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ON SUPERRECURRENCE

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ABSTRACT. Let T be a non-singular, conservative, ergodic automorphism of a Lebesgue space. We study a kind of weighted cocycles called H-cocycles. We introduce the notions of H-superrecurrence and H-supertransience. We use skew products to give necessary and sufficient conditions for H-superrecurrence.

1. **Introduction.** In studying cocyles Klaus Schmidt [4] proved that a cocycle of f is *recurrent* if and only if it is *superrecurrent*. In this paper, we study a kind of weighted cocycles [6] called *H-cocycles.* A natural problem is to try to generalize Schmidt's results [4] to *H-cocycles.* It is still unknown whether *H-recurrence* is equivalent to *H-superrecurrence*; however, we have made some progress toward a general understanding of the problem. In Section 3 we use skew products to obtain necessary and sufficient conditions of H-superrecurrence of H-cocycles. We also define the notion of H-supertransience and prove the following dichotomy: an H-cocycle is either H-superrecurrent or H-supertransient. In the remainder of Section 3, we show that the sufficient conditions which we obtained for H-superrecurrence can be relaxed. Finally, in Section 4 we give a few examples.

2. **Definitions and preliminaries.** Let $(X, \mathcal{B}, \mu)$ be a Lebesgue probability space. Let $T: X \longrightarrow X$ be a non-singular automorphism of X: that is, T is a measurable bijection of X such that for $A \in \mathcal{B}$,

$$\mu(TA) = 0 \text{ if and only if } \mu(A) = 0.$$

We also assume that the transformation T is conservative: for all $B \in \mathcal{B}$ with $\mu(B) > 0$, there exists $n \neq 0$ such that $\mu(B \cap T^{-n}B) > 0$, and aperiodic: $\mu(\cup_{n>0}\{x : T^n x = x\}) = 0$.

The non-singularity of T allows us to define for an integer $n \in Z$ a measure $\mu \circ T^n$ on X defined by $\mu \circ T^n(A) = \mu(T^n A)$ for $A \in \mathcal{B}$. These measures are equivalent to μ. For $n \in Z$, let $\omega_n(x) = \frac{d\mu\circ T^n}{d\mu}(x)$ be the Radon-Nikodym derivative of $\mu \circ T^n$ with respect to μ. Thus $\frac{d\mu\circ T^n}{d\mu}(x)$ is the almost everywhere unique function satisfying

$$\mu \circ T^n(A) = \int_A \frac{d\mu \circ T^n}{d\mu(x)}\, d\mu(x).$$

It is easy to see that

$$\omega_n(x) = \omega_1(x)\omega_1(Tx)\ldots\omega_1(T^{n-1}x),$$

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and

$$\omega_{n+m}(x) = \omega_n(x)\omega_m(T^n x) \text{ for all } n, m \in Z.$$

Let $f: X \longrightarrow R$ be any measurable function.

DEFINITIONS.

(1) The *cocycle* of f is defined to be the function $f^*: Z \times X \longrightarrow R$ given by

$$f^*(n,x) = \begin{cases} \sum_{i=0}^{n-1} f(T^i x), & \text{if } n > 0; \\ 0, & \text{if } n = 0; \\ -f^*(-n, T^n x), & \text{if } n < 0. \end{cases}$$

We have the following *cocycle* identity:

$$f^*(n+m,x) = f^*(n,x) + f^*(m, T^n x), \text{ for all } n, m \in Z, \text{ and for almost all } x \in X.$$

(2) The *H-cocycle* of f is defined to be the function $f_*: Z \times X \longrightarrow R$ given by

$$f_*(n,x) = \begin{cases} \sum_{i=0}^{n-1} f(T^i x)\omega_i(x), & \text{if } n > 0; \\ 0, & \text{if } n = 0; \\ -\omega_n(x) f_*(-n, T^n x), & \text{if } n < 0. \end{cases}$$

f_* satisfies the *H-cocycle* identity;

$$f_*(n+m,x) = f_*(n,x) + \omega_n(x) f_*(m, T^n x), \text{ for all } n, m \in Z, \text{ and for almost all } x \in X.$$

Observe that when T is measure preserving, the *cocycle* of f coincides with the *H-cocycle* of f.

(3) The *H-cocycle* (or *cocycle*) of f is said to be *H-recurrent* (or *recurrent*) if for every $\varepsilon > 0$, and for every $B \in \mathcal{B}$ with $\mu(B) > 0$, there exists $n \neq 0$ such that

$$\mu[B \cap T^{-n}B \cap \{x : |f_*(n,x)| < \varepsilon\}] > 0.$$

(or $\mu[B \cap T^{-n}B \cap \{x : |f^*(n,x)| < \varepsilon\}] > 0$).

(4) The *H-cocycle* (or *cocycle*) of f is to be *H-superrecurrent* (or *superrecurrent*) if for every $\varepsilon > 0$ and for every $B \in \mathcal{B}$, there exists $n \neq 0$ such that

$$\mu[B \cap T^{-n}B \cap \{x : |f_*(n,x)| + |\log \omega_n(x)| < \varepsilon\}] > 0.$$

(or $\mu[B \cap T^{-n}B \cap \{x : f^*(n,x)| + |\log \omega_n(x)| < \varepsilon\}] > 0$.

(5) A measurable function f on X is said to be an *H-coboundary* if $f(x) = g(x) - \omega_1(x)(Tx)$ for some measurable function g on X.

(6) Two functions f, g on X are said to be *H-cohomologous* if their difference is an H-coboundary.

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3. ***H*-Superrecurrence and skew products.** Let $(R, \mathcal{C}, \lambda)$ be the real line with the Lebesgue *σ-field* and Lebesgue measure. With every measurable function $f: X \longrightarrow R$ we associate the *skew product* $\bar{T}_f$ (built from f) defined on $X \times R \times R$ by

$$\bar{T}_f(x, r, s) = \left(Tx, \frac{r + f(x)}{\omega_1(x)}, s + \log \omega_1(x)\right).$$

where $X \times R \times R$ is given the product *σ-field* and the product measure $\bar{\mu} = \mu \times \lambda \times \lambda$. We also see that

$$\bar{T}_f^n(x, r, s) = \left(T^n x, \frac{r + f_*(n, x)}{\omega_n(x)}, s + \log \omega_n(x)\right).$$

PROPOSITION 1. *$\bar{\mu}$ is invariant under $\bar{T}_f$.*

PROOF. We only need to show that $\bar{\mu}$-measure of measurable rectangles is invariant under $\bar{T}_f$. To this end, let $A \in \mathcal{B}$ and $U, V \in \mathcal{C}$; observe that

$$\bar{T}_f^{-1}(A \times U \times V) = \{(x, r, s) : x \in T^{-1}A,\ r \in \omega_1(x)U - f(x),\ s \in V - \log \omega_1(x)\}$$

so that,

$$\begin{aligned}
\bar{\mu}[\bar{T}_f^{-1}(A \times U \times V)] &= \int_{T^{-1}A} \int_{\omega_1(x)U - f(x)} \int_{V - \log \omega_1(x)} d\mu(x)\, d\lambda(r)\, d\lambda(s) \\
&= \int_{T^{-1}A} \omega_1(x)\, d\mu(x)\, \lambda(U)\, \lambda(V) \\
&= \int_A d\mu(x)\, \lambda(U)\, \lambda(V) \\
&= \int_A \int_U \int_V d\mu(x)\, d\lambda(r)\, d\lambda(s) \\
&= \bar{\mu}(A \times U \times V).
\end{aligned}$$

Let f, g be two measurable functions on X. Denote by $\bar{T}_f, \bar{T}_g$, the skew product of f and g respectively as defined above.

PROPOSITION 2. *If f is H-cohomologous to g then $\bar{T}_f$ is isomorphic to $\bar{T}_g$.*

PROOF. Let $h: X \longrightarrow R$ be such that $f(x) - g(x) = h(x) - \omega_1(x)h(Tx)$.

Define $\lambda: X \times R \times R \longrightarrow X \times R \times R$ by $\lambda(x, r, s) = (x, r + h(x), s)$. It is easy to check that λ is the required isomorphism.

Now, suppose that μ is equivalent to the measure ν. Denote by $\frac{d\mu}{d\nu}$ the Radon-Nikodym derivative of μ with respect to ν. We have for every $n \in Z$ the following relationship:

$$(1) \qquad \frac{d\mu \circ T^n}{d\mu}(x) = \frac{d\mu \circ T^n}{d\nu \circ T^n}(x)\, \frac{d\nu \circ T^n}{d\nu}(x)\, \frac{d\nu}{d\mu}(x).$$

Since in the next Proposition we will be considering two different equivalent measures in order to avoid confusion when the H-cocycle of a function f is taken with respect to a specific measure μ we denote it by f_*^μ. Observe that for every $n \in Z$ equation (1) gives:

$$(2) \qquad f_*^\mu(n, x) = \frac{d\nu}{d\mu}(x)\left(f \cdot \frac{d\mu}{d\nu}\right)_*^\nu(n, x).$$

PROPOSITION 3. *If μ is equivalent to ν then $\bar{T}_f$ is isomorphic to $\bar{T}_{f\cdot\frac{d\mu}{d\nu}}$.*

PROOF. Define $\lambda: X \times R \times R \longrightarrow X \times R \times R$ by $\lambda(x, r, s) = \left(x, r \cdot \frac{d\mu}{d\nu}, s - \log \frac{d\mu}{d\nu}\right)$. Then λ is the required isomorphism.

LEMMA 1. *Let T be a measure preserving automorphism of a Lebesgue space $(X, \mathcal{B}, \mu)$. If there exists two sets E and F of positive measure such that $\mu(F) < \infty$ and a.e. $x \in E$ visits F infinitely often under the action of T, then E is contained in the conservative part of X.*

PROOF. Assume not: then there exists a wandering set $D \subset X$ such that $\mu(D) > 0$ and $\mu[E \cap (\cup_{-\infty}^{\infty} T^n D)] > 0$. We shall assume with no loss of generality that $\mu(E \cap D) > 0$. Then

$$\begin{aligned}
\infty > \mu(F) \geq \mu\Big[\bigcup_{-\infty}^{\infty} T^n(E \cap D) \cap F\Big] \\
&= \int \sum_{n=-\infty}^{\infty} \chi_{T^n(E \cap D) \cap F}(x)\, d\mu(x) \\
&= \int \sum_{n=-\infty}^{\infty} \chi_{(E \cap D) \cap T^{-n}F}(x)\, d\mu(x) \\
&= \int_{E \cap D} \sum_{n=-\infty}^{\infty} \chi_F(T^n x)\, d\mu(x) \\
&= \infty,
\end{aligned}$$

by hypothesis which is a contradiction.

LEMMA 2. *Let T be non-singular, ergodic automorphism of a Lebesgue space and $\bar{T}_f, \bar{\mu}$ as defined before. Suppose there exist two sequences of sets E_m and F_m in $X \times R \times R$ and $A \subset X$ with $\mu(A) > 0$ such that:*

(a) $\bar{\mu}(F_m) < \infty$ for all m,

(b) $\bar{\mu}$ a.e. $(x, r, s) \in E_m$ visits F_m infinitely often under the action of $\bar{T}_f$, and

(c) $A \times R \times R \subset \cup_m E_m$.

Then $\bar{T}_f$ is conservative.

THEOREM 1. *Let T be an ergodic, conservative, non-singular automorphism of a non-atomic Lebesgue probability space $(X, \mathcal{B}, \mu)$, and $f: X \longrightarrow R$ be a measurable function. Let $\bar{T}_f$ be the skew product on $X \times R \times R$ built from f. Then $\bar{T}_f$ is conservative if and only if the H-cocycle of f is H-superrecurrent.*

PROOF. Suppose $\bar{T}_f$ is conservative. Let $A \subset X$ with $\mu(A) > 0$. Let $U = \left(\frac{\varepsilon}{2e^{\varepsilon}}, \frac{\varepsilon}{2e^{\varepsilon}}\right)$ and $V = \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$. Then $A \times U \times V \subset X \times R \times R$ such that $\bar{\mu}(A \times U \times V) > 0$. By Conservativity of $\bar{T}_f$ there exists $n \neq 0$ such that

$$\bar{\mu}\Big[(A \times U \times V) \bigcap \bar{T}_f^n(A \times U \times V)\Big] > 0.$$

But $(x, r, s) \in (A \times U \times V) \cap \bar{T}_f^{-n}(A \times U \times V)$ implies that $x \in A \cap T^{-n}A, |r| < \frac{\varepsilon}{2e^{\varepsilon}}, |s| < \frac{\varepsilon}{2}, \left|\frac{r+f_*(n,x)}{\omega_n(x)}\right| < \frac{\varepsilon}{2e^{\varepsilon}}$, and $|s + \log \omega_n(x)| < \frac{\varepsilon}{2}$. Then $|\log \omega_n(x)| \leq |s| + \frac{\varepsilon}{2} < \varepsilon$, or that $e^{-\varepsilon} < \omega_n(x) < e^{\varepsilon}$. Also $|r + f_*(n, x)| < \frac{\varepsilon}{2e^{\varepsilon}} \cdot \omega_n(x) < \frac{\varepsilon}{2}$ which implies that $|f_*(n, x)| < \frac{\varepsilon}{2} + |r| < \varepsilon$. Thus

$$(A \times U \times V) \bigcap \bar{T}_f^{-n}(A \times U \times V) \subset A \cap T^{-n}A \cap \{x : |f_*(n,x)| + |\log \omega_n(x)| < 2\varepsilon\} \times U \times V.$$

Since $\bar{\mu}$ is the product measure it follows that

$$\mu(A \cap T^{-n}A \cap \{x : |f_*(n,x)| + \log \omega_n(x)| < 2\varepsilon\}) > 0.$$

Therefore f_* is H-superrecurrent.

Conversely, suppose f_* is H-superrecurrent. Let $\varepsilon > 0$ and $A \subset X$ with $\mu(A) > 0$. For $m \in N$ let $E_m = A \times B_m \times B_m$, and $F_m = A \times B_{(m+\varepsilon)e^{\varepsilon}} \times B_{m+\varepsilon}$, where $B_m = \{r \in R : |r| < m\}$. Then, $\bar{\mu}(F_m) < \infty$ for all m, and $A \times R \times R \subset \bigcup_m E_m$. By superrecurrence of f_* a.e. $x \in A$ has infinitely many non-zero integers n such that $x \in T^{-n}A \cap \{x : |f_*(n,x)| + |\log \omega_n(x)| < \varepsilon\}$. Call such an integer n *good* for x. Now, let $(x, r, s) \in E_m$ and let n be good for x. Then,

$$\bar{T}_f(x, r, s) = \left(T^n x, \frac{r + f_*(n,x)}{\omega_n(x)}, s + \log \omega_n(x)\right)$$

is such that $T^n x \in A$, $\frac{|r+f_*(n,x)|}{\omega_n(x)} \leq \frac{1}{\omega_n(x)}\left(|r| + |f_*(n,x)|\right) < e^{\varepsilon}(m + \varepsilon)$, and $|s + \log \omega_n(x)| \leq |s| + |\log \omega_n(x)| < m + \varepsilon$. Thus $\bar{T}_f(x, r, s) \in F_m$. Since a.e. $x \in A$ has infinitely many good n it follows by Lemma 2 that $\bar{T}_f$ is conservative.

COROLLARY 1. *If μ is equivalent to ν then f_*^{μ} is H-superrecurrent if and only if $\left(f \cdot \frac{d\mu}{d\nu}\right)_*^{\nu}$ is.*

Let $A \subset X$ be given and consider the induced transformation $T_A : A \longrightarrow A$ given by $T_A x = T^{r(x)} x$ where $r(x) = \min\{n > 0 : T^n x \in A\}$. With an H-cocycle f_* under the action of T we associate an H-cocycle f_*^A under the action of T_A given by

$$f_*^A(n, x) = f_*\left(\sum_{i=0}^{n-1} r(T_A^i), x\right).$$

In particular, $f^A = f_*^A(1, x) = f_*\big(r(x), x\big)$. Also $\omega_1^A(x) = \omega_{r(x)}(x) = \frac{d\mu \circ T_A}{d\mu}(x)$.

With f_*^A we associate the skew product $\bar{T}_{f^A} : A \times R \times R \longrightarrow A \times R \times R$ defined by

$$\bar{T}_{f^A}(x, r, s) = \left(T_A x, \frac{r + f^A(x)}{\omega_1^A(x)}, s + \log \omega_1^A(x)\right).$$

Now for $(x, r, s) \in A \times R \times R$ the first return time of (x, r, s) to $A \times R \times R$ is the same as the first return times, $r(x)$, of x to A. Thus,

$$\begin{aligned}\bar{T}_{f^A}(x,r,s) &= \left(T_A x, \frac{r+f^A(x)}{\omega_1^A(x)}, s+\log\omega_1^A(x)\right)\\ &= \left(T^{r(x)}x, \frac{r+f_*\big(r(x),x\big)}{\omega_{r(x)}(x)}, s+\log\omega_{r(x)}(x)\right)\\ &= \bar{T}_f^{r(x)}(x,r,s)\\ &= \left(\bar{T}_f\right)_{A\times R\times R}(x,r,s).\end{aligned}$$

Since conservativity is preserved under inducing, it follows that $\bar{T}_{f^A}$ is conservative if and only if $\bar{T}_f$ is conservative.

DEFINITION. The H-cocycle of f is said to be *H-supertransient* if and only if for every $B \in \mathcal{B}$ with positive measure and for all real numbers $M > 0$,

$$\mu\left[\limsup_{n\to\infty} B \cap T^{-n}B \cap \{x : |f_*(n,x)| + |\log\omega_n(x)| < M\}\right] = 0.$$

PROPOSITION 4. *Either f_* is H-superrecurrent or is H-supertransient.*

PROOF. Assume that f_* is not H-superrecurrent, then the skew product $\bar{T}_f$ is not conservative by Theorem 1. Let $B \subset X$ be any set of positive measure and let $M > 0$ be any real number. For $x \in B$, call n *good* for x if $x \in B \cap T^{-n}B \cap \{x : |f_*(n,x)| + |\log\omega_n(x)| < M\}$. Let

$$A_1 = \{x \in B : x \text{ has infinitely many good } n\}, \text{ and}$$
$$A_2 = B \backslash A_1.$$

If $\mu(A_1) > 0$, then for $m \in N$, let

$$E_m = A_1 \times B_m \times B_m,$$

and

$$F_m = A_1 \times B_{(M+m)e^M} \times B_{m+M},$$

where $B_l = \{r \in R : |r| < l\}$. Then $\cup_{m\in N} E_m = A_1 \times R \times R$, $\bar{\mu}(F_m) < \infty$ for all m and for any $(x,r,s) \in E_m$ and n good we have

$$T^n(x) \in B,$$

$$\frac{|f_*(n,x)+r|}{\omega_n(x)} \le \frac{|f_*(n,x)| + (x)|r|}{\omega_n(x)} < (M+m)e^M,$$

and

$$|s + \log\omega_n(x)| \le |s| + |\log\omega_n(x)| < m + M.$$

That is, $\bar{T}_f^n(x,r,s) \in F_m$ for all good n. By Lemma 2 $\bar{T}_f$ is conservative, which is a contradiction since f_* was assumed not to be H-superrecurrent. Thus $\mu(A_2) = 1$, and f_* is H-supertransient.

In the remainder of this section we show that we can characterize the H-superrecurrence of an H-cocycle by means of the asymptotic behaviour of $|f_*(n,x)| + |\log \omega_n(x)|$ for points $x \in X$. Precisely, we will show that f_* is H-superrecurrent if and only if $\liminf_{n\to\infty} |f_*(n,x)| + |\log \omega_n(x)| = 0$, and f_* is H-supertransient if and only if $\liminf_{n\to\infty} |f_*(n,x)| + |\log \omega_n(x)| = \infty$.

PROPOSITION 5. *The H-cocycle of f is H-superrecurrent if and only if*

$$\liminf_{n\to\infty} |f_*(n,x)| + |\log \omega_n(x)| = 0 \text{ a.e.}$$

PROOF. Assume f_* is H-superrecurrent. Let $\varepsilon > 0$ and let

$$D = \{x \in X : \liminf_{n\to\infty} |f_*(n,x)| + |\log \omega_n(x)| > 0\}.$$

We claim that $\mu(D) = 0$. For if $\mu(D) > 0$, then there exists an integer $N > 0$ so large such that,

$$C = \{x \in D : |f_*(n,x)| + |\log \omega_n(x)| > 2\varepsilon \text{ for all } |n| > N\}$$

has positive measure. Using Rokhlin's lemma we can find $B \subset C$ of positive meaure such that $B \cap T^n B = \phi$ for all $0 \neq |n| \leq N$. Also for each $x \in B$ and each $|n| > N$ either $|f_*(n,x)| \geq \varepsilon$ or $|\log \omega_n(x)| \geq \varepsilon$, otherwise $|f_*(n,x)| + |\log \omega_n(x)| < 2\varepsilon$ with $|n| > N$, a contradiction. Hence,

$$\mu\Big[B \cap T^{-n} B \cap \{x : |f_*(n,x)| < \varepsilon\} \cap \{x : |\log \omega_n(x)| < \varepsilon\}\Big] = 0 \text{ for } n \neq 0,$$

but this contradicts H-superrecurrence of f_*. Thus, $\liminf_{n\to\infty} |f_*(n,x)| + |\log \omega_n(x)| = 0$ a.e.

Conversely, suppose $\liminf_{n\to\infty} |f_*(n,x)| + |\log \omega_n(x)| = 0$ a.e. We want to show that f_* is H-superrecurrent. For this we show that $\bar{T}_f$ is conservative. Given $\varepsilon > 0$, by hypothesis, for a.e. $x \in X$ there exist infinitely many non-zero integers n such that $|f_*(n,x)| + |\log \omega_n(x)| < \varepsilon$. Call such an n *good* for x. For $m \in N$, let $E_m = X \times B_m \times B_m$ and $F_m = X \times B_{(m+\varepsilon)e^{\varepsilon}} \times B_{m+\varepsilon}$. Since $\mu(X) = 1$ it follows that $\bar{\mu}(F_m) < \infty$ for all m and $X \times R \times R \subset \bigcup_m E_m$. Now let $(x,r,s) \in E_m$ and let n be good for x. It is easy to see that $\bar{T}_f^n(x,r,s) \in F_m$. Since x has infinitely many good n it follows by Lemma 2 that $\bar{T}_f$ is conservative and hence by Theorem 1 f_* is H-superrecurrent.

PROPOSITION 6. *The H-cocycle f_* is H-supertransient if and only if*

$$\liminf_{n\to\infty} |f_*(n,x)| + |\log \omega_n(x)| = \infty \text{ a.e.}$$

PROOF. Clearly, if

$$\liminf_{n\to\infty} |f_*(n,x)| + |\log \omega_n(x)| = \infty \text{ a.e.}$$

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then f_* is H-supertransient.

For the converse we shall prove the contrapositive. Assume that the set $B = \{x \in X : \liminf_{n\to\infty} |f_*(n,x)| + |\log \omega_n(x)| < \infty\}$ has positive measure. Choose $N > 0$ so that the set

$$C = \{x \in B : \liminf_{n\to\infty} |f_*(n,x)| + |\log \omega_n(x)| < N\}$$

has positive measure. Let $E_m = C \times B_m \times B_m$ and $F_m = X \times B_{(N+m)e^N} \times B_{m+N}$ where the sets B_l as defined above. It is easy to see that conditions (a), (b) and (c) of Lemma 2 are satisifed, which implies that $\bar{T}_f$ is conservative, a contradiction.

4. **Examples.** Example 1: If μ is equivalent to ν where ν is a finite invariant measure, and $f: X \longrightarrow R$ a measurable function, then the H-cocycle of f is H-recurrent if and only if it is H-superrecurrent.

PROOF. Clearly, f_* H-superrecurrent implies f_* H-recurrent. Now, assume f_* is H-recurrent and observe that for $n \in Z$ and $x \in X, f_*(n,x) = \frac{d\nu}{d\mu}(x)\left(f\frac{d\mu}{d\nu}\right)^*(n,x)$. Let $B \in \mathcal{B}$ with $\mu(B) > 0$. There exists $M > 0$ such that the set $C = \{x \in B : 1/M < \frac{d\mu}{d\nu}(x) < M\}$ has positive measure. By H-recurrence of f_*, there exists $n \neq 0$ such that

$$\mu\left[C \cap T^{-n}C \cap \{x : |f_*(n,x)| < \varepsilon/M\}\right] > 0,$$

which implies

$$\mu(C \cap T^{-n}C \cap \{x : \left|\left(f\frac{d\mu}{d\nu}\right)^*(n,x)\right| < \varepsilon\} > 0.$$

This implies that $(f\frac{d\mu}{d\nu})^*$ is recurrent and hence superrecurrent (Schmidt [4]), so that there exists $m \neq 0$ such that,

$$\mu\left[C \cap T^{-n}C \cap \left\{x : \left|\left(f\frac{d\mu}{d\nu}\right)^*(n,x)\right| < \varepsilon/M\right\} \cap \{x : |\log \omega_n(x)| < \varepsilon\}\right] > 0,$$

or,

$$\mu\left[C \cap T^{-n}C \cap \{x : |f_*(n,x)| < \varepsilon\} \cap \{x : |\log \omega_n(x)| < \varepsilon\}\right] > 0.$$

That is, f_* is H-superrecurrent.

Example 2: Let $f(x) = g(x) - \omega_1(x)g(Tx)$, that is f is an H-coboundary. Then f is H-superrecurrent.

PROOF. Let $h(x) = g(x) - g(Tx)$, then $h(x)$ is a coboundary, hence recurrent, and by [4] h^* is superrecurrent. Let $\varepsilon > 0$ be given and let $B \in \mathcal{B}$ be such $\mu(B) > 0$. Choose M sufficiently large so that the set $C = \{x \in B : |g(x)| < M\}$ has positive measure. Also, there exists $n \neq 0$ such that

$$\mu\left[C \cap T^{-n}C \cap \{x : |h^*(n,x)| < \varepsilon/2\} \cap \{x : \log \omega_n(x)| < \log(1 + \varepsilon/2M)\}\right] > 0.$$

But $x \in C \cap T^{-n}C \cap \{x : |h^*(n,x)| < \varepsilon/2\} \cap \{x : |\log \omega_n(x)| < \log(1+\varepsilon/2M)\}$, implies

$$\begin{aligned} x &\in C \cap T^{-n}C, \\ |g(T^n x)| &< M, \\ |\omega_n(x) - 1| &< \varepsilon/2M, \text{ and} \\ |f_*(n,x)| &= |g(x) - \omega_n(x)g(T^n x)| \\ &\leq |g(x) - g(T^n x)| + |g(T^n x) - \omega_n(x)g(T^n x)| \\ &= |g(x) - g(T^n(x))| + |g(T^n x)|\,|\omega_n(x) - 1| \\ &= |h^*(n,x)| + |g(T^n x)|\,|\omega_n(x) - 1| \\ &< \varepsilon/2 + M\varepsilon/2M \end{aligned}$$

Hence, $\mu\left[C \cap T^{-n}C \cap \{x : |f_*(n,x)| < \varepsilon\} \cap \{x : |\log \omega_n(x)| < \log(1+\varepsilon/2M)\}\right] > 0$. Therefore, f is H-superrecurrent.

REMARKS.

(a) If f_* is H-superrecurrent and b is an H-coboundary then $(f+b)_*$ is H-superrecurrent.

PROOF. Let $1 > \varepsilon > 0$ be given, and let $b(x) = g(x) - \omega_1(x)g(Tx)$. For each $n \in Z$, let $A_n = \{x \in X : \varepsilon n < g(x) \leq \varepsilon(n+1)\}$. Then $\cup_{n=-\infty}^{\infty} A_n = X$. Let $B \in \mathcal{B}$ with $\mu(B) > 0$. It is easy to see that there exist $m \neq 0$ and an integer n such that $\mu(A_n \cap B) > 0$ and

$$\begin{aligned} &\mu\left[B \cap T^{-m}B \cap \{x : |(f+b)_*(m,x)| < 3\varepsilon\} \cap \{x : |\omega_m(x) - 1| < \varepsilon\}\right. \\ &\quad \geq \mu\left[(B \cap A_n) \cap T^{-m}(B \cap A_n) \cap \{x : |f_*(m,x)| < \varepsilon\}\right. \\ &\qquad\qquad \cap \{x : |\omega_m(x) - 1| < \varepsilon/(|n|+1)\} \\ &\quad > 0. \end{aligned}$$

Hence, $(f+b)_*$ is H-superrecurrent.

(b) If for almost every x, the sequence $f_*(n,x)$ is bounded then f is an H-coboundary.

PROOF. Let $g(x) = \limsup_{n\to\infty} f_*(n,x)$, then

$$g(Tx) = \limsup_{n\to\infty} f_*(n, Tx) = \limsup_{n\to\infty} \frac{f_*(n+1,x) - f_*(1,x)}{\omega_1(x)},$$

which implies,

$$\omega_1(x)g(Tx) = \limsup_{n\to\infty} f_*(n+1,x) - f_*(1,x) = g(x) - f(x).$$

That is $f(x) = g(x) - \omega_1(x)g(Tx)$, i.e., f is an H-coboundary.

H-recurrence of H-cocycles was studied by Dan Ullman [5,6]. In [5] he showed that for $f \in L^1(X)$, f_* is H-recurrent if and only if $\int f\, d\mu = 0$. The question is whether the result is still true if H-recurrence is replaced by H-superrecurrence? More generally whether H-recurrence is equivalent to H-superrecurrence, even in the case where $\int f$ does not exist.

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