

SOME TRIPLE TRIGONOMETRICAL SERIES

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1. Introduction. This paper considers the determination of the coefficients in two sets of triple trigonometrical series and shows that these can be obtained in closed form. The series considered are special cases of some triple series in Jacobi polynomials studied by K. N. Srivastava [1]. Srivastava, however, shows that the problem for the more general series can be reduced to the solution of a Fredholm integral equation of the second kind and he does not discuss special cases which may lead to closed form solutions.

By employing an idea due to Williams [2], I reduce the problem to one in triple integral equations and I first discuss the determination of the coefficients A_n in the triple series

$$\sum_{n=1}^{\infty} A_n \sin nx = 0 \quad (0 \leq x < \alpha \quad \text{and} \quad \beta < x \leq \pi), \quad (1)$$

$$\sum_{n=1}^{\infty} nA_n \sin nx = f(x) \quad (\alpha < x < \beta), \quad (2)$$

where $f(x)$ is a prescribed function of x . I find that these coefficients are given by

$$A_n = 2 \int_0^{\infty} A(t) J_{2n}(t) dt, \quad (3)$$

where the function $A(t)$ satisfies the triple integral equations

$$\int_0^{\infty} A(t) \sin ut dt = 0 \quad (0 \leq u < \sin \frac{1}{2}\alpha \quad \text{and} \quad \sin \frac{1}{2}\beta < u < \infty), \quad (4)$$

$$\int_0^{\infty} tA(t) \sin ut dt = \frac{2}{\sqrt{(1-u^2)}} f(2 \sin^{-1} u) \quad (\sin \frac{1}{2}\alpha < u < \sin \frac{1}{2}\beta). \quad (5)$$

As I have already given a method [3] for finding $A(t)$ from these triple integral equations, the determination of the coefficients A_n in the triple trigonometrical series (1) and (2) is theoretically possible.

In boundary value problems in which the triple series (1) and (2) occur, interest often lies more in the determination of the function $V(x)$ given by

$$V(x) = \sum_{n=1}^{\infty} A_n \sin nx \quad (\alpha < x < \beta) \quad (6)$$

than in that of the actual coefficients A_n . It is therefore worth noting that this function can be found from the relation

$$V(2 \sin^{-1} u) = \int_0^{\infty} A(t) \sin ut dt \quad (\sin \frac{1}{2}\alpha < u < \sin \frac{1}{2}\beta), \quad (7)$$

and that, in the analysis of the triple integral equations (4) and (5), the integral in equation (7) is rather more easily found than the function $A(t)$ itself.

The analysis which leads to the above results is given in Section 2 below and this is followed in Section 3 by a worked example. The final section of this note gives the corresponding results when the sines in series (1) and (2) are replaced by cosines and when n is replaced by $n - \frac{1}{2}$.

2. The analysis. From equations (1) and (6),

$$A_n = \frac{2}{\pi} \int_{\alpha}^{\beta} V(x) \sin nx \, dx = \frac{4}{\pi} \int_{\sin \frac{1}{2}\alpha}^{\sin \frac{1}{2}\beta} V(2 \sin^{-1} u) \frac{\sin(2n \sin^{-1} u)}{\sqrt{(1-u^2)}} \, du,$$

where $u = \sin \frac{1}{2}x$. Now [4, p. 405]

$$\int_0^{\infty} J_{2n}(t) \sin ut \, dt = \frac{\sin(2n \sin^{-1} u)}{\sqrt{(1-u^2)}} \quad (0 < u < 1),$$

so that the relation (3) is established by writing

$$A(t) = \frac{2}{\pi} \int_{\sin \frac{1}{2}\alpha}^{\sin \frac{1}{2}\beta} V(2 \sin^{-1} u) \sin tu \, du. \tag{8}$$

Equation (8) implies the validity of relations (4) and (7) so that it only remains to establish equation (5) from equation (2).

This can be done by using Jacobi's expansion in series of Bessel coefficients; this is [4, p. 22]

$$\cos(t \sin \frac{1}{2}x) = J_0(t) + 2 \sum_{n=1}^{\infty} J_{2n}(t) \cos nx$$

and differentiation with respect to x gives

$$\frac{1}{2}t \cos \frac{1}{2}x \sin(t \sin \frac{1}{2}x) = 2 \sum_{n=1}^{\infty} nJ_{2n}(t) \sin nx. \tag{9}$$

Hence, from equations (2), (3) and (9), when $\alpha < x < \beta$,

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} nA_n \sin nx = 2 \int_0^{\infty} A(t) \sum_{n=1}^{\infty} nJ_{2n}(t) \sin nx \, dt \\ &= \int_0^{\infty} \frac{1}{2}tA(t) \cos \frac{1}{2}x \sin(t \sin \frac{1}{2}x) \, dt, \end{aligned}$$

and this is equation (5) when $\sin \frac{1}{2}x$ is replaced by u .

3. An example. I take $f(x) = \sin x$, so that the triple integral equations (4) and (5) are

$$\left. \begin{aligned} \int_0^\infty A(t) \sin ut \, dt &= 0 && (0 \leq u < \sin \frac{1}{2}\alpha \text{ and } \sin \frac{1}{2}\beta < u < \infty), \\ \int_0^\infty tA(t) \sin ut \, dt &= 4u && (\sin \frac{1}{2}\alpha < u < \sin \frac{1}{2}\beta). \end{aligned} \right\} \quad (10)$$

These equations can be solved by writing

$$A(t) = t^{-1} \sum_{r=1}^\infty (-1)^{r-1} c_r J_{2r}(t \sin \frac{1}{2}\beta) \quad (11)$$

and using the results [4, p. 405]

$$\int_0^\infty J_{2r}(t \sin \frac{1}{2}\beta) \sin ut \frac{dt}{t} = \begin{cases} \frac{1}{2r} \sin \{2r \sin^{-1}(u \operatorname{cosec} \frac{1}{2}\beta)\} & (u < \sin \frac{1}{2}\beta), \\ 0 & (u > \sin \frac{1}{2}\beta), \end{cases} \quad (12)$$

and

$$\int_0^\infty J_{2r}(t \sin \frac{1}{2}\beta) \sin ut \, dt = \frac{\sin \{2r \sin^{-1}(u \operatorname{cosec} \frac{1}{2}\beta)\}}{\sqrt{(\sin^2 \frac{1}{2}\beta - u^2)}} \quad (u < \sin \frac{1}{2}\beta). \quad (13)$$

Thus we find that the form chosen in (11) for $A(t)$ automatically gives $\int_0^\infty A(t) \sin ut \, dt = 0$ when $\sin \frac{1}{2}\beta < u < \infty$ and that the other two of equations (10) require that the coefficients c_r should satisfy the dual relations

$$\left. \begin{aligned} \sum_{r=1}^\infty c_r \sin r\theta &= 2 \sin^2 \frac{1}{2}\beta \sin \theta && (0 \leq \theta < \gamma), \\ \sum_{r=1}^\infty r^{-1} c_r \sin r\theta &= 0 && (\gamma < \theta \leq \pi), \end{aligned} \right\} \quad (14)$$

where

$$u = \cos \frac{1}{2}\theta \sin \frac{1}{2}\beta, \quad \cos \frac{1}{2}\gamma = \sin \frac{1}{2}\alpha \operatorname{cosec} \frac{1}{2}\beta. \quad (15)$$

It is also found that equation (7) becomes

$$V(2 \sin^{-1} u) = \int_0^\infty A(t) \sin ut \, dt = \frac{1}{2} \sum_{r=1}^\infty r^{-1} c_r \sin r\theta \quad (0 \leq \theta < \gamma). \quad (16)$$

Using the method given by Sneddon [5, p. 152] for the solution of the dual series (14), we find that, when $0 \leq \theta < \gamma$,

$$\sum_{r=1}^\infty r^{-1} c_r \sin r\theta = \sin \frac{1}{2}\theta \int_\theta^\gamma \frac{h(t) \, dt}{\sqrt{(\cos \theta - \cos t)}}, \quad (17)$$

where

$$h(t) = \frac{2}{\pi} \cot \frac{1}{2}t \int_0^t \frac{2 \sin^2 \frac{1}{2}\beta \sin \theta \sin \frac{1}{2}\theta}{\sqrt{(\cos \theta - \cos t)}} d\theta. \tag{18}$$

Since $2 \sin \theta \sin \frac{1}{2}\theta = \cos \frac{1}{2}\theta - \cos \frac{3}{2}\theta$, use of Mehler's integral for $P_n(\cos t)$ shows that

$$h(t) = \frac{2}{\pi} \cot \frac{1}{2}t \sin^2 \frac{1}{2}\beta \frac{\pi}{\sqrt{2}} \{P_0(\cos t) - P_1(\cos t)\} = \sqrt{2} \sin^2 \frac{1}{2}\beta \sin t.$$

Hence, when $0 \leq \theta < \gamma$,

$$\sum_{r=1}^{\infty} r^{-1} c_r \sin r\theta = \sqrt{2} \sin^2 \frac{1}{2}\beta \sin \frac{1}{2}\theta \int_0^\gamma \frac{\sin t dt}{\sqrt{(\cos \theta - \cos t)}} = 2 \sqrt{2} \sin^2 \frac{1}{2}\beta \sin \frac{1}{2}\theta \sqrt{(\cos \theta - \cos \gamma)}.$$

Using equations (15) and (16), we find that, when $\sin \frac{1}{2}\alpha < u < \sin \frac{1}{2}\beta$,

$$V(2 \sin^{-1} u) = 2(u^2 - \sin^2 \frac{1}{2}\alpha)^{\frac{1}{2}} (\sin^2 \frac{1}{2}\beta - u^2)^{\frac{1}{2}}.$$

Hence, when $\alpha < x < \beta$,

$$V(x) = 2(\sin^2 \frac{1}{2}x - \sin^2 \frac{1}{2}\alpha)^{\frac{1}{2}} (\sin^2 \frac{1}{2}\beta - \sin^2 \frac{1}{2}x)^{\frac{1}{2}} = (\cos \alpha - \cos x)^{\frac{1}{2}} (\cos x - \cos \beta)^{\frac{1}{2}}$$

and equations (1) and (6) then give

$$A_n = \frac{2}{\pi} \int_\alpha^\beta (\cos \alpha - \cos x)^{\frac{1}{2}} (\cos x - \cos \beta)^{\frac{1}{2}} \sin nx dx.$$

4. Triple cosine series. The coefficients B_n in the triple cosine series

$$\sum_{n=1}^{\infty} B_n \cos(n - \frac{1}{2})x = 0 \quad (0 \leq x < \alpha \quad \text{and} \quad \beta < x \leq \pi), \tag{19}$$

$$\sum_{n=1}^{\infty} (n - \frac{1}{2})B_n \cos(n - \frac{1}{2})x = f(x) \quad (\alpha < x < \beta), \tag{20}$$

can be found by analysis similar to that given in Section 2 above. The result is

$$B_n = 2 \int_0^\infty B(t) J_{2n-1}(t) dt, \tag{21}$$

where $B(t)$ satisfies the triple integral equations

$$\int_0^\infty B(t) \cos ut dt = 0 \quad (0 \leq u < \sin \frac{1}{2}\alpha \quad \text{and} \quad \sin \frac{1}{2}\beta < u < \infty), \tag{22}$$

$$\int_0^\infty t B(t) \cos ut dt = \frac{2}{\sqrt{(1-u^2)}} f(2 \sin^{-1} u) \quad (\sin \frac{1}{2}\alpha < u < \sin \frac{1}{2}\beta). \tag{23}$$

It can also be shown that, if

$$W(x) = \sum_{n=1}^{\infty} B_n \cos(n - \frac{1}{2})x \quad (\alpha < x < \beta), \quad (24)$$

then

$$W(2 \sin^{-1} u) = \int_0^{\infty} B(t) \cos ut \, dt \quad (\sin \frac{1}{2}\alpha < u < \sin \frac{1}{2}\beta). \quad (25)$$

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