# A CLASS OF SOLUTIONS OF EINSTEIN-MAXWELL EQUATIONS WITH THE COSMOLOGICAL CONSTANT

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**Abstract.** Working in the signature $(+++-)$ and units such that $G=1=c$, it was found a solution of Einstein-Maxwell equations with $\lambda$ (without current and pseudo-current). In real coordinates $x^\mu=(p,\sigma,q,\tau)$ the solutions is:

$$\omega=:\tfrac{1}{2}(f_{\mu\nu}+\breve{f}_{\mu\nu})\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu=-\mathrm{d}\left\{\frac{e_0+ig_0}{q+ip}(\mathrm{d}\tau-ipq\,\mathrm{d}\sigma)\right\}, \tag{1}$$

$$\begin{aligned}\mathrm{d}s^2=:&\frac{p^2+q^2}{P}\,\mathrm{d}p^2+\frac{P}{p^2+q^2}(\mathrm{d}\tau+q^2\,\mathrm{d}\sigma)^2+\\&+\frac{p^2+q^2}{Q}\,\mathrm{d}q^2-\frac{Q}{p^2+q^2}(\mathrm{d}\tau-p^2\,\mathrm{d}\sigma)^2,\end{aligned} \tag{2}$$

where

$$\begin{aligned}P&=:b-g_0^2+2n_0p-\varepsilon p^2-\tfrac{\lambda}{3}p^4,\\Q&=:b+e_0^2-2m_0q+\varepsilon q^2-\tfrac{\lambda}{3}q^4,\end{aligned} \tag{3}$$

$[\breve{f}^{\mu\nu}=:(i/2\sqrt{-g})\,\varepsilon^{\mu\nu\varrho\sigma}f_{\varrho\sigma}$ is pure imaginary; in (1) 'd' denotes the external differential]. Not all constants $m_0$, $n_0$, $e_0$, $g_0$, $b$, $\varepsilon$, $\lambda$ are physically significant: by re-scaling coordinates $\varepsilon$ can be made equal to $+1, 0,$ or $-1$. The solution is of the type D: the double Debever-Penrose vectors

ff

$$\pm k_\mu^{(\pm)}\,\mathrm{d}x^\mu=:\mathrm{d}\left(\tau\pm\int\frac{q^2\,\mathrm{d}q}{Q}\right)-p^2\,\mathrm{d}\left(\sigma\mp\int\frac{\mathrm{d}q}{Q}\right) \tag{4}$$

have the common complex expansion $Z=(q+ip)^{-1}$. Among $C^{(a)}$'s only $C^{(3)}$ given by:

$$C^{(3)}=\frac{-2}{(q+ip)^2}\left\{\frac{m_0+in_0}{q+ip}-\frac{e_0^2+g_0^2}{q^2+p^2}\right\} \tag{5}$$

is in general $\neq 0$. The invariants of the electromagnetic field are:

$$F=:\tfrac{1}{4}f_{\mu\nu}f^{\mu\nu}+\tfrac{1}{4}f_{\mu\nu}\breve{f}^{\mu\nu}=-\frac{1}{2}\frac{(e_0+ig_0)^2}{(q+ip)^4}. \tag{6}$$

The constants contained in (1)–(6) have the interpretation of: (1) $e_0$ and $g_0$ are the electric and magnetic monopoles charges respectively, (2) $m_0$ and $n_0$ are the mass and NUT parameters (3) $b$ is related to the Kerr constant (4) $\lambda$ is cosmologic constant (5) the sign $\varepsilon$ in the sub-family of solutions which contains Kerr metric

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is equal to $+1$. [With $\varepsilon=1$, $\lambda=0$ the result described above amounts to the charged Kerr-Newman-NUT metric generalized by the presence of the magnetic monopole; here $b=g_0^2-n_0^2+a_0^2$ where $a_0$ is the Kerr constant.]

For a test particle of mass $\Delta m$ which carries electric and magnetic charges $\Delta e$, $\Delta g$ the Hamilton-Jacobi equation is separable: The solution of this equation is:

$$W=C_\tau\cdot\tau+C_\sigma\cdot\sigma+\varepsilon_1\int\frac{\mathrm{d}p}{\sqrt{P}}\left[C_0-(\Delta m)^2\,p^2-\frac{1}{P}(p^2C_\tau-p\Delta_g+C_\sigma)\right]^{1/2}$$
$$+\varepsilon_2\int\frac{\mathrm{d}q}{\sqrt{Q}}\left[-C_0-(\Delta m)^2\,q^2+\frac{1}{Q}(q^2C_\tau-q\Delta_e-C_\sigma)\right]^{1/2} \tag{7}$$

where

$$\varepsilon_1^2=1=\varepsilon_2^2,\qquad \Delta_e+i\Delta_g=:(\Delta e-i\Delta g)(e_0+ig_0). \tag{8}$$

and $C_\tau$, $C_\sigma$, $C_0$ are the separation constants.

Working together with M. Demiański we generalized these results as follows: we have a solution of Maxwell-Einstein equations with $\lambda$ described by:

$$\omega=\mathrm{d}\left\{\frac{e+ig}{1-ipq}(q\,\mathrm{d}\tau+ip\,\mathrm{d}\sigma)\right\} \tag{9}$$

$$\mathrm{d}s^2=\frac{1}{(p+q)^2}\cdot\left\{\frac{1+(pq)^2}{P}\,\mathrm{d}p^2+\frac{P}{1+(pq)^2}(\mathrm{d}\sigma+q^2\,\mathrm{d}\tau)^2+\right.$$
$$\left.\neq\frac{1+(pq)^2}{Q}\,\mathrm{d}q^2\equiv\frac{Q}{1\neq \mathrm{j}6ql^2}\,\mathrm{jdJ}\equiv 6^2\,\mathrm{d}\sigma l^2\right\} \tag{10}$$

$$P=:\left(\frac{-\lambda}{6}-g^2+\gamma\right)+2np-\varepsilon p^2+2mp^3+\left(\frac{-\lambda}{6}-e^2-\gamma\right)p^4$$
$$Q=:\left(\frac{-\lambda}{6}+g^2-\gamma\right)+2nq+\varepsilon q^2+2mq^3+\left(\frac{-\lambda}{6}+e^2+\gamma\right)q^4 \tag{11}$$

endowed in continuous constants $m$, $n$, $e$, $g$, $\varepsilon$, $\gamma$, $\lambda$. This is also a solution of the type D with twisting double Debever-Penrose directions.

We have here:

$$C^{(3)}=:2(m+in)\left(\frac{p+q}{1-ipq}\right)^3-2(e^2+g^2)\left(\frac{p+q}{1-ipq}\right)^3\frac{p-q}{1+ipq} \tag{12}$$

$$F=:-\tfrac{1}{2}(e+ig)^2\left(\frac{p+q}{1-ipq}\right)^4. \tag{13}$$

The transformation $q\to-1/q$, then $(p,q)\to(1/e)(p,q)$, $\tau\to e\tau$, $\sigma\to e^3\sigma$; $P\to e^4P$, $Q\to e^4Q$, $e+ig\to e^{-2}(e_0+ig_0)$, $m+in\to e^{-3}(m_0+in_0)$, $\varepsilon\to e^{-2}\varepsilon$, $\gamma\to e^{-4}b+(\lambda/6)$, $\lambda\to\lambda$ yields in the limit $e\to\infty$ the solution previously described by (1)–(6). Another contraction: $(p,q,\sigma,\tau)\to e^{-1}(p,q,\sigma,\tau)$, $n\to ne$, $\varepsilon\to\varepsilon e^2$, $m\to me^3$, $e+ig\to(e_0+ig_0)\,e^2$, $\gamma\to\gamma+e^4g^2$,

$\lambda \to \lambda$, and then $e \to \infty$ brings the solution to the Kinerseley-Walker family of solutions.

The solution described by (9)–(13) in general is not separable. Constants $e$, $g$, $m$, $n$ are related to electric and magnetic charges, mass and NUT parameters; $\lambda$ is the cosmological constant; it is conjected that 'kinematical constants' $\gamma$ and $\varepsilon$ are related to uniform acceleration and rotation parameters ($\gamma$ in contractions corresponds to the Kerr constant).