**], [**12**]. For certain mappings of 2-complexes $\phi: \mathcal{K} \to \mathcal{L}$ ("strong mappings") one gets an induced mapping $\phi^{st}: \mathcal{K}^{st} \to \mathcal{L}^{st}$ of the associated star-complexes. Then st is a covariant functor from the category of 2-complexes (where the morphisms are strong mappings) to the category of 1-complexes, and this functor behaves very nicely with respect to coverings (Theorem 1).

Hyperbolic complexes arise when one considers assigning numbers ("weights") to the edges of the star-complex of a 2-complex. The most well-known hyperbolic complexes are the surface presentations

$$\left\langle x_1, y_1, \ldots, x_n, y_n; \prod_{i=1}^{n} [x_i, y_i] \right\rangle \quad (n \geq 2),$$

$$\left\langle x_1, \ldots, x_n; \prod_{i=1}^{n} x_i^2 \right\rangle \quad (n \geq 3),$$

and the presentations of triangle groups

$$\langle a, b, c; a^p, b^q, c^r, abc \rangle, \qquad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1.$$

(More generally, as M. El-Mosalamy has remarked, an F-presentation (see [**11**, p. 126]) is hyperbolic (in the sense of this paper) if and only if its measure is positive.)

Gersten [**5**] proved that finite hyperbolic complexes have solvable word and conjugacy problems. In this paper we give a simpler proof of a much more general result. If $\mathcal{K}$ is a 2-complex and n is a positive integer we define a decision problem which we call the *dependence problem* DP(n) for $\mathcal{K}$. (The word problem is DP(1) and the conjugacy problem is DP(2).) We show that if $\mathcal{K}$ is a finite hyperbolic complex then DP(n) is solvable for all n (Theorem 2). (Actually, we prove a somewhat stronger result, namely that the "union" DP(∞) of all the problems DP(n) is solvable.)

It turns out that hyperbolic complexes are related to small cancellation complexes. The "standard" small cancellation conditions are $C(p)$, $T(q)$, where $1/p + 1/q \leq \frac{1}{2}$. In [**6**] we gave a formulation of the $T(q)$ condition in terms of star-complexes. Here we introduce a condition $\tilde{T}(q)$, and we show that $C(p)$, $\tilde{T}(q)$-complexes are hyperbolic if $1/p + 1/q < \frac{1}{2}$ (Theorem 4).

In Theorem 5 we show that hyperbolicity is preserved when one subdivides a non-periodic defining path of a hyperbolic complex.

† A presentation can be regarded as a 2-complex with a single vertex.

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We wish to make this paper reasonably self-contained and accessible to combinatorial group-theorists; so we give below a fairly comprehensive collection of definitions and background information regarding 1- and 2-complexes. This material is entirely standard and can be found elsewhere, though not always in the form we require.

The headings of the sections of the paper are as follows.

1. Definitions and elementary concepts.
2. Star-complexes of 2-complexes.
3. Hyperbolic complexes.
4. The dependence problems for hyperbolic complexes.
5. Small cancellation complexes.
6. Subdividing defining paths.

In a sequel **[13]** to this paper, I will discuss, among other things, the dependence problems for groups with "many" generators of order 2.

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1. Definitions and elementary concepts. A *1-complex* $\mathscr{X}$ consists of two disjoint sets $V(\mathscr{X})$ (vertices), $E(\mathscr{X})$ (edges) together with three functions $\iota: E(\mathscr{X}) \to V(\mathscr{X})$, $\tau: E(\mathscr{X}) \to V(\mathscr{X})$, ${}^{-1}: E(\mathscr{X}) \to E(\mathscr{X})$ satisfying: $\iota(e^{-1}) = \tau(e)$, $(e^{-1})^{-1} = e$, $e^{-1} \neq e$ for all $e \in E(\mathscr{X})$. A *non-empty path* α in $\mathscr{X}$ is a sequence $e_1 e_2 \ldots e_n$ $(n \geq 1)$ of edges with $\tau(e_i) = \iota(e_{i+1})$ $(1 \leq i < n)$. We define $\iota(\alpha)$, $\tau(\alpha)$ to be $\iota(e_1)$, $\tau(e_n)$ respectively. The path is said to be *closed* if $\iota(\alpha) = \tau(\alpha)$. The *length* $L(\alpha)$ of α is n. The *inverse* α^{-1} of α is the path $e_n^{-1} \ldots e_2^{-1} e_1^{-1}$. We say that α is *reduced* if $e_i \neq e_{i+1}^{-1}$ for $i = 1, \ldots, n-1$. Moreover, if α is closed then we say that α is *cyclically reduced* if all its cyclic permutations are reduced.

With each vertex v of $\mathscr{X}$ we associate an *empty path* 1_v (or simply 1). This path has no edges (thus $L(1_v) = 0$). Moreover, $\iota(1_v) = \tau(1_v) = v$ and $1_v^{-1} = 1_v$.

We say that the *product* $\beta\gamma$ of two paths β, γ *is defined* if $\tau(\beta) = \iota(\gamma)$. Then $\beta\gamma$ is the path consisting of the edges of β followed by the edges of γ.

A 1-complex is said to be *connected* if, given any two vertices u, v there is a path α with $\iota(\alpha) = u$, $\tau(\alpha) = v$. A *subcomplex* of a 1-complex $\mathscr{X}$ is a subset of $V(\mathscr{X}) \cup E(\mathscr{X})$ which is closed under ι, τ, ${}^{-1}$. If $V \subseteq V(\mathscr{X})$ then the *full subcomplex* on V consists of V together with *all* edges e of $\mathscr{X}$ where both $\iota(e)$, $\tau(e)$ belong to V. A maximal connected subcomplex of a 1-complex is called a *component*.

If v is a vertex in a 1-complex $\mathscr{X}$ then

$$\operatorname{Star}(v) = \{e : e \in E(\mathscr{X}),\ \iota(e) = v\}.$$

A *circle* is a connected 1-complex with only finitely many vertices, and such that $|\operatorname{Star}(v)| = 2$ for each vertex v. A *line* is a 1-complex obtained from a circle by removing one edge pair e, e^{-1}.

Let $\mathscr{X}$, $\mathscr{Y}$ be 1-complexes. A *mapping* (*of 1-complexes*) $\phi: \mathscr{X} \to \mathscr{Y}$ is a function sending vertices of $\mathscr{X}$ to vertices of $\mathscr{Y}$, and paths in $\mathscr{X}$ to paths in $\mathscr{Y}$, and satisfying: $\phi(1_v) = 1_{\phi(v)}$ for all $v \in V(\mathscr{X})$; $\phi(\alpha^{-1}) = \phi(\alpha)^{-1}$ for all paths α in $\mathscr{X}$; whenever $\alpha_1\alpha_2$ is defined (α_1, α_2 paths in $\mathscr{X}$) $\phi(\alpha_1)\phi(\alpha_2)$ is defined, and $\phi(\alpha_1\alpha_2) = \phi(\alpha_1)\phi(\alpha_2)$. The mapping is said to be *rigid* if it preserves length, that is, $L(\phi(\alpha)) = L(\alpha)$ for all paths α (in other words, ϕ takes edges to edges).

Suppose $\phi : \tilde{\mathscr{X}} \to \mathscr{X}$ is rigid. Then

$$\phi \operatorname{Star}(\tilde{v}) \subseteq \operatorname{Star}(\phi(\tilde{v})) \quad \text{for all} \quad \tilde{v} \in V(\tilde{\mathscr{X}}).$$

We say that ϕ is *locally injective* (resp. *locally surjective, locally bijective*) if

$$\phi : \operatorname{Star}(\tilde{v}) \to \operatorname{Star}(\phi(\tilde{v}))$$

is injective (resp. surjective, bijective) for all $\tilde{v} \in V(\tilde{\mathscr{X}})$. These concepts are directly connected to *lifts. Let* u be a vertex of $\mathscr{X}$ and let $\tilde{u}$ be a vertex of $\tilde{\mathscr{X}}$ such that $\phi(\tilde{u}) = u$ (we often say, in this regard, that $\tilde{u}$ *lies over* u). If α is a path in $\mathscr{X}$ with $\iota(\alpha) = u$ then a path $\tilde{\alpha}$ in $\tilde{\mathscr{X}}$ such that $\iota(\tilde{\alpha}) = \tilde{u}$ and $\phi(\tilde{\alpha}) = \alpha$ is called a *lift* of α at $\tilde{u}$. Now it is not difficult to show that all possible lifts of paths of $\mathscr{X}$ *exist* if and only if ϕ is locally surjective. Also, one can show that lifts, when they exist, are *unique,* if and only if ϕ is locally injective. Thus the local bijectivity of ϕ is equivalent to the condition that all possible lifts of paths of $\mathscr{X}$ exist and are unique.

A 2-*complex* $\mathscr{K}$ is an object $\langle \mathscr{X}; \rho_\lambda (\lambda \in \Lambda) \rangle$, where $\mathscr{X}$ is a 1-complex (called the 1-*skeleton* of $\mathscr{K}$, and often denoted by $\mathscr{K}^{(1)}$) and each ρ_λ is a closed path in $\mathscr{X}$. The ρ_λ are called *defining paths*. The elements of Λ are called *indices. We will always assume in this paper that the defining paths are non-empty and cyclically reduced.*

When working with a 2-complex $\mathscr{K}$ as above, any use of terms such as "vertex", "edge", "path", "connectedness", and so on, refers to the 1-skeleton of $\mathscr{K}$. In particular, $V(\mathscr{K}) = V(\mathscr{X})$, $E(\mathscr{K}) = E(\mathscr{X})$. We say that $\mathscr{K}$ is *finite* if $V(\mathscr{K})$, $E(\mathscr{K})$, Λ are all finite.

For a 2-complex $\mathscr{K}$, we define $R(\mathscr{K})$ to be the set of cyclic permutations of defining paths and their inverses.

A 2-complex with a single vertex is called a *presentation*. If $\langle \mathscr{Y}; \beta_i\, (i \in I) \rangle$ is a presentation, and if the edges of $\mathscr{Y}$ are y_1, y_1^{-1}, y_2, $y_2^{-1}, \ldots$ then we will often use the more standard notation

$$\langle y_1, y_2, \ldots; \beta_i\, (i \in I) \rangle$$

for the presentation.

Let $\mathscr{K}$ be a 2-complex as above. We define an equivalence relation $\sim_{\mathscr{K}}$ (or simply $\sim$) on paths in $\mathscr{K}$ as follows. An *elementary reduction* of a path α is a transformation of α to $\alpha_1\alpha_2$ if α has one of the forms $\alpha_1\gamma\gamma^{-1}\alpha_2$ (γ any path), $\alpha_1\rho\alpha_2$ ($\rho \in R(\mathscr{K})$). Then for two paths α, α' we define $\alpha \sim_{\mathscr{K}} \alpha'$ if and only if there is a sequence of paths $\alpha = \alpha_0$, $\alpha_1, \ldots, \alpha_m = \alpha'$, where for $i = 0, \ldots, m-1$ one of α_i, α_{i+1} is obtained from the other by an elementary reduction. The $\sim$-equivalence class containing α is denoted by $[\alpha]_{\mathscr{K}}$ (or simply by $[\alpha]$). A path which is $\sim$-equivalent to an empty path is said to be *contractible*. If α, β are paths such that $\alpha\beta$ is defined, then we define $[\alpha][\beta]$ to be $[\alpha\beta]$. This partial product is easily seen to be well-defined. For a vertex v of $\mathscr{K}$, the *fundamental group* $\pi_1(\mathscr{K}, v)$ of $\mathscr{K}$ at v has underlying set

$$\{[\alpha] : \iota(\alpha) = \tau(\alpha) = v\}$$

and the binary operation is the product defined above. If $\mathcal{K}$ is connected then the isomorphism type of this group is independent of the choice of v. We can then refer to *the fundamental group of* $\mathcal{K}$.

Let $\mathcal{K}$, $\mathcal{L}$ be 2-complexes. A *mapping* (*of* 2-*complexes*) from $\mathcal{K}$ to $\mathcal{L}$ consists of a mapping of 1-complexes from $\mathcal{K}^{(1)}$ to $\mathcal{L}^{(1)}$ which takes contractible paths in $\mathcal{K}$ to contractible paths in $\mathcal{L}$. If $\phi:\mathcal{K}\to\mathcal{L}$ is a mapping of 2-complexes then we will usually use the same letter ϕ to denote the underlying mapping of 1-complexes. However, occasionally for emphasis we will use the symbol $\phi^{(1)}$ to denote the underlying mapping of 1-complexes.

We say that a mapping of 2-complexes $\phi:\tilde{\mathcal{K}}\to\mathcal{K}$ is *strong* if it does not map any edge to an empty path, and if $\phi R(\tilde{\mathcal{K}})\subseteq R(\mathcal{K})$. We say that ϕ is *locally bijective* if $\phi^{(1)}$ is locally bijective and $\phi^{-1}R(\mathcal{K})=R(\tilde{\mathcal{K}})$. If, in addition, $\tilde{\mathcal{K}}$ and $\mathcal{K}$ are connected then ϕ is called a *covering*. The importance of coverings stems from their connection with subgroups of fundamental groups. We briefly record the main results.

(1.1) *Let $\phi:\tilde{\mathcal{K}}\to\mathcal{K}$ be locally bijective, let α, β be paths in $\mathcal{K}$ with $\iota(\alpha)=\iota(\beta)$, and let $\tilde{\alpha}$, $\tilde{\beta}$ be lifts of α, β with $\iota(\tilde{\alpha})=\iota(\tilde{\beta})$. If $\alpha\sim\beta$ then $\tilde{\alpha}\sim\tilde{\beta}$.*

To prove this it suffices to deal with the case when β is obtained from α by an elementary reduction. The general case then follows by induction. Thus suppose α has one of the forms $\alpha_1\gamma\gamma^{-1}\alpha_2$, $\alpha_1\rho\alpha_2$ $(\rho\in R(\mathcal{K}))$ and $\beta=\alpha_1\alpha_2$. If α has the first form then the local bijectivity of $\phi^{(1)}$ guarantees that the subpath of $\tilde{\alpha}$ mapping onto $\gamma\gamma^{-1}$ will be $\tilde{\gamma}\tilde{\gamma}^{-1}$ for some lift $\tilde{\gamma}$ of γ, while if α has the second form then, since $\phi^{-1}R(\mathcal{K})=R(\tilde{\mathcal{K}})$, the subpath of $\tilde{\alpha}$ mapping onto ρ will belong to $R(\tilde{\mathcal{K}})$. Thus in either case $\tilde{\beta}$ is obtained from $\tilde{\alpha}$ by an elementary reduction.

(1.2) *If $\phi:\tilde{\mathcal{K}}\to\mathcal{K}$ is locally bijective and $\tilde{v}$ is a vertex of $\tilde{\mathcal{K}}$ then the induced homomorphism $\phi_*:\pi_1(\tilde{\mathcal{K}},\tilde{v})\to\pi_1(\mathcal{K},\phi(\tilde{v}))$, defined by*

$$\phi_*[\tilde{\alpha}]=[\phi(\tilde{\alpha})] \qquad ([\tilde{\alpha}]\in\pi_1(\tilde{\mathcal{K}},\tilde{v})),$$

is injective.

This follows from (1.1).

(1.3) *Let $\mathcal{K}$ be a connected 2-complex and let v be a vertex of $\mathcal{K}$. Let H be a subgroup of $\pi_1(\mathcal{K},v)$. Then there is a covering $\phi_H:\mathcal{K}_H\to\mathcal{K}$ and a vertex v_H of $\mathcal{K}_H$ such that $\phi_{H*}\pi_1(\mathcal{K}_H,v_H)=H$.*

The construction is as follows.

Suppose $\mathcal{K}=\langle\mathcal{X};\rho_\lambda(\lambda\in\Lambda)\rangle$. Let

$$X=\{[\alpha]:\iota(\alpha)=v\}.$$

We say that two elements $[\alpha]$, $[\beta]$ of X are *equivalent mod H* if $\tau(\alpha)=\tau(\beta)$ and $[\alpha\beta^{-1}]\in H$. The equivalence class containing $[\alpha]$ is

$$\{[\gamma][\alpha]:[\gamma]\in H\},$$

which we denote by $H[\alpha]$. We define the 1-skeleton $\mathscr{X}_H$ of $\mathscr{K}_H$ as follows:

vertices $\quad H[\alpha],\ [\alpha]\in X,$

edges $\quad (H[\alpha], e),\ e\in E(\mathscr{X}),\ [\alpha]\in X,\ \tau(\alpha)=\iota(e).$

For an edge $(H[\alpha], e)$, we set

$$\iota(H[\alpha], e)=H[\alpha],\ \tau(H[\alpha], e)=H[\alpha e],\ (H[\alpha], e)^{-1}=(H[\alpha e], e^{-1}).$$

We take v_H to be the vertex $H[1_v]$. There is an obvious locally bijective mapping of 1-complexes $\phi_H^{(1)}:\mathscr{X}_H\to\mathscr{X}$ which takes $H[\alpha]$ to $\tau(\alpha)$ $(H[\alpha]\in V(\mathscr{X}_H))$ and $(H[\alpha], e)$ to e $((H[\alpha], e)\in E(\mathscr{X}_H))$. For a defining path $\rho_\lambda=e_1e_2\ldots e_n$ of $\mathscr{K}$, and a vertex $H[\alpha]$ of $\mathscr{X}_H$ lying over $\iota(\rho_\lambda)$, we let

$$\rho_{(\lambda,H[\alpha])}=(H[\alpha], e_1)(H[\alpha e_1], e_2)(H[\alpha e_1e_2], e_3)\ldots(H[\alpha e_1e_2\ldots e_{n-1}], e_n).$$

The defining paths of $\mathscr{K}_H$ are then all the $\rho_{(\lambda,H[\alpha])}$ $(\lambda\in\Lambda,\ [\alpha]\in X,\ \tau(\alpha)=\iota(\rho_\lambda))$.

2. Star-complexes of 2-complexes. Let $\mathscr{K}$ be a 2-complex. We can associate with $\mathscr{K}$ a 1-complex $\mathscr{K}^{\mathrm{st}}$, called the *star-complex* of $\mathscr{K}$, as follows:

vertex set of $\mathscr{K}^{\mathrm{st}}$ $\quad E(\mathscr{K}),$

edge set of $\mathscr{K}^{\mathrm{st}}$ $\quad R(\mathscr{K}).$

If γ is an edge of $\mathscr{K}^{\mathrm{st}}$, then we define the inverse edge to be simply the inverse path γ^{-1}. We also need to stipulate the endpoints of γ. Ordinarily these would be denoted by $\iota(\gamma)$, $\tau(\gamma)$. However, $\iota(\gamma)$, $\tau(\gamma)$ already have another meaning (§1). We thus use the notation $\iota^{\mathrm{st}}(\gamma)$, $\tau^{\mathrm{st}}(\gamma)$. We define $\iota^{\mathrm{st}}(\gamma)$ to be the first edge of γ and $\tau^{\mathrm{st}}(\gamma)$ to be the *inverse* of the last edge of γ.

It is easy to see that two vertices e, f of $\mathscr{K}^{\mathrm{st}}$ are in the same component only if $\iota(e)=\iota(f)$, that is, only if there is a vertex v of $\mathscr{K}$ such that $e, f\in\mathrm{Star}(v)$. For a vertex v of $\mathscr{K}$, we denote the full subcomplex of $\mathscr{K}^{\mathrm{st}}$ on $\mathrm{Star}(v)$ by $\mathscr{K}^{\mathrm{st}}(v)$.

Let $\phi:\mathscr{K}\to\mathscr{L}$ be a strong mapping of 2-complexes. Then we have an induced (rigid) mapping of 1-complexes

$$\phi^{\mathrm{st}}:\mathscr{K}^{\mathrm{st}}\to\mathscr{L}^{\mathrm{st}}$$

defined as follows:

on vertices of $\mathscr{K}^{\mathrm{st}}$, $\quad \phi^{\mathrm{st}}(e)$ is the first edge of $\phi(e)$,

on edges of $\mathscr{K}^{\mathrm{st}}$, $\quad \phi^{\mathrm{st}}(\gamma)=\phi(\gamma).$

It is easy to show that if v is a vertex of $\mathscr{K}$ then ϕ^{st} maps $\mathscr{K}^{\mathrm{st}}(v)$ into $\mathscr{L}^{\mathrm{st}}(\phi(v))$.

It is also easy to show that if $\psi:\mathscr{L}\to\mathscr{M}$ is another strong mapping then $(\psi\phi)^{\mathrm{st}}=\psi^{\mathrm{st}}\phi^{\mathrm{st}}$. Thus $^{\mathrm{st}}$ is a covariant functor from the category of 2-complexes (where the mappings are strong mappings) to the category of 1-complexes.

We will say that a strong mapping ϕ is *reduced* if ϕ^{st} is locally injective.

In §1, we defined a mapping $\phi:\tilde{\mathscr{K}}\to\mathscr{K}$ to be locally bijective if $\phi^{(1)}$ is locally

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bijective and $\phi^{-1}R(\mathscr{K}) = R(\tilde{\mathscr{K}})$. The next theorem shows that we could equally well have defined ϕ to be locally bijective if both $\phi^{(1)}$ and ϕ^{st} are locally bijective.

THEOREM 1. *Let $\phi: \tilde{\mathscr{K}} \to \mathscr{K}$ be a strong mapping, and suppose that $\phi^{(1)}$ is locally bijective. Then the following are equivalent*:

(2.1) *ϕ^{st} is locally bijective*;
(2.2) $\phi^{-1}R(\mathscr{K}) = R(\tilde{\mathscr{K}})$;
(2.3) *for each vertex $\tilde{v}$ of $\tilde{\mathscr{K}}$, ϕ^{st} maps $\tilde{\mathscr{K}}^{\text{st}}(\tilde{v})$ isomorphically onto $\mathscr{K}^{\text{st}}(\phi(\tilde{v}))$.*

Proof. (2.1)$\Rightarrow$(2.2). Let $\gamma \in R(\mathscr{K})$ and suppose that $\phi(\tilde{\gamma}) = \gamma$. Let $\iota^{\text{st}}(\gamma) = e$, and let $\tilde{e}$ be the unique edge of Star$(\iota(\tilde{\gamma}))$ lying over e. By the local surjectivity of ϕ^{st}, there is an element $\tilde{\delta} \in R(\tilde{\mathscr{K}})$ with $\iota^{\text{st}}(\tilde{\delta}) = \tilde{e}$ and $\phi^{\text{st}}(\tilde{\delta}) = \gamma$. Thus $\iota(\tilde{\gamma}) = \iota(\tilde{\delta})$ and $\phi(\tilde{\gamma}) = \phi(\tilde{\delta}) = \gamma$. Hence $\tilde{\gamma} = \tilde{\delta}$ by uniqueness of lifts, so $\tilde{\gamma} \in R(\tilde{\mathscr{K}})$.

(2.2)$\Rightarrow$(2.3). Since $\phi^{(1)}$ is locally bijective, ϕ^{st} maps the vertex set Star$(\tilde{v})$ of $\tilde{\mathscr{K}}^{\text{st}}(\tilde{v})$ bijectively onto the vertex set Star$(\phi(\tilde{v}))$ of $\mathscr{K}^{\text{st}}(\phi(\tilde{v}))$. Now consider edges. Let $\tilde{\gamma}_1, \tilde{\gamma}_2$ be edges of $\mathscr{K}^{\text{st}}(\tilde{v})$ such that $\phi^{\text{st}}(\tilde{\gamma}_1) = \phi^{\text{st}}(\tilde{\gamma}_2)$. Then $\iota(\tilde{\gamma}_1) = \iota(\tilde{\gamma}_2)$ $(= \tilde{v})$ and $\phi(\tilde{\gamma}_1) = \phi(\tilde{\gamma}_2)$, so $\tilde{\gamma}_1 = \tilde{\gamma}_2$ by uniqueness of lifts. Thus ϕ^{st} is injective on edges of $\mathscr{K}^{\text{st}}(\tilde{v})$. To see that $\phi^{\text{st}}: \mathscr{K}^{\text{st}}(\tilde{v}) \to \mathscr{K}^{\text{st}}(\phi(\tilde{v}))$ is surjective on edges, let γ be an edge of $\mathscr{K}^{\text{st}}(\phi(\tilde{v}))$. By (2.2), the (unique) lift $\tilde{\gamma}$ of γ at $\tilde{v}$ belongs to $R(\tilde{\mathscr{K}})$ and is thus an edge of $\tilde{\mathscr{K}}^{\text{st}}(\tilde{v})$. Moreover, $\phi^{\text{st}}(\tilde{\gamma}) = \gamma$.

(2.3)$\Rightarrow$(2.1). This is a consequence of the following obvious result. Let $\mathscr{A}$ and $\mathscr{B}$ be 1-complexes, each expressed as a disjoint union of subcomplexes:

$$\mathscr{A} = \dot{\bigcup_{i \in I}} \mathscr{A}_i, \qquad \mathscr{B} = \dot{\bigcup_{j \in J}} \mathscr{B}_j.$$

Let $\theta: \mathscr{A} \to \mathscr{B}$ be a rigid mapping such that for each $i \in I$, θ maps $\mathscr{A}_i$ isomorphically onto some $\mathscr{B}_{j(i)}$. Then θ is locally bijective.

This proves Theorem 1.

It should be remarked that, under the assumption that $\phi^{(1)}$ is locally bijective, (2.1) is equivalent to

$$\phi^{\text{st}} \quad \textit{is locally surjective.}$$

This is a consequence of the following easily proved fact: if $\psi: \mathscr{L} \to \mathscr{M}$ is a strong mapping, and if $\psi^{(1)}$ is locally injective, then ψ^{st} is locally injective.

Mappings $\phi: \tilde{\mathscr{K}} \to \mathscr{K}$ for which ϕ^{st} is locally bijective (but where $\phi^{(1)}$ may not be locally bijective) have been considered by S. M. Gersten [**5**].

3. Hyperbolic complexes. A *weight function* on a 1-complex is a mapping m from the edge set into $\mathbb{R}$ such that $m(e^{-1}) = m(e)$ for all edges e. If $e_1 e_2 \ldots e_n$ is a path in the 1-complex (where the e_i are edges) then the *weight* $m(e_1 e_2 \ldots e_n)$ of the path is defined to be

$$\sum_{i=1}^{n} m(e_i).$$

The situation we will be interested in is when we have a 2-complex $\mathcal{K}$ and a weight m on $\mathcal{K}^{st}$. We will use the notation $(\mathcal{K}, m)$ to denote this situation. For an element $\gamma = e_1 e_2 \ldots e_n$ of $R(\mathcal{K})$ (where the e's are edges of $\mathcal{K}$), we define $m^*(\gamma)$ to be

$$\sum_{i=1}^{n} m(e_i \ldots e_n e_1 \ldots e_{i-1}).$$

If $\phi : \tilde{\mathcal{K}} \to \mathcal{K}$ is a strong mapping then we obtain from m an *induced weight function* $\tilde{m}$ on $\tilde{\mathcal{K}}^{st}$ via ϕ^{st}.

Following Gersten [**4**], we say that $(\mathcal{K}, m)$ is *hyperbolic* if:

(3.1) $m^*(\gamma) < L(\gamma) - 2$ *for all* $\gamma \in R(\mathcal{K})$,
(3.2) *the weight of any non-empty cyclically reduced closed path in* $\mathcal{K}^{st}$ *is at least* 2,
(3.3) *there exists a non-negative real number* N *such that every reduced path in* $\mathcal{K}^{st}$ *has weight greater than or equal to* $-N$.

Note that in Gersten's definition of hyperbolic, m is required to take non-negative values, but we impose the weaker condition (3.3) here. This will enable us to obtain a subdivision theorem (Theorem 5).

Note further that Gersten also considers pairs $(\mathcal{K}, m)$, where m satisfies (3.2) and a weakening of (3.1) where the strict inequality is replaced by $\leq$ (in addition, m is no longer required to take only non-negative values). As Gersten shows (and as one can very easily prove), under these conditions there can be no reduced rigid mapping from a sphere into $\mathcal{K}$ (Gersten's "weight test" for diagrammatic reducibility).

We will say that a complex $\mathcal{K}$ is hyperbolic if there is an m such that $(\mathcal{K}, m)$ is hyperbolic. We will say that a group is hyperbolic if it is isomorphic to the fundamental group of a connected hyperbolic complex.

It is easy to see that if $(\mathcal{K}, m)$ is hyperbolic and if $\phi : \tilde{\mathcal{K}} \to \mathcal{K}$ is a strong mapping which is rigid and reduced (for example, ϕ could be a covering) then $(\tilde{\mathcal{K}}, \tilde{m})$ is hyperbolic, where $\tilde{m}$ is the induced weight function. We deduce, in particular (making use of (1.2), (1.3) and Theorem 1) that the class of hyperbolic groups is closed under taking subgroups.

4. The dependence problems for hyperbolic complexes. Let $\mathcal{K}$ be a 2-complex and let $(\omega_0, \omega_1, \ldots, \omega_k)$ be a sequence of non-empty cyclically reduced closed paths in $\mathcal{K}$. We write

$$(\omega_1, \ldots, \omega_k) \vdash \omega_0$$

if there is a subset $\{i_1, \ldots, i_l\}$ of $\{1, \ldots, k\}$ and paths $\eta_1, \ldots, \eta_l$ such that

$$\omega_0 \sim_{\mathcal{K}} (\eta_1^{-1} \omega_{i_1}^{-1} \eta_1)(\eta_2^{-1} \omega_{i_2}^{-1} \eta_2) \ldots (\eta_l^{-1} \omega_{i_l}^{-1} \eta_l).$$

If n is a positive integer or ∞, then the *dependence problem* DP(n) asks for an algorithm to decide for any sequence $(\omega_0, \omega_1, \ldots, \omega_k)$ $(0 \leq k < n)$ whether or not $(\omega_1, \ldots, \omega_k) \vdash \omega_0$.

Note that DP(1) and DP(2) are just the *word problem* and *conjugacy problem* for $\mathscr{K}$, respectively.

THEOREM 2. *Suppose* $(\mathscr{K}, m)$ *is hyperbolic and* $\mathscr{K}$ *is finite. Then* DP(∞) *is solvable for* $\mathscr{K}$.

In order to prove this result we introduce the concept of a *diagram*.

Let S be a tesselated sphere. We associate with S a 2-complex $\mathscr{C}_S$ as follows. The vertices of $\mathscr{C}_S$ are the vertices of S. Let e be an edge of S with endpoints u, v, and choose an orientation of e, say running from u to v. Then e gives rise to an inverse pair e_+, e_- of edges of $\mathscr{C}_S$ with $\iota(e_+) = u$, $\tau(e_-) = v$. Let D be a region of S and choose a fixed but arbitrary vertex w on ∂D. Read around ∂D in the clockwise direction starting at w. If we traverse an edge e in the direction of its orientation we write down the symbol e_+, and if we traverse e against its orientation we write down e_-. In this way we obtain a closed path β_D. The defining paths of $\mathscr{C}_S$ are all the paths β_D as D ranges over the regions of S. (Note that $\mathscr{C}_S$ is not uniquely defined; it depends on the choice of orientation of the edges of S, and also on which vertices we choose to start reading around the boundaries of regions. However, this ambiguity is of no great consequence.)

A *diagram* over a 2-complex $\mathscr{L}$ is a triple (S, Θ, ϕ), where S is a finitely tesselated sphere, Θ is a subset of the set of regions of S, ϕ is a strong rigid mapping from

$$\mathring{\mathscr{C}}_S = \langle \mathscr{C}_S^{(1)}; \beta_D(D \notin \Theta)\rangle$$

to $\mathscr{L}$. The diagram is said to be *reduced* if ϕ^{st} is locally injective. The image of a path under ϕ is called the *label* of the path. The regions in Θ are called *distinguished regions*.

(4.1) $(\omega_1, \omega_2, \ldots, \omega_k) \vdash \omega_0$ *if and only if there is a (reduced) diagram* $(S, \{D_0, D_1, \ldots, D_l\}, \phi)$ *over* $\mathscr{L}$ *with* $\phi(\beta_{D_0}) = \omega_0$ *and* $(\phi(\beta_{D_1}), \ldots, \phi(\beta_{D_l}))$ *a subsequence of* $(\omega_1, \ldots, \omega_k)$.

This result is proved by arguments like those in Section V.1 of [**11**]. We sketch the proof briefly. The "if" part is proved in a similar way to the proof of Lemma V.1.2 of [**11**]. To prove the "only if" part, suppose $(\omega_1, \ldots, \omega_k) \vdash \omega_0$. Then ω_0^{-1} is equivalent in $\mathscr{L}^{(1)}$ to a product

$$\prod_{i=1}^{r} \gamma_i \delta_i \gamma_i^{-1},$$

where some subsequence $(\delta_{j_1}, \ldots, \delta_{j_l})$ of $(\delta_1, \ldots, \delta_r)$ is a subsequence of $(\omega_1, \ldots, \omega_k)$ and for $i \neq j_1, \ldots, j_l$, $\delta_i \in R(\mathscr{L})$. Among all products of the above form which are equivalent to ω_0^{-1} in $\mathscr{L}^{(1)}$, choose one with r minimal. Then, starting with the usual "bunch of lollipops" labelled as indicated and "sewing up the boundary" (see [**11**, pp. 237–238]), we obtain a diagram as required in (4.1). See overleaf.

As a consequence of (4.1) we have the following easily proved result.

(4.2) *Assume* $\mathscr{L}$ *is finite. Suppose that for* $0 \leq l < n$ *there are recursive functions* g_l *of* $l+1$ *variables such that whenever* $(S, \{D_0, D_1, \ldots, D_l\}, \phi)$ *is a reduced diagram over* $\mathscr{L}$,

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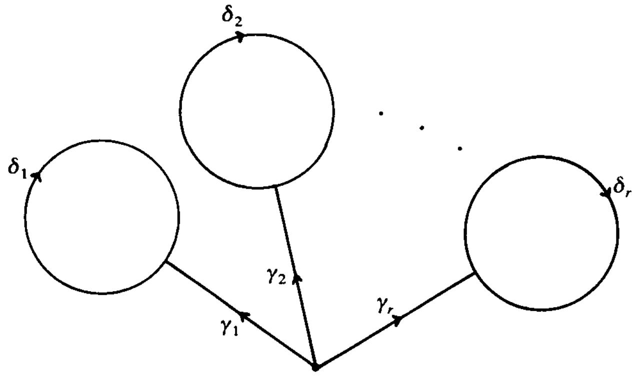

the number of regions of S is bounded above by $g_l(L(\beta_{D_0}), \ldots, L(\beta_{D_l}))$. *Then* DP($n$) *is solvable for* $\mathscr{L}$.

We now prove Theorem 2 by verifying that (4.2) holds. We will show that we can take

$$g_l(z_0, z_1, \ldots, z_l) = 1 + \frac{3+\varepsilon+N}{\varepsilon} \sum_{i=0}^{l} z_i \quad (l = 0, 1, 2, \ldots),$$

where

$$\varepsilon = \min\{L(\gamma) - m^*(\gamma) - 2 : \gamma \in R(\mathscr{K})\}.$$

Thus we must show that if S is a finitely tesselated sphere with regions $D_0, D_1, \ldots, D_r$, and if $(S, \{D_0, \ldots, D_l\}, \phi)$ is a reduced diagram over $\mathscr{K}$, then

$$r \leq \frac{3+\varepsilon+N}{\varepsilon} \sum_{i=0}^{l} L(\beta_{D_i}). \tag{4.3}$$

We can assume that S has more than two regions, otherwise the result is trivial. Let

$$\mathring{\mathscr{C}}_S = \langle \mathscr{C}_S^{(1)} : \beta_{D_{l+1}}, \ldots, \beta_{D_r} \rangle,$$

and let $\mathring{m}$ be the induced weight function. Let

$$2s = \sum \mathring{m}(\delta),$$

where δ ranges over $R(\mathring{\mathscr{C}}_S)$. Now since S has more than two regions, each component of $\mathscr{C}_S^{\mathrm{st}}$ is a circle. If we remove all the edges of $\mathscr{C}_S^{\mathrm{st}}$ which are cyclic permutations of $\beta_{D_0}^{\pm 1}, \ldots, \beta_{D_l}^{\pm 1}$, we obtain a collection of circles and lines. Suppose there are c circles and d lines. Then

$$c \geq |V(\mathscr{C}_S)| - \sum_{i=0}^{l} L(\beta_{D_i}),$$

$$d \leq \sum_{i=0}^{l} L(\beta_{D_i}).$$

For each circle, the weight of the closed path going once around the circle is at least 2. For each line, the weight of the path determined by the line is at least $-N$. Thus

$$\begin{aligned} s &\geqslant 2c - Nd \\ &\geqslant 2\Big(|V(\mathscr{C}_S)| - \sum_{i=0}^{l} L(\beta_{D_i})\Big) - N\sum_{i=0}^{l} L(\beta_{D_i}). \end{aligned}$$

But we also have

$$\begin{aligned} s &= \sum_{i=l+1}^{r} \mathring{m}^*(\beta_{D_i}) \\ &\leqslant \sum_{i=l+1}^{r} (L(\beta_{D_i}) - (2+\varepsilon)) \\ &= |E(\mathscr{C}_S)| - \sum_{i=0}^{l} L(\beta_{D_i}) - (r-l)(2+\varepsilon). \end{aligned}$$

Eliminating s between these two inequalities, and using the fact that $2\,|V(\mathscr{C}_S)| - |E(\mathscr{C}_S)| + 2(r+1) = 2\chi(S) = 4$, we obtain

$$\begin{aligned} r\varepsilon &\leqslant -2 + (1+N)\sum_{i=0}^{l} L(\beta_{D_i}) + l(2+\varepsilon) \\ &= -4 - \varepsilon + (1+N)\sum_{i=0}^{l} L(\beta_{D_i}) + (2+\varepsilon)(l+1) \\ &< (3+\varepsilon+N)\sum_{i=0}^{l} L(\beta_{D_i}), \end{aligned}$$

which gives (4.3).

5. Small cancellation complexes. Throughout this section we will assume for convenience that $\mathscr{K}$ is a 2-complex satisfying the condition that each defining path has length greater than 1. This assumption will not be repeated.

A non-empty path π in $\mathscr{K}$ is called a *piece* if there are distinct elements $\pi\alpha$, $\pi\beta \in R(\mathscr{K})$. The complex $\mathscr{K}$ satisfies the small cancellation condition $C(p)$ (p a positive integer) if no element of $R(\mathscr{K})$ is the product of less than p pieces. This condition is only of use when considered in conjunction with another small cancellation condition $T(q)$, where q is an integer greater than 2 and $1/p + 1/q \leqslant \frac{1}{2}$. In **[6]** we gave a formulation of the $T(q)$ condition in terms of star-complexes, namely, $\mathscr{K}$ satisfies $T(q)$ if and only if there are no cyclically reduced closed paths in $\mathscr{K}^{\mathrm{st}}$ of length l with $3 \leqslant l < q$. Let us now say that $\mathscr{K}$ satisfies the $\tilde{T}(q)$ condition if there are no non-empty cyclically reduced closed paths in $\mathscr{K}^{\mathrm{st}}$ of length less than q.

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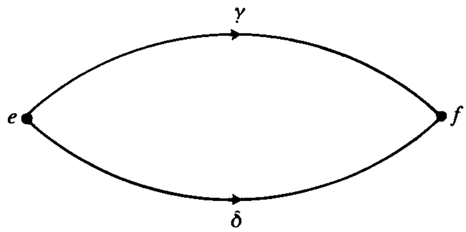
If $\mathscr{K}$ satisfies $T(q)$ then it will also satisfy $\bar{T}(q)$ provided there are not distinct edges γ, δ in $\mathscr{K}^{st}$ with $\iota^{st}(\gamma)=\iota^{st}(\delta)$, $\tau^{st}(\gamma)=\tau^{st}(\delta)$.



Now observe that if we have γ, δ as above, then we have a piece of length 2, namely $f^{-1}e$. For there are distinct paths α, β with $\gamma = e\alpha f^{-1}$, $\delta = e\beta f^{-1}$. Conversely, if we have a piece $f^{-1}e$ of length 2 then we have edges γ, δ in $\mathscr{K}^{st}$ with $\iota^{st}(\gamma)=\iota^{st}(\delta)=e$, $\tau^{st}(\gamma)=\tau^{st}(\delta)=f$. Thus *a $T(q)$-complex satisfies $\bar{T}(q)$ if and only if there are no pieces of length* 2.

Now if $\mathscr{K}$ satisfies $\bar{T}(q)$ then, since there are no pieces of length 2, *$\mathscr{K}$ will satisfy $C(p)$ if and only if, for each defining path ρ, $L(\rho)\geq p$ unless ρ contains an edge which is not a piece*. We will say that a $C(p)$, $\bar{T}(q)$-complex is *non-degenerate* if $L(\rho)\geq p$ for each defining path p.

Having introduced the $\bar{T}(q)$ condition, let us now observe that *if $q\geq 5$ then $T(q)$ and $\bar{T}(q)$* are the same property. For suppose $\mathscr{K}$ satisfied $T(q)$ but not $\bar{T}(q)$ $(q\geq 5)$. Then in $\mathscr{K}^{st}$ there would be a reduced closed path of length 2. The square of this would then be a reduced closed path of length 4, contradicting $T(q)$.

We see in particular from the previous paragraphs that *in a $T(6)$-complex there are no pieces of length* 2, *and the complex satisfies $C(3)$ if and only if for each defining path ρ, $L(\rho)\geq 3$ unless ρ contains an edge which is not a piece*. Some results concerning $C(3)$, $T(6)$-groups can be found in [**3**].

It is interesting to observe how the properties $C(p)$, $T(q)$, $\bar{T}(q)$ behave with respect to mappings of 2-complexes.

THEOREM 3. *Let $\phi:\bar{\mathscr{K}}\to\mathscr{K}$ be a strong mapping.*

(i) *If $\phi^{(1)}$ is locally injective and if $\mathscr{K}$ satisfies $C(p)$ then $\bar{\mathscr{K}}$ satisfies $C(p)$.*

(ii) *If ϕ^{st} is locally injective and if $\mathscr{K}$ satisfies $T(q)$ (or $\bar{T}(q)$) then $\bar{\mathscr{K}}$ satisfies $T(q)$ (or $\bar{T}(q)$).*

Proof. (i) First observe that if $\bar{\pi}$ is a piece in $\bar{\mathscr{K}}$ then $\phi(\bar{\pi})$ is a piece in $\mathscr{K}$. For there are distinct elements $\bar{\pi}\bar{\alpha}$, $\bar{\pi}\bar{\beta}\in R(\bar{\mathscr{K}})$, and then $\phi(\bar{\pi})\phi(\bar{\alpha})$, $\phi(\bar{\pi})\phi(\bar{\beta})$ are elements of $R(\mathscr{K})$, which are distinct (by uniqueness of lifts).

It now follows that if $\bar{\pi}_1\bar{\pi}_2\ldots\bar{\pi}_r$ is a factorization of an element $\bar{\gamma}$ of $R(\bar{\mathscr{K}})$ into pieces then $\phi(\bar{\pi}_1)\phi(\bar{\pi}_2)\ldots\phi(\bar{\pi}_r)$ is a factorization of $\phi(\bar{\gamma})$ into pieces. Thus $r\geq p$.

(ii) It is a general result that the image of a cyclically reduced closed path under a locally injective mapping is a cyclically reduced closed path. Hence if ϕ^{st} is locally injective and if there is a cyclically reduced closed path in $\tilde{\mathcal{K}}^{st}$ of a certain length l, then there will be a cyclically reduced closed path in $\mathcal{K}^{st}$ of length l.

Let us say that a group is a *$C(p)$, $T(q)$-group* (or a *$C(p)$, $\tilde{T}(q)$-group*) if it is isomorphic to the fundamental group of a connected $C(p)$, $T(q)$-complex (or a $C(p)$, $\tilde{T}(q)$-complex). Then using Theorem 3 together with (1.2), (1.3) and Theorem 1, we obtain the following result.

COROLLARY. *The class of $C(p)$, $T(q)$-groups, and the class of $C(p)$, $\tilde{T}(q)$-groups, is closed under taking subgroups.*

This result is essentially that of Comerford [**1**].

We now relate small cancellation complexes to hyperbolic complexes.

THEOREM 4. *If $\mathcal{K}$ is a non-degenerate $C(p)$, $\tilde{T}(q)$-complex, where $1/p+1/q<\frac{1}{2}$, then $\mathcal{K}$ is hyperbolic.*

Proof. Define a weight function m on $\mathcal{K}^{st}$ by setting $m(\gamma)=2/q$ for each edge $\gamma\in R(\mathcal{K})$ of $\mathcal{K}^{st}$. Then the weight of any non-empty cyclically reduced closed path in $\mathcal{K}^{st}$ is at least 2, since such a path has length at least q. Also, for any $\gamma\in R(\mathcal{K})$, since $L(\gamma)\geqslant p$, we have:

$$
\begin{aligned}
L(\gamma)-m^*(\gamma) &= L(\gamma)-L(\gamma)\frac{2}{q}\\
&\geqslant p\left(1-\frac{2}{q}\right)\\
&> p\frac{2}{p}\\
&=2.
\end{aligned}
$$

Thus $(\mathcal{K}, m)$ is hyperbolic.

COROLLARY. *Any non-degenerate $C(7)$, $\tilde{T}(3)$-, $C(5)$, $\tilde{T}(4)$-, $C(4)$, $T(5)$-, or $C(3)$, $T(7)$-complex is hyperbolic.*

We remark that, in general, non-degenerate $C(6)$, $\tilde{T}(3)$-, $C(4)$, $\tilde{T}(4)$-, and $C(3)$, $T(6)$-complexes need not be hyperbolic, as the presentations

$$\langle a, b, c; a^{-1}b^{-1}c^{-1}abc\rangle,\ \langle a, b; a^{-1}b^{-1}ab\rangle,\ \langle a, b, t; a^{-1}b^{-1}t^{-1}, tab\rangle$$

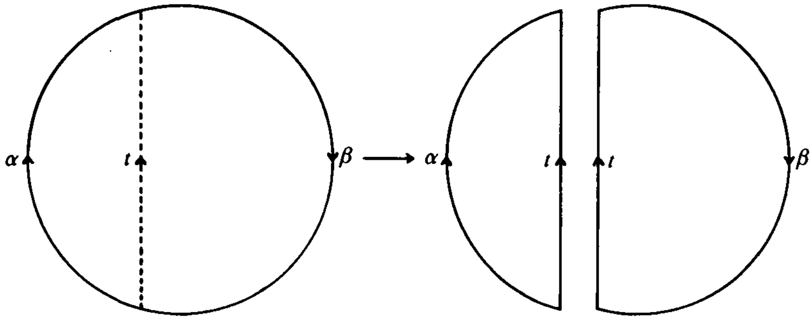
show.

6. Subdividing defining paths. Let $\mathcal{K}=\langle\mathcal{X}; \rho_\lambda(\lambda\in\Lambda)\rangle$, where we assume that no ρ_λ is a cyclic permutation of $\rho_\nu^{\pm1}$ $(\nu\neq\lambda)$. Let μ be an element of Λ and suppose that $\alpha\beta$ is a cyclic permutation of ρ_μ. Adjoin to $\mathcal{X}$ a new edge pair t, t^{-1} with $\iota(t)=\iota(\alpha)$,

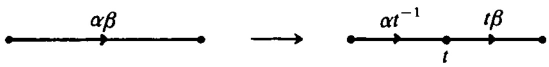
$\tau(t) = \tau(\alpha)$ giving a 1-complex $\mathscr{X}_0$, and let

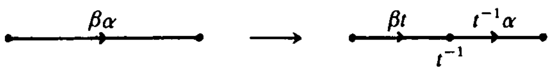
$$\mathscr{K}_0 = \langle \mathscr{X}_0; \rho_\lambda\ (\lambda \in \Lambda,\ \lambda \neq \mu),\ \alpha t^{-1},\ t\beta \rangle.$$

We say that $\mathscr{K}_0$ is obtained from $\mathscr{K}$ by *subdividing the defining path* ρ_μ.

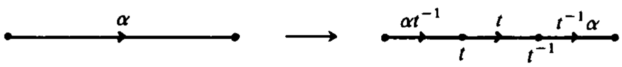


We will be particularly interested in the above operation in the case when ρ_μ is not a proper power. In this case, $\mathscr{K}_0^{st}$ is obtained from $\mathscr{K}^{st}$ in a particularly simple way. Each of the edges $\alpha\beta$, $\beta\alpha$ is subdivided into two:

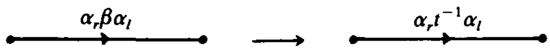




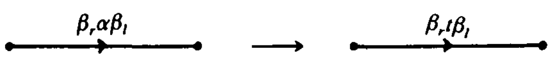
The only exception to this is when one of α, β is empty. If β (say), is empty, then the edge α gets subdivided into three:



The remaining edges of $\mathscr{K}^{st}$ which are cyclic permutations of ρ_μ are changed as follows:



$(\alpha = \alpha_l\alpha_r,\ \alpha_l,\ \alpha_r$ non-empty)



$(\beta = \beta_l\beta_r,\ \beta_l,\ \beta_r$ non-empty)

The edges of $\mathscr{K}^{st}$ which are not cyclic permutations of $\rho_\mu^{\pm 1}$ are left unchanged.

THEOREM 5. *If $\mathscr{K}_0$ is obtained from $\mathscr{K}$ by subdividing a defining path which is not a proper power, then $\mathscr{K}_0$ is hyperbolic if and only if $\mathscr{K}$ is hyperbolic.*

We remark that if $\mathscr{K}$ is hyperbolic and $\mathscr{K}_0$ is obtained from $\mathscr{K}$ by subdividing a defining path which is a proper power then $\mathscr{K}_0$ may not be hyperbolic. For example, let $\mathscr{K}$

be the presentation $\langle a, b; (a^{-1}b^{-1}ab)^2\rangle$. Then $\mathcal{K}$ is hyperbolic (it satisfies $C(8)$, $\tilde{T}(4)$). However, if $\mathcal{K}_0 = \langle a, b, t; a^{-1}b^{-1}abt^{-1}, ta^{-1}b^{-1}ab\rangle$ then $\mathcal{K}_0$ is not hyperbolic.

In our proof of Theorem 5 we will take $\mathcal{K}$ and $\mathcal{K}_0$ to be as in the discussion preceding the statement of the theorem, and we will assume that ρ_μ is not a proper power. If one of α, β is empty then, as above, we will assume that β is empty.

Suppose that m and m_0 are weight functions on $\mathcal{K}^{\mathrm{st}}$, $\mathcal{K}_0^{\mathrm{st}}$ respectively which are related to each other as follows. For each edge γ which is not a cyclic permutation of $\rho_\mu^{\pm 1}$, $m(\gamma) = m_0(\gamma)$. For an edge $\alpha_r\beta\alpha_l$ (resp. $\beta_r\alpha\beta_l$) of $\mathcal{K}^{\mathrm{st}}$ and the corresponding edge $\alpha_r t^{-1}\alpha_l$ (resp. $\beta_r t\beta_l$) of $\mathcal{K}_0^{\mathrm{st}}$, $m(\alpha_r\beta\alpha_l) = m_0(\alpha_r t^{-1}\alpha_l)$ (resp. $m(\beta_r\alpha\beta_l) = m_0(\beta_r t\beta_l)$). (With an eye to future computations we let c (resp. d) denote the sum of the m-values of all edges of $\mathcal{K}^{\mathrm{st}}$ of the form $\alpha_r\beta\alpha_l$ (resp. $\beta_r\alpha\beta_l$).) Finally

$$\begin{aligned} &m(\alpha\beta) = m_0(\alpha t^{-1}) + m_0(t\beta) \quad \text{and} \\ &m(\beta\alpha) = m_0(\beta t) + m_0(t^{-1}\alpha) \quad \text{if } \beta \text{ is non-empty}, \\ &m(\alpha) = m_0(\alpha t^{-1}) + m_0(t) + m_0(t^{-1}\alpha) \quad \text{if } \beta \text{ is empty}. \end{aligned} \tag{6.1}$$

Then it is clear that (3.2) and (3.3) hold for $(\mathcal{K}_0, m_0)$ if and only if they hold for $(\mathcal{K}, m)$.

Moreover, suppose that (3.1) holds for $(\mathcal{K}_0, m_0)$. Then (3.1) holds for $(\mathcal{K}, m)$. To establish this we must show that

$$m^*(\rho_\mu) < L(\rho_\mu) - 2.$$

If β is non-empty we have

$$\begin{aligned} m^*(\rho_\mu) &= m(\alpha\beta) + c + d + m(\beta\alpha) \\ &= (m_0(\alpha t^{-1}) + c + m_0(t^{-1}\alpha)) + (m_0(\beta t) + d + m_0(t\beta)) \\ &= m_0^*(\alpha t^{-1}) + m_0^*(t\beta) \\ &< (L(\alpha) + 1 - 2) + (L(\beta) + 1 - 2) \\ &= L(\rho_\mu) - 2, \end{aligned}$$

while if β is empty we have

$$\begin{aligned} m^*(\rho_\mu) &= m(\alpha) + c \\ &= m_0^*(\alpha t^{-1}) + m_0^*(t) \\ &< (L(\alpha) + 1 - 2) + (1 - 2) \\ &= L(\rho_\mu) - 2. \end{aligned}$$

Now suppose that (3.1) holds for $(\mathcal{K}, m)$. Then (3.1) will hold for $(\mathcal{K}_0, m_0)$ provided the following conditions are satisfied.

$$m_0(\alpha t^{-1}) + c + m_0(t^{-1}\alpha) < L(\alpha) - 1. \tag{6.2}$$

$$\begin{aligned} &m_0(\beta t) + d + m_0(t\beta) < L(\beta) - 1 \quad \text{if } \beta \text{ is non-empty}, \\ &m_0(t) < -1 \quad \text{if } \beta \text{ is empty}. \end{aligned} \tag{6.3}$$

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These conditions are compatible with those already imposed. To see this, let

$$-\theta = m^*(\rho_\mu) - L(\rho_\mu) + 2$$

(note then that $\theta > 0$), and let κ be any real number. Put

$$m_0(\alpha t^{-1}) = \kappa, \quad m_0(t^{-1}\alpha) = L(\alpha) - 1 - \kappa - c - \frac{\theta}{2},$$

$$m_0(t\beta) = m(\alpha\beta) - \kappa \quad \text{and}$$

$$m_0(\beta t) = m(\beta\alpha) - L(\alpha) + 1 + \kappa + c + \frac{\theta}{2} \quad \text{if } \beta \text{ is non-empty,}$$

$$m_0(t) = m(\alpha) - L(\alpha) + 1 + c + \frac{\theta}{2} \qquad \text{if } \beta \text{ is empty.}$$

Then clearly (6.1) and (6.2) are satisfied. Also, (6.3) is satisfied. For, if β is non-empty, we have

$$\begin{aligned} m_0(\beta t) &+ d + m_0(t\beta) - L(\beta) + 1 \\ &= m(\beta\alpha) - L(\alpha) + 1 + \kappa + c + \frac{\theta}{2} + d + m(\alpha\beta) - \kappa - L(\beta) + 1 \\ &= (m(\beta\alpha) + c + d + m(\alpha\beta)) - L(\alpha\beta) + 2 + \frac{\theta}{2} \\ &= (m^*(\rho_\mu) - L(\rho_\mu) + 2) + \frac{\theta}{2} \\ &= -\theta + \frac{\theta}{2} \\ &< 0. \end{aligned}$$

On the other hand, if β is empty then

$$\begin{aligned} m_0(t) + 1 &= m(\alpha) + c - L(\alpha) + 2 + \frac{\theta}{2} \\ &= (m^*(\alpha) - L(\alpha) + 2) + \frac{\theta}{2} \\ &= -\theta + \frac{\theta}{2} \\ &< 0. \end{aligned}$$

Note added in proof. Concepts of hyperbolicity in group theory are discussed in great detail in a recently published paper by M. Gromov [**14**].

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