Ergod. Th. & Dynam. Sys., page 1 of 24 © The Author(s), 2024. Published by Cambridge University Press.
doi:10.1017/etds.2024.88

Quantitative reducibility of C^k quasi-periodic cocycles

AO CAI, HUIHUI LV and ZHIGUO WANG
Soochow University School of Mathematical Sciences, Suzhou, Jiangsu, China
(e-mail: acai@suda.edu.cn, 20224207041@stu.suda.edu.cn, zgwang@suda.edu.cn)

(*Received* 12 *April* 2024 *and accepted in revised form* 13 *September* 2024)

Abstract. This paper establishes an extreme C^k reducibility theorem of quasi-periodic $SL(2, \mathbb{R})$ cocycles in the local perturbative region, revealing both the essence of Eliasson [Floquet solutions for the 1-dimensional quasi-periodic Schrödinger equation. *Comm. Math. Phys.* **146** (1992), 447–482], and Hou and You [Almost reducibility and non-perturbative reducibility of quasi-periodic linear systems. *Invent. Math.* **190** (2012), 209–260] in respectively the non-resonant and resonant cases. By paralleling further the reducibility process with the almost reducibility, we are able to acquire the least initial regularity as well as the least loss of regularity for the whole Kolmogorov–Arnold–Moser (KAM) iterations. This, in return, makes various spectral applications of quasi-periodic Schrödinger operators wide open.

Key words: discrete, finitely differentiable, quantitative reducibility, cocycle
2020 Mathematics Subject Classification: 37J40 (Primary)

1. *Introduction*

Assume that $(M, \mathcal{B}, \mu)$ is a probability space and $f : M \to M$ is an invertible map which preserves the measure μ and is ergodic with respect to μ. Let $A : M \to SL(2, \mathbb{R})$ be a measurable function. The linear cocycle defined by A over the base dynamics f is the transformation:

$$(f, A) : M \times \mathbb{R}^2 \to M \times \mathbb{R}^2; \quad (\theta, v) \mapsto (f(\theta), A(\theta) \cdot v).$$

Note that the iterates of (f, A) have the form $(f, A)^n = (f^n, A_n)$, where $A_n(\theta) = A(f^{n-1}(\theta)) \cdots A(f(\theta))A(\theta)$, $n \in \mathbb{N}$ and $A_{-n}(\theta) = A_n(f^{-n}(\theta))^{-1}$. In particular, a C^k quasi-periodic linear cocycle (α, A) consists of a rationally independent $\alpha \in \mathbb{T}^d$, which determines an ergodic torus translation on the base and $A \in C^k(\mathbb{T}^d, SL(2, \mathbb{R}))$ which is a k times differentiable matrix-valued function with continuous kth derivatives.

In this paper, we present a purified quantitative reducibility theorem for finitely differentiable quasi-periodic cocycles. In particular, it applies to the C^k quasi-periodic Schrödinger cocycles (α, A), where

$$A(\theta) = S_E^V(\theta) = \begin{pmatrix} E - V(\theta) & -1 \\ 1 & 0 \end{pmatrix}.$$

This effectively extends the spectral applications in the one-dimensional discrete quasi-periodic Schrödinger operator with a C^k potential:

$$(H_{V,\alpha,\theta}x)_n = x_{n+1} + x_{n-1} + V(\theta + n\alpha)x_n, \quad n \in \mathbb{Z}, \tag{1}$$

as any formal solution (not necessarily in ℓ^2) of $H_{V,\alpha,\theta}x = Ex$ satisfies

$$A(\theta + n\alpha)\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix}.$$

In equation (1), $V \in C^k(\mathbb{T}^d, \mathbb{R})$ is called the potential, $\alpha \in \mathbb{R}^d$ is called the frequency satisfying $\langle n, \alpha\rangle \notin \mathbb{Z}$ for any $n \in \mathbb{Z}^d\backslash\{0\}$ and $\theta \in \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ (or $\mathbb{R}^d/(2\pi\mathbb{Z})^d$ if preferable) is called the initial phase, $k, d \in \mathbb{N}^+$. The spectrum of $H_{V,\alpha,\theta}$ is denoted as $\Sigma_{V,\alpha,\theta}$, which is independent of the phase due to the minimality of dynamics and strong operator convergence [17]. Since V is bounded by compactness, $H_{V,\alpha,\theta}$ is a bounded self-adjoint operator on $\ell^2(\mathbb{Z})$ and $\Sigma_{V,\alpha,\theta} \subset \mathbb{R}$ is a compact perfect set. See the nice survey of Damanik [16] for more details of the Schrödinger operator unity. Moreover, readers are invited to consult the excellent book of Damanik and Fillman [17] which is significant and timely for the community.

1.1. *Quantitative reducibility.* The reducibility of quasi-periodic cocycles aims to conjugate the original quasi-periodic cocycles to constant ones via transformations that are essentially coordinate changes. However, due to topological obstructions, reducibility may fail and the concept of almost reducibility naturally arises, which settles for almost conjugating to constant cocycles but has proven to be very powerful in studying the spectral theory of quasi-periodic Schrödinger operators [38]. Readers are referred to §2 for precise definitions of (almost) reducibility.

In the previous literature, almost reducibility is regarded as a prerequisite for reducibility, especially in the analytic region, due to the great flexibility of shrinking the analytic radius arbitrarily. Nevertheless, in the C^k topology, the least loss of regularity is a fixed number. To establish a C^k reducibility theorem with least initial regularity and most remainder, we perform a process parallel to the process of almost reducibility. In other words, reducibility is built not after almost reducibility, but at the same time with extra assumptions of the fibered rotation number $\rho(\alpha, A)$, see its definition in §2.2.

Recall that $\alpha \in \mathbb{R}^d$ is called *Diophantine* if there exist $\kappa > 0$ and $\tau > d$ such that $\alpha \in \mathrm{DC}_d(\kappa, \tau)$, where

$$\mathrm{DC}_d(\kappa, \tau) := \left\{\alpha \in \mathbb{R}^d : \inf_{j\in\mathbb{Z}} |\langle m, \alpha\rangle - j| > \frac{\kappa}{|m|^\tau} \text{ for all } m \in \mathbb{Z}^d\backslash\{0\}\right\}. \tag{2}$$

https://doi.org/10.1017/etds.2024.88 Published online by Cambridge University Press

Here, we denote

$$|m| = |m_1| + |m_2| + \cdots + |m_d|$$

and

$$\langle m, \alpha\rangle = m_1\alpha_1 + m_2\alpha_2 + \cdots + m_d\alpha_d.$$

Denote $\mathrm{DC}_d = \bigcup_{\kappa,\tau} \mathrm{DC}_d(\kappa, \tau)$, which is of full Lebesgue measure.

Here, $\phi \in \mathbb{R}$ is called *Diophantine* with respect to α if it satisfies the condition $\phi \in \mathrm{DC}_d^{\alpha}(\gamma, \tau)$, where $\gamma > 0$, $\tau > d$, and

$$\mathrm{DC}_d^{\alpha}(\gamma, \tau) := \left\{\phi \in \mathbb{R} : \inf_{j\in\mathbb{Z}} |2\phi - \langle m, \alpha\rangle - j| > \frac{\gamma}{(|m| + 1)^{\tau}} \text{ for all } m \in \mathbb{Z}^d\right\}. \quad (3)$$

Additionally, $\phi \in \mathbb{R}$ is called rational with respect to α if $2\phi = \langle m_0, \alpha\rangle \mod \mathbb{Z}$ for some $m_0 \in \mathbb{Z}^d$.

Our main theorem is as follows.

THEOREM 1.1. *Let $A \in SL(2, \mathbb{R}), \alpha \in \mathrm{DC}_d(\kappa, \tau)$ and $f \in C^k(\mathbb{T}^d, sl(2, \mathbb{R}))$ with $k > 14\tau + 2$. There exists $\epsilon = \epsilon(\kappa, \tau, d, k, \|A\|)$ such that if $\|f(\theta)\|_k \leqslant \epsilon$ and $\rho(\alpha, Ae^f)$ is Diophantine or rational with respect to α, then $(\alpha, Ae^{f(\theta)})$ is C^{k,k_0} reducible with $k_0 < k - 10\tau - 3$.*

Remark 1.2. The quantitative version of Theorem 1.1 can be found in §3, specifically in Theorem 3.8. Our theorem is a significant improvement over the results presented in [12]. Notably, the loss of regularity $10\tau + 3$ does not depend on the parameter k.

As there are few reducibility results in the C^k topology, we only briefly review the development of analytic reducibility. Let us first discuss it for local cocycles of the form $(\alpha, Ae^{f(\theta)})$. An achievement was initiated by Dinaburg and Sinai [22], who applied the classical Kolmogorov–Arnold–Moser (KAM) scheme. They established the positive measure reducibility in terms of the rotation number for continuous quasi-periodic Schrödinger equations featuring small analytic potentials in the perturbative regime. Moser and Pöschel [32] later expanded on this achievement by using a resonance-cancellation technique, which extended the positive measure reducibility to a class of rotation numbers that are rational with respect to α. Moreover, the breakthrough came from Eliasson [23], who established weak almost reducibility for all energies E and full measure reducibility for Diophantine frequencies and small analytic potentials. For further insights into strong almost reducibility results, readers can refer to the works of Chavaudret [14] and Leguil *et al* [31]. In the non-perturbative regime, Puig [33] employed the localization method to derive a non-perturbative version of Eliasson's reducibility theorem. As for continuous linear systems, Hou and You [26] proved weak almost reducibility results for all rotation numbers and frequencies $\omega = (1, \alpha) \in \mathbb{T}^2$ under small analytic perturbations.

In the case of global cocycle $(\alpha, A(\theta))$ with $A \in C^{\omega}(\mathbb{T}, SL(2, \mathbb{R}))$ and $\alpha \in \mathbb{T}$, Avila and Krikorian [5] applied the renormalization scheme to show that, for α satisfying certain recurrent Diophantine conditions and almost every (a.e.) E, the quasi-periodic Schrödinger cocycle is either reducible or non-uniform hyperbolic. Additionally, Avila, Fayad, and Krikorian [3] proved that for irrational α and a.e. E, the quasi-periodic Schrödinger cocycle is either rotations reducible or non-uniformly hyperbolic.

In the realm of finitely differentiable topology, reducibility results are scarce and the existing ones are rough in the sense of very large initial regularity and very high loss. We note that our philosophy is that nicer reducibility implies nicer spectral applications. In this direction, Theorem 1.1, which was established via keeping all the parameters, allows us to obtain the most general applications according to the technique.

1.2. *Spectral type and structure.* Over the past forty years, advances have been made in studying the spectral theory of the Schrödinger operator, focusing on understanding the spectral type and the structure of the spectrum. In the 21st century, people found that quantitative dynamical estimates lead to quantitative spectral applications [**4**]. In particular, quantitative (almost) reducibility is one of the most powerful techniques, with abundant fruitful spectral results [**2**, **10**–**13**, **18**, **27**, **28**, **33**, **37**]. One is invited to consult You's 2018 ICM survey [**38**]. Generally, spectral type results refer to absolutely continuous spectrum, singular continuous spectrum, pure point spectrum, ballistic transport, Anderson localization, etc. while regarding spectral structure, common results often involve the Cantor spectrum and homogeneous spectrum. For the sake of concision, we only list one spectral type application and one spectral structure application, though our Theorem 1.1 along with its quantitative version is promisingly applicable to many other spectral results in relation.

THEOREM 1.3. (Spectral type application) *Assume $\alpha \in \mathrm{DC}_d(\kappa, \tau)$, $V \in C^k(\mathbb{T}^d, \mathbb{R})$ with $k > 14\tau + 2$. If there exists $\epsilon' = \epsilon'(\kappa, \tau, k, d)$ such that $\|V\|_k \leqslant \epsilon'$ and $\rho(\alpha, S_E^V)$ is Diophantine or rational with respect to α for $E \in \Sigma_{V,\alpha}$, then $H_{V,\alpha,\theta}$ has strong ballistic transport for all $\theta \in \mathbb{T}^d$.*

Remark 1.4. The definition of (global) strong ballistic transport is too lengthy, so we prefer to put it in §4 rather than here. This theorem compensates for the work of Ge and Kachkovskiy [**25**] by providing the precise requirement of k for a C^k quasi-periodic Schrödinger family to be reducible.

Indeed, there is a qualitative connection between spectral type and transport properties, which can be understood through the RAGE theorem [**16**]. However, quantitative results also illustrate the connection between them. For example, except for purely absolutely continuous spectra, we generally do not expect to observe ballistic transport phenomena. In particular, it has been proven in [**34**] that point spectra cannot support any ballistic motion. However, when the operator is restricted to a subspace that supports purely absolutely continuous spectrum, we can still expect to observe ballistic transport phenomena. Additionally, the works of Guarneri, Combes, and Last [**15**, **30**] have also made contributions to the connection. In particular, the Guarneri–Combes–Last theorem [**30**] quantitatively provides insights into transport phenomena in one-dimensional systems with absolutely continuous spectrum. In subsequent studies, Asch and Knauf [**1**], as well as Damanik [**20**], demonstrated the occurrence of strong ballistic transport in periodic continuous Schrödinger operators, which are widely known to have absolutely continuous spectra. Later, Zhang and Zhao [**42**] explicitly established a connection between the values of transport exponents and absolutely continuous spectra in the setting of discrete single-frequency quasi-periodic operators. More recently, based on the results and techniques in the discrete setting of Fillman [24], Young's research [39] discovered strong ballistic transport in a class of continuum limit-periodic operators known to possess absolutely continuous spectra.

Let us now state our spectral structure result. First, recall the following definition.

Definition 1.5. [13] Let $\nu > 0$. A closed set $\mathfrak{B} \subset \mathbb{R}$ is called ν-homogeneous if

$$|\mathfrak{B} \cap (E - \epsilon, E + \epsilon)| > \nu\epsilon \quad \text{for all } E \in \mathfrak{B}, \text{ for all } 0 < \epsilon < \operatorname{diam}\mathfrak{B}.$$

As another corollary of Theorem 1.1, we have the following.

THEOREM 1.6. (Spectral structure application) *Assume $\alpha \in \mathrm{DC}_d(\kappa, \tau)$, $V \in C^k(\mathbb{T}^d, \mathbb{R})$ with $k > 17\tau + 2$. If there exists $\epsilon'' = \epsilon''(\kappa, \tau, k, d)$ such that $\|V\|_k \leqslant \epsilon''$, then $\Sigma_{V,\alpha}$ is ν-homogeneous for some $\nu \in (0, 1)$.*

Remark 1.7. We know that the homogeneity of the spectrum is related to polynomial decay of gap length and Hölder continuity of the integrated density of states. Therefore, we need to change the assumption to $k > 17\tau + 2$ so that we can further ensure the $\frac{1}{2}$-Hölder continuity of IDS (see [10, Theorems 3.3 and 3.4]). This theorem greatly reduces the initial regularity assumption in the C^k case.

The spectrum's homogeneity plays a crucial role in the inverse spectral theory, as demonstrated in the seminal works of Sodin and Yuditskii [35, 36]. Under the assumption of a finite total gap length and a reflectionless condition on the spectrum, it has been proved that the homogeneity of the spectrum implies the almost periodicity of the associated potentials [35]. Note that the assumption of having a finite total gap length is trivial in the discrete case since the gap length is always bounded by the diameter of the spectrum. In particular, the homogeneity of the spectrum is closely linked to Deift's conjecture, which investigates whether the solutions of the Korteweg–De Vries (KdV) equation exhibit quasi-periodicity when the initial data are quasi-periodic [9, 19]. In the continuous setting, Binder *et al* [9] demonstrated that when considering small analytic quasi-periodic initial data with Diophantine frequency, the solution of the KdV equation exhibits almost periodicity in the temporal variable. In the discrete setting, Leguil *et al* [31] demonstrated that for the subcritical potential $V \in C^\omega(\mathbb{T}, \mathbb{R})$, the Toda flow is almost periodic in the time variable when considering initial data that are also almost periodic with $\beta(\alpha) = 0$. Recently, Avila *et al* [6] also constructed an intriguing counter-example that even for the AMO, its spectrum is not homogeneous if $e^{-2/3\beta(\alpha)} < \lambda < e^{2/3\beta(\alpha)}$. In the C^k case, Cai and Wang [13] have recently proved the homogeneity of the spectrum. They achieved this through a not so refined quantitative C^k reducibility theorem for quasi-periodic $SL(2, \mathbb{R})$ cocycles, as well as by employing the Moser–Pöschel argument for the associated Schrödinger cocycles.

2. *Preliminaries*

2.1. *Conjugation and reducibility.* For a bounded analytic (possibly matrix valued) function $F(\theta)$ defined on $\mathcal{S}_h := \{\theta = (\theta_1, \ldots, \theta_d) \in \mathbb{C}^d \mid |\Im\theta_i| < h \text{ for all } 1 \leqslant i \leqslant d\}$,

https://doi.org/10.1017/etds.2024.88 Published online by Cambridge University Press

let $|F|_h = \sup_{\theta \in \mathcal{S}_h} \|F(\theta)\|$ and denote by $C_h^\omega(\mathbb{T}^d, *)$ the set of all these $*$-valued functions ($*$ will usually denote $\mathbb{R}$, $sl(2, \mathbb{R})$, $SL(2, \mathbb{R})$). Let $C^\omega(\mathbb{T}^d, *) = \bigcup_{h>0} C_h^\omega(\mathbb{T}^d, *)$ and $C^k(\mathbb{T}^d, *)$ be the space of k times differentiable with continuous kth derivatives functions. Define the norm as

$$\|F\|_k = \sup_{|k'| \leqslant k, \theta \in \mathbb{T}^d} \|\partial^{k'} F(\theta)\|.$$

For two cocycles (α, A_1), $(\alpha, A_2) \in \mathbb{T}^d \times C^*(\mathbb{T}^d, SL(2, \mathbb{R}))$, '$*$' represents '$\omega$' or '$k$', we can say that they are C^* conjugated if there exists $Z \in C^*(2\mathbb{T}^d, SL(2, \mathbb{R}))$, such that

$$Z(\theta + \alpha)A_1(\theta)Z^{-1}(\theta) = A_2(\theta).$$

Notably, we want to define Z on the $2\mathbb{T}^d = \mathbb{R}^d/(2\mathbb{Z})^d$ for the purpose of making it still real-valued.

An analytic cocycle $(\alpha, A) \in \mathbb{T}^d \times C_h^\omega(\mathbb{T}^d, SL(2, \mathbb{R}))$ is called almost reducible if there exist a sequence of constant matrices $A_j \in SL(2, \mathbb{R})$, a sequence of conjugations $Z_j \in C_{h_j}^\omega(2\mathbb{T}^d, SL(2, \mathbb{R}))$, and a sequence of small perturbation $f_j \in C_{h_j}^\omega(\mathbb{T}^d, sl(2, \mathbb{R}))$ such that

$$Z_j(\theta + \alpha)A(\theta)Z_j^{-1}(\theta) = A_j e^{f_j(\theta)}$$

with

$$|f_j(\theta)|_{h_j} \to 0, \quad j \to \infty.$$

Furthermore, it is said to be weak (C^ω) almost reducible if $h_j \to 0$ and it is said to be strong ($C^\omega_{h_j,h'}$) almost reducible if $h_j \to h' > 0$. We also claim (α, A) is $C^\omega_{h,h'}$ reducible if there exist a constant matrix $\widetilde{A} \in SL(2, \mathbb{R})$ and a conjugation map $\widetilde{Z} \in C_{h'}^\omega(2\mathbb{T}^d, SL(2, \mathbb{R}))$ such that

$$\widetilde{Z}(\theta + \alpha)A(\theta)\widetilde{Z}^{-1}(\theta) = \widetilde{A}(\theta).$$

To avoid repetition, we have provided an equivalent definition of C^k (almost) reducibility as follows.

One can say that a finitely differentiable cocycle (α, A) is C^{k,k_1} almost reducible if $A \in C^k(\mathbb{T}^d, SL(2, \mathbb{R}))$ and the C^{k_1}-closure of its C^{k_1} conjugacies contains a constant. In addition, we say (α, A) is C^{k,k_1} reducible if $A \in C^k(\mathbb{T}^d, SL(2, \mathbb{R}))$ and its C^{k_1} conjugacies contain a constant.

2.2. *Rotation number and degree.* Suppose $A \in C^0(\mathbb{T}^d, SL(2, \mathbb{R}))$ is homotopic to identity. Then we show the projective skew-product $F_A : \mathbb{T}^d \times \mathbb{S}^1 \to \mathbb{T}^d \times \mathbb{S}^1$ with

$$F_A(x, \omega) := \left(x + \alpha, \frac{A(x) \cdot \omega}{|A(x) \cdot \omega|}\right),$$

which is homotopic to identity as well. Thus, we will lift F_A to a map $\widetilde{F}_A : \mathbb{T}^d \times \mathbb{R} \to \mathbb{T}^d \times \mathbb{R}$ with $\widetilde{F}_A(x, y) = (x + \alpha, y + \psi(x, y))$, where for every $x \in \mathbb{T}^d$, $\psi(x, y)$ is $\mathbb{Z}$-periodic in y. The map $\psi : \mathbb{T}^d \times \mathbb{R} \to \mathbb{R}$ is said to be a lift of A. Assume μ, which is

https://doi.org/10.1017/etds.2024.88 Published online by Cambridge University Press

invariant by $\widetilde{F}_A$, is any probability measure on $\mathbb{T}^d \times \mathbb{R}$. Its projection on the first coordinate is provided by the Lebesgue measure. The number

$$\rho_{(\alpha,A)} := \int_{\mathbb{T}^d \times \mathbb{R}} \psi(x, y)\, d\mu(x, y) \mod \mathbb{Z} \tag{4}$$

has nothing to do with the choices of the lift ψ or the measure μ. One calls it the *fibered rotation number* of cocycle (α, A) (readers can refer to [28] for more details).

Assume

$$R_\phi := \begin{pmatrix} \cos 2\pi\phi & -\sin 2\pi\phi \\ \sin 2\pi\phi & \cos 2\pi\phi \end{pmatrix}$$

if $A \in C^0(\mathbb{T}^d, SL(2, \mathbb{R}))$ is homotopic to $\theta \to R_{\langle n,\theta\rangle}$ for some $n \in \mathbb{Z}^d$, then we call n the *degree* of A and denote it by deg$A$. Furthermore,

$$\deg(AB) = \deg A + \deg B. \tag{5}$$

Please note that the fibered rotation number remains invariant under real conjugacies that are homotopic to the identity map. Generally speaking, when the cocycle (α, A_1) is conjugated to (α, A_2) by $B \in C^0(2\mathbb{T}^d, SL(2, \mathbb{R}))$, that is, $B(\cdot + \alpha)A_1(\cdot)B^{-1}(\cdot) = A_2(\cdot)$, we have

$$\rho_{(\alpha,A_2)} = \rho_{(\alpha,A_1)} - \frac{\langle \deg B, \alpha\rangle}{2}. \tag{6}$$

2.3. *Hyperbolicity and integrated density of states.* We call the cocycle (α, A) *uniformly hyperbolic* if for every $\theta \in \mathbb{T}^d$, there exists a continuous decomposition $\mathbb{C}^2 = E^s(\theta) \oplus E^u(\theta)$ such that for some constants $C > 0$, $c > 0$, and every $n \geqslant 0$,

$$|A_n(\theta)v| \leqslant Ce^{-cn}|v|, \quad v \in E^s(\theta),$$

$$|A_{-n}(\theta)v| \leqslant Ce^{-cn}|v|, \quad v \in E^u(\theta).$$

This decomposition is invariant by the dynamics, which means that for any $\theta \in \mathbb{T}^d$, $A(\theta)E^*(\theta) = E^*(\theta + \alpha)$ for $* = s, u$. In the C^0 topology, the set of uniformly hyperbolic cocycles is an open set. Specifically, in the case of quasi-periodic Schrödinger operators, the cocycle (α, S_E^V) is uniformly hyperbolic if and only if $E \notin \Sigma_{V,\alpha}$, or in other words, if the energy lies within a spectral gap [29].

Let us consider the Schrödinger operators $H_{V,\alpha,\theta}$, where an important concept is the integrated density of states (IDS). The IDS is a function $N_{V,\alpha} : \mathbb{R} \to [0, 1]$ that can be defined by

$$N_{V,\alpha}(E) = \int_{\mathbb{T}^d} \mu_{V,\alpha,\theta}(-\infty, E]\, d\theta,$$

where $\mu_{V,\alpha,\theta} = \mu^{e_{-1}}_{V,\alpha,\theta} + \mu^{e_0}_{V,\alpha,\theta}$ is the universal spectral measure of $H_{V,\alpha,\theta}$ and $\{e_i\}_{i\in\mathbb{Z}}$ is the canonical basis of $\ell^2(\mathbb{Z})$. We say $\{e_{-1}, e_0\}$ is the pair of cyclic vectors of $H_{V,\alpha,\theta}$ here.

There exist alternative approaches to defining the IDS by counting eigenvalues of the truncated Schrödinger operator. For further details, readers may consult [7]. Furthermore,

https://doi.org/10.1017/etds.2024.88 Published online by Cambridge University Press

there is a connection between $\rho(\alpha, S_E^V)$ and the IDS, which can be expressed as follows:

$$N_{V,\alpha}(E) = 1 - 2\rho(\alpha, S_E^V) \text{ mod } \mathbb{Z}. \tag{7}$$

To gain a better understanding of the various types of spectral measures, I recommend referring to the book [16].

2.4. *Analytic approximation.* Let $f \in C^k(\mathbb{T}^d, sl(2,\mathbb{R}))$. By Zehnder [40], there is a sequence $\{f_j\}_{j\geqslant 1}$, $f_j \in C^\omega_{1/j}(\mathbb{T}^d, sl(2,\mathbb{R}))$ and a universal constant C', such that

$$\begin{aligned} \|f_j - f\|_k &\to 0, \quad j \to +\infty, \\ |f_j|_{1/j} &\leqslant C'\|f\|_k, \\ |f_{j+1} - f_j|_{1/(j+1)} &\leqslant C'\bigg(\frac{1}{j}\bigg)^k \|f\|_k. \end{aligned} \tag{8}$$

Furthermore, if $k \leqslant \tilde{k}$ and $f \in C^{\tilde{k}}$, the properties in equation (8) still hold with $\tilde{k}$ instead of k. This implies that the sequence can be constructed from f irrespective of its regularity (since f_j is achieved by convolving f with a map that does not depend on k).

3. *Dynamical estimates: full measure reducibility*

In this section, the main emphasis is on investigating the reducibility property of the following C^k quasi-periodic $SL(2,\mathbb{R})$ cocycle:

$$(\alpha, Ae^{f(\theta)}) : \mathbb{T}^d \times \mathbb{R}^2 \to \mathbb{T}^d \times \mathbb{R}^2; (\theta, v) \mapsto (\theta + \alpha, Ae^{f(\theta)} \cdot v),$$

where $\alpha \in \mathrm{DC}_d(\kappa, \tau)$, $A \in SL(2,\mathbb{R})$, $f \in C^k(\mathbb{T}^d, sl(2,\mathbb{R}))$, and $d \in \mathbb{N}^+$. Our approach involves initially analyzing the approximating analytic cocycles $\{(\alpha, Ae^{f_j(\theta)})\}_{j\geqslant 1}$ and subsequently transferring the obtained estimates to the targeted C^k cocycle $(\alpha, Ae^{f(\theta)})$ through analytic approximation techniques.

3.1. *Preparations.* In the following subsections, we will consider fixed parameters $\rho, \epsilon, N, \sigma$; we refer to the situation in which there exists m_* satisfying $0 < |m_*| \leqslant N$ such that

$$\inf_{j\in\mathbb{Z}} |2\rho - \langle m_*, \alpha\rangle - j| < \epsilon^\sigma,$$

as the 'resonant case' (for simplicity, we will use the notation '$|2\rho - \langle m_*, \alpha\rangle|$' to represent the left side and similarly, '$|\langle m_*, \alpha\rangle|$' to represent the right side). The integer vector m_* will be referred to as a 'resonant site'. This type of small divisor problem commonly arises when attempting to solve the cohomological equation at each step of the KAM procedure. Resonances are connected to a useful decomposition of the space $\mathcal{B}_r := C_r^\omega(\mathbb{T}^d, su(1,1))$. For further details and the precise definition of $su(1,1)$ and $SU(1,1)$, please refer to the proof of Proposition 3.3.

https://doi.org/10.1017/etds.2024.88 Published online by Cambridge University Press

Given $\alpha \in \mathbb{R}^d$, $A \in SU(1,1)$, and $\eta > 0$, there exists a decomposition $\mathcal{B}_r = \mathcal{B}_r^{nre}(\eta) \bigoplus \mathcal{B}_r^{re}(\eta)$ satisfying that for any $Y \in \mathcal{B}_r^{nre}(\eta)$,

$$A^{-1}Y(\theta+\alpha)A \in \mathcal{B}_r^{nre}(\eta), \ |A^{-1}Y(\theta+\alpha)A - Y(\theta)|_r \geqslant \eta |Y(\theta)|_r. \tag{9}$$

Additionally, denote $\mathbb{P}_{nre}$, $\mathbb{P}_{re}$ as the standard projections from $\mathcal{B}_r$ onto $\mathcal{B}_r^{nre}(\eta)$ and $\mathcal{B}_r^{re}(\eta)$, respectively.

Next, we have a crucial lemma that plays a key role in eliminating all the non-resonant terms. This lemma will be used in the proof of the resonant conditions.

LEMMA 3.1. [11, 26] *Assume that $A \in SU(1,1)$, $\epsilon \leqslant (4\|A\|)^{-4}$, and $\eta \geqslant 13\|A\|^2\epsilon^{1/2}$. For any $g \in \mathcal{B}_r$ with $|g|_r \leqslant \epsilon$, there exist $Y \in \mathcal{B}_r$ and $g^{re} \in \mathcal{B}_r^{re}(\eta)$ such that*

$$e^{Y(\theta+\alpha)}(Ae^{g(\theta)})e^{-Y(\theta)} = Ae^{g^{re}(\theta)},$$

with estimates

$$|Y|_r \leqslant \epsilon^{1/2}, \quad |g^{re}|_r \leqslant 2\epsilon.$$

Remark 3.2. For the inequality '$\eta \geqslant 13\|A\|^2\epsilon^{1/2}$', '$\frac{1}{2}$' is sharp because of the quantitative implicit function theorem [8, 21]. The proof relies solely on the fact that $\mathcal{B}_r$ is a Banach space. Therefore, it is applicable not only to the C^ω topology but also to the C^k and C^0 topologies. For more detailed information, please refer to [11, appendix].

3.2. *Analytic KAM theorem.* Following our plan, our first objective is to establish the KAM theorem for the analytic quasi-periodic $SL(2,\mathbb{R})$ cocycle $(\alpha, Ae^{f(\theta)})$, where A possesses eigenvalues $e^{i\rho}, e^{-i\rho}$ with $\rho \in 2\pi\mathbb{R} \cup 2\pi i\mathbb{R}$. We present our quantitative analytic KAM theorem in the following.

PROPOSITION 3.3. [10, 11] *Let $\alpha \in \mathrm{DC}_d(\kappa,\tau)$, $\kappa, r > 0$, $\tau > d$, $\sigma < \frac{1}{6}$. Suppose that $A \in SL(2,\mathbb{R})$ satisfying $\|A\|$ bounded, $f \in C_r^\omega(\mathbb{T}^d, sl(2,\mathbb{R}))$. Then for any $0 < r' < r$, there exist constants $c = c(\kappa,\tau,d)$, $D > 2/\sigma$ and $\tilde{D} = \tilde{D}(\sigma)$ such that if*

$$|f|_r \leqslant \epsilon \leqslant \frac{c}{\|A\|^{\tilde{D}}}(r-r')^{D\tau}, \tag{10}$$

then there exist $B \in C_{r'}^\omega(2\mathbb{T}^d, SL(2,\mathbb{R}))$, $A_+ \in SL(2,\mathbb{R})$ and $f_+ \in C_{r'}^\omega(\mathbb{T}^d, sl(2,\mathbb{R}))$ such that

$$B(\theta+\alpha)(Ae^{f(\theta)})B^{-1}(\theta) = A_+e^{f_+(\theta)}.$$

More precisely, let $N = (2/(r-r'))|\ln\epsilon|$, then we can distinguish two cases:

- *(Non-resonant case) if for any $m \in \mathbb{Z}^d$ with $0 < |m| \leqslant N$, we have*

$$|2\rho - \langle m, \alpha\rangle| \geqslant \epsilon^\sigma,$$

then

$$|B(\theta) - \mathrm{Id}|_{r'} \leqslant \epsilon^{1-8/D}, |f_+|_{r'} \leqslant \epsilon^{2-8/D}$$

https://doi.org/10.1017/etds.2024.88 Published online by Cambridge University Press

and

$$\|A_+ - A\| \leqslant 2\|A\|\epsilon;$$

- *(Resonant case) if there exists* $m_* \in \mathbb{Z}^d$ *with* $0 < |m_*| \leqslant N$ *such that*

$$|2\rho - \langle m_*, \alpha\rangle| < \epsilon^{\sigma},$$

then

$$|B|_{r'} \leqslant 8\bigg(\frac{\|A\|}{\kappa}\bigg)^{1/2}\bigg(\frac{2}{r-r'}|\ln\epsilon|\bigg)^{\tau/2} \times \epsilon^{-r'/(r-r')},$$

$$\|B\|_0 \leqslant 8\bigg(\frac{\|A\|}{\kappa}\bigg)^{1/2}\bigg(\frac{2}{r-r'}|\ln\epsilon|\bigg)^{\tau/2},$$

$$|f_+|_{r'} \leqslant \frac{2^{5+\tau}\|A\||\ln\epsilon|^{\tau}}{\kappa(r-r')^{\tau}}\epsilon e^{-N'(r-r')}(N')^d e^{Nr'} \ll \epsilon^{100},\ \ N' > 2N^2.$$

Moreover, $A_+ = e^{A''}$ *with* $\|A''\| \leqslant 2\epsilon^{\sigma}$, $A'' \in sl(2,\mathbb{R})$. *More accurately, we have*

$$MA''M^{-1} = \begin{pmatrix} it & v \\ \bar{v} & -it \end{pmatrix}$$

with $|t| \leqslant \epsilon^{\sigma}$ *and*

$$|v| \leqslant \frac{2^{4+\tau}\|A\||\ln\epsilon|^{\tau}}{\kappa(r-r')^{\tau}}\epsilon e^{-|m_*|r}.$$

Proof. We will only prove estimates for the non-resonant case because it is more delicate and the proof of the resonant case is the same compared with those in [10].

Let us recall that $sl(2,\mathbb{R})$ is isomorphic to $su(1,1)$, which is a Lie algebra consisting of matrices of the form

$$\begin{pmatrix} it & v \\ \bar{v} & -it \end{pmatrix}$$

with $t \in \mathbb{R}$, $v \in \mathbb{C}$. The isomorphism between them is given by the map $A \to MAM^{-1}$, where

$$M = \frac{1}{1+i}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

and a straightforward calculation yields

$$M\begin{pmatrix} x & y+z \\ y-z & -x \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} iz & x-iy \\ x+iy & -iz \end{pmatrix}$$

where $x, y, z \in \mathbb{R}$. Here, $SU(1,1)$ is the corresponding Lie group of $su(1,1)$. We will prove this theorem within $SU(1,1)$, which is isomorphic to $SL(2,\mathbb{R})$.

We consider the non-resonant case as follows.

For $0 < |m| \leqslant N = (2/(r-r'))|\ln\epsilon|$, we have

$$|2\rho - \langle m, \alpha\rangle| \geqslant \epsilon^{\sigma}, \tag{11}$$

https://doi.org/10.1017/etds.2024.88 Published online by Cambridge University Press

by equation (10) and $D > 2/\sigma$, we get

$$|\langle m, \alpha\rangle| \geqslant \frac{\kappa}{|m|^\tau} \geqslant \frac{\kappa}{|N|^\tau} \geqslant \epsilon^{\sigma/2}. \tag{12}$$

Indeed, it is well known that the conditions in equations (11) and (12) play a role in addressing the small denominator problem in KAM theory.

We now define $g \in C_r^\omega(\mathbb{T}^d, su(1,1))$ such that

$$g(\theta) = \sum_{n\in\mathbb{Z}^d, 0<|n|\leqslant N} \hat{f}(n) e^{2\pi i\langle n,\theta\rangle}.$$

From Schur's theorem, we might as well assume $A = \left(\begin{smallmatrix} e^{i\rho} & p \\ 0 & e^{-i\rho}\end{smallmatrix}\right) \in SU(1,1)$.The condition $\|A\|$ satisfying bounded gives a bound for p and that is the only reason for this condition.

Now we want to solve the cohomological equation

$$Y(\theta+\alpha)A - AY(\theta) = -Ag(\theta). \tag{13}$$

Let

$$Y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}, \quad g = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix},$$

then we can obtain the equations

$$\begin{cases} e^{i\rho} y_3(\theta+\alpha) - e^{-i\rho} y_3(\theta) = -e^{-i\rho} g_3(\theta), \\ e^{i\rho} y_1(\theta+\alpha) - e^{i\rho} y_1(\theta) - p y_3(\theta) = -e^{i\rho} g_1(\theta) - p g_3(\theta), \\ p y_3(\theta+\alpha) + e^{-i\rho} y_4(\theta+\alpha) - e^{-i\rho} y_4(\theta) = -e^{-i\rho} g_4(\theta), \\ p y_1(\theta+\alpha) + e^{-i\rho} y_2(\theta+\alpha) - e^{i\rho} y_2(\theta) - p y_4(\theta) = -e^{i\rho} g_2(\theta) - p g_4(\theta). \end{cases}$$

These equations can be solved by Fourier transform. Compare the corresponding Fourier coefficients of the two sides, and this shows the existence of Y. Apply equation (11) twice to solve the off-diagonal and apply equation (12) once to solve the diagonal, we can get

$$|Y|_{r'} \leqslant c\epsilon^{-3\sigma} |g|_r, \quad 0 < r' < r, \tag{14}$$

where the constant only depends on κ, τ.

By the cohomological equation (13), we obtain

$$Y(\theta+\alpha) = AY(\theta)A^{-1} - Ag(\theta)A^{-1}.$$

Then, we can get

$$\begin{aligned} &e^{Y(\theta+\alpha)}(Ae^{f(\theta)})e^{-Y(\theta)} \\ &\quad = e^{AY(\theta)A^{-1} - Ag(\theta)A^{-1}}(Ae^{f(\theta)})e^{-Y(\theta)} \\ &\quad = Ae^{\hat{f}(0) - (\mathcal{T}_N f)(\theta) + Y(\theta)} e^{f(\theta)} e^{-Y(\theta)} \\ &\quad = A[e^{\hat{f}(0)} + \mathcal{O}(f(\theta) - (\mathcal{T}_N f)(\theta) + f(\theta)Y(\theta))] \\ &\quad = Ae^{\hat{f}(0)}[\mathrm{Id} + e^{-\hat{f}(0)}\mathcal{O}((\mathcal{R}_N f)(\theta) + f(\theta)Y(\theta))] \\ &\quad = A_+ e^{f_+(\theta)}. \end{aligned}$$

https://doi.org/10.1017/etds.2024.88 Published online by Cambridge University Press

Here, $\mathcal{T}_N$ is the truncation operator such that

$$(\mathcal{T}_N f)(\theta) = \sum_{n\in\mathbb{Z}^d, |n|\leqslant N} \hat{f}(n)e^{2\pi i\langle n,\theta\rangle}$$

and

$$(\mathcal{R}_N f)(\theta) = \sum_{n\in\mathbb{Z}^d, |n|> N} \hat{f}(n)e^{2\pi i\langle n,\theta\rangle}.$$

Therefore, we define

$$\begin{cases} B(\theta) = e^{Y(\theta)}, \\ A_+ = Ae^{\hat{f}(0)}, \\ f_+ = \mathcal{O}((\mathcal{R}_N f)(\theta) + f(\theta)Y(\theta)). \end{cases}$$

Note that although we write $D > 2/\sigma$, that is, $\sigma > 2/D$, we consider D to be very close to $2/\sigma$ in the actual process, such as $7/D > 3\sigma > 6/D$. Now we have estimates

$$|g|_r \leqslant \sum_{n\in\mathbb{Z}^d, 0<|n|\leqslant N} |\hat{f}(n)e^{2\pi i\langle n,\theta\rangle}| \leqslant c\epsilon N^{d-1} < \epsilon N^{\tau-1} < \epsilon^{1-1/D},$$
$$|Y|_{r'} \leqslant c\epsilon^{-3\sigma}|g|_r < \epsilon^{-7/D}\cdot\epsilon^{1-1/D} < \epsilon^{1-8/D},$$
$$|B-\mathrm{Id}|_{r'} \leqslant |Y|_{r'} < \epsilon^{1-8/D}.$$

$$\begin{aligned} |(\mathcal{R}_N f)(\theta)|_{r'} &\leqslant \sum_{n\in\mathbb{Z}^d, |n|>N} |\hat{f}(n)e^{2\pi i\langle n,\theta\rangle}|_{r'} \\ &\leqslant c|f|_r e^{-N(r-r')}\bigg(N+\frac{1}{r-r'}\bigg)^d \\ &\leqslant c\epsilon e^{-2\log(1/\epsilon)}(N)^d \\ &< \epsilon\cdot\epsilon^2\cdot\epsilon^{-1/D} \\ &< \epsilon^{3-1/D}. \end{aligned}$$

$$|f_+|_{r'} \leqslant |fY|_{r'} + |(\mathcal{R}_N f)|_{r'} \leqslant c\epsilon\cdot\epsilon^{1-8/D} + \epsilon^{3-1/D} < \epsilon^{2-8/D}.$$
$$\|A_+ - A\| \leqslant \|A\|\|\mathrm{Id} - e^{\hat{f}(0)}\| \leqslant 2\|A\|\epsilon.$$

This finishes the proof of Proposition 3.3. □

Remark 3.4. This version of analytic KAM theorem is a perfect combination of Eliasson [23] and Hou and You [26]. While the resonant case which absorbs the essence of Hou and You stays the same, the refined non-resonant case avoids eliminating irrelevant non-resonant terms via Eliasson's way compared with Cai [10]. This essentially reduces the norm of conjugation maps, which is later vital for us to ensure that the final loss of regularity is independent of the initial k for our C^k reducibility theorem. Note that this is not at all considered by Cai [10] as almost reducibility does not really care about the convergence of the conjugation maps, but reducibility does.

https://doi.org/10.1017/etds.2024.88 Published online by Cambridge University Press

3.3. *C^k reducibility.* As planned, we will present the quantitative C^k reducibility via analytic approximation [40].

The crucial improvement here compared with [10] lies at the point where we are able to obtain reducibility results in a KAM step. In fact, when the resonant steps are well separated from each other, it allows for improved control over the conjugation maps during the inductive argument. Then, after we apply certain conditions to the rotation number of cocycle $(\alpha, Ae^{f(\theta)})$, we can find that there are only finite resonant steps in the whole KAM iteration process. This, in return, will give us the best possible initial regularity k through the technique.

Before we prove the main theorem, we will first cite [10, Proposition 3.1] to simplify the main proof. We first recall some notation given in [10].

Let $\{f_j\}_{j\geqslant 1}$, $f_j \in C^{\omega}_{1/j}(\mathbb{T}^d, sl(2,\mathbb{R}))$ be the analytic sequence approximating $f \in C^k(\mathbb{T}^d, sl(2,\mathbb{R}))$ which satisfies equation (8).

For $0 < r' < r$, denote

$$\epsilon_0'(r, r') = \frac{c}{(2\|A\|)^{\tilde{D}}}(r - r')^{D\tau} \tag{15}$$

and

$$\epsilon_m = \frac{c}{(2\|A\|)^{\tilde{D}} m^{D\tau+1/2}}, \quad m \in \mathbb{Z}^+, \tag{16}$$

where c depends on κ, τ, d, and $D, \tilde{D} \in \mathbb{Z}$ depend on σ.

Then for any $0 < s \leqslant 1/(6D\tau + 3)$ fixed, there exists m_0 such that for any $m \geqslant m_0$, we can get

$$\frac{c}{(2\|A\|)^{\tilde{D}} m^{D\tau+1/2}} \leqslant \epsilon_0'\left(\frac{1}{m}, \frac{1}{m^{1+s}}\right). \tag{17}$$

We will begin with $M > \max\{(2\|A\|)^{\tilde{D}}/c, m_0\}$, $M \in \mathbb{N}^+$. Let $l_j = M^{(1+s)^{j-1}}$, $j \in \mathbb{N}^+$. Because l_j is not an integer, we pick $[l_j] + 1$ instead of l_j.

Now, let $\Omega = \{n_1, n_2, n_3, \ldots\}$ denote the sequence of all resonant steps. In other words, the (n_j)th step is obtained by the resonant case. By the analytic approximation in equation (8) and Proposition 3.3 in each iteration step, we can establish the following almost reducibility result concerning each $(\alpha, Ae^{f_{l_j}(\theta)})$ by applying induction.

PROPOSITION 3.5. [10, 12] *Let $\alpha \in \mathrm{DC}_d(\kappa, \tau)$, $\sigma < \frac{1}{6}$. Assume that $A \in SL(2,\mathbb{R})$ satisfying A bounded, $f \in C^k(\mathbb{T}^d, sl(2,\mathbb{R}))$ with $k > (D+2)\tau + 2$ and $\{f_j\}_{j\geqslant 1}$ be as in §2.4. There exists $\bar{\epsilon} = \bar{\epsilon}(\kappa, \tau, d, k, \|A\|, \sigma)$ such that if*

$$\|f\|_k \leqslant \bar{\epsilon} \leqslant \epsilon_0'\left(\frac{1}{l_1}, \frac{1}{l_2}\right), \tag{18}$$

then there exist $B_{l_j} \in C^{\omega}_{1/l_{j+1}}(2\mathbb{T}^d, SL(2,\mathbb{R}))$, $A_{l_j} \in SL(2,\mathbb{R})$, and $f'_{l_j} \in C^{\omega}_{1/l_{j+1}}(\mathbb{T}^d, sl(2,\mathbb{R}))$ such that

$$B_{l_j}(\theta+\alpha)(Ae^{f_{l_j}(\theta)})B_{l_j}^{-1}(\theta) = A_{l_j}e^{f'_{l_j}(\theta)}, \tag{19}$$

https://doi.org/10.1017/etds.2024.88 Published online by Cambridge University Press

with estimates

$$|B_{l_j}(\theta)|_{1/l_{j+1}} \leqslant 64\bigg(\frac{\|A\|}{\kappa}\bigg)\bigg(\frac{2}{(1/l_j)-(1/l_{j+1})}\ln\frac{1}{\epsilon_{l_j}}\bigg)^{\tau} \times \epsilon_{l_j}{}^{-(2/l_{j+1})/((1/l_j)-(1/l_{j+1}))} \leqslant \epsilon_{l_j}^{-\sigma/2-s}, \tag{20}$$

$$\|B_{l_j}(\theta)\|_0 \leqslant 64\bigg(\frac{\|A\|}{\kappa}\bigg)\bigg(\frac{2}{(1/l_j)-(1/l_{j+1})}\ln\frac{1}{\epsilon_{l_j}}\bigg)^{\tau} \leqslant \epsilon_{l_j}^{-\sigma/2}, \tag{21}$$

$$|\deg B_{l_j}| \leqslant 4l_j \ln\frac{1}{\epsilon_{l_j}}, \tag{22}$$

$$|f'_{l_j}(\theta)|_{1/l_{j+1}} \leqslant \epsilon_{l_j}^{2-8/D}, \quad \|A_{l_j}\| \leqslant 2\|A\|. \tag{23}$$

Moreover, there exists $\overline{f}_{l_j} \in C^{k_0}(\mathbb{T}^d, sl(2,\mathbb{R}))$ *with* $k_0 \in \mathbb{N}$, $k_0 \leqslant (k-10\tau-3)/(1+s)$ *such that*

$$B_{l_j}(\theta+\alpha)(Ae^{f(\theta)})B_{l_j}^{-1}(\theta) = A_{l_j}e^{\overline{f}_{l_j}(\theta)}, \tag{24}$$

with estimate

$$\|\overline{f}_{l_j}(\theta)\|_{k_0} \leqslant \epsilon_{l_j}^{3/D}. \tag{25}$$

Remark 3.6. By the new analytic KAM scheme and the original proof method in [10, 12], we can also get the results of the proposition. The fact that the existing estimates in equations (23) and (25) are more delicate than the original is also due to the use of the new C^{ω} KAM theorem.

Remark 3.7. The estimates of equation (22) can be done better in the non-resonant case because the non-resonant step does not change the degree, thus the estimates of the degree in the non-resonant case can be considered as the estimates of the resonant step closest to this step.

With Proposition 3.5 in hand, we are going to transfer all the estimates from $(\alpha, Ae^{f_{l_j}(\theta)})$ to $(\alpha, Ae^{f(\theta)})$ through analytic approximation. We establish the following quantitative C^k reducibility theorem.

THEOREM 3.8. *Let* $\alpha \in \mathrm{DC}_d(\kappa,\tau)$, $\sigma < \frac{1}{6}$, $A \in SL(2,\mathbb{R})$, *and* $f \in C^k(\mathbb{T}^d, sl(2,\mathbb{R}))$ *with* $k > (D+2)\tau + 2$. *Then there exists* $\epsilon_0 = \epsilon_0(\kappa,\tau,d,k,\|A\|,\sigma)$ *such that if*

$$\|f\|_k \leqslant \epsilon_0 \leqslant \epsilon_0'\bigg(\frac{1}{l_1},\frac{1}{l_2}\bigg) \tag{26}$$

and:

- *if* $\rho(\alpha, Ae^f)$ *is Diophantine with respect to* α: $\rho(\alpha, Ae^f) \in \mathrm{DC}_d^{\alpha}(\gamma,\tau)$, *then there exists two constants* $C_1 = C_1(\gamma,\kappa,\tau,d,k,\|A\|,\sigma)$, $C_2 = C_2(\gamma,\kappa,\tau,d,k,\|A\|,\sigma)$ *and* $B_1 \in C^{k_0}(2\mathbb{T}^d, SL(2,\mathbb{R}))$ *with* $k_0 \in \mathbb{N}$, $k_0 \leqslant (k-10\tau-3)/(1+s)$ *such that*

$$B_1(\theta+\alpha)(Ae^{f(\theta)})B_1^{-1}(\theta) = R_{\phi} \in SL(2,\mathbb{R}), \quad \phi \notin \mathbb{Z}, \tag{27}$$

  with estimates

$$\|B_1\|_{k_0} \leqslant C_1, \quad |\deg B_1| \leqslant C_2; \tag{28}$$

https://doi.org/10.1017/etds.2024.88 Published online by Cambridge University Press

- *if $\rho(\alpha, Ae^f)$ is rational with respect to α: $2\rho(\alpha, Ae^f) = \langle m_0, \alpha\rangle \bmod \mathbb{Z}$ for some $m_0 \in \mathbb{Z}^d$, then there exists $B_2 \in C^{k_0}(2\mathbb{T}^d, SL(2,\mathbb{R}))$ with $k_0 \in \mathbb{N}$, $k_0 \leqslant (k - 10\tau - 3)/(1+s)$ such that*

$$B_2(\theta+\alpha)(Ae^{f(\theta)})B_2^{-1}(\theta) = \tilde{A}_2 \in SL(2,\mathbb{R}), \tag{29}$$

with

$$\rho(\alpha, \tilde{A}_2) = 0. \tag{30}$$

Remark 3.9. In the inequality $k_0 \leqslant (k - 10\tau - 3)/(1+s)$, due to the precise choice of k, τ, s, we can also pick $k_0 = [k - 10\tau - 3]$. Here, '$[x]$' stands for the integer part of x.

Proof. (Diophantine case) By equation (26) and Proposition 3.5, take $\epsilon_0 = \bar{\epsilon}$ as in Proposition 3.5 and apply it to cocycle $(\alpha, Ae^{f(\theta)})$. Then there exists $B_{l_j} \in C^{\omega}_{1/l_{j+1}}(2\mathbb{T}^d, SL(2,\mathbb{R}))$, $A_{l_j} \in SL(2,\mathbb{R})$, and $\overline{f}_{l_j} \in C^{k_0}(\mathbb{T}^d, sl(2,\mathbb{R}))$ with $k_0 \in \mathbb{N}$, $k_0 \leqslant (k - 10\tau - 3)/(1+s)$ such that

$$B_{l_j}(\theta+\alpha)(Ae^{f(\theta)})B_{l_j}^{-1}(\theta) = A_{l_j}e^{\overline{f}_{l_j}(\theta)}, \tag{31}$$

with estimates

$$|B_{l_j}(\theta)|_{1/l_{j+1}} \leqslant \epsilon_{l_j}^{-\sigma/2-s}, \quad \|B_{l_j}(\theta)\|_0 \leqslant \epsilon_{l_j}^{-\sigma/2}, \quad |\deg B_{l_j}| \leqslant 4l_j \ln\frac{1}{\epsilon_{l_j}}, \tag{32}$$

$$\|A_{l_j}\| \leqslant 2\|A\|, \quad \|\overline{f}_{l_j}(\theta)\|_{k_0} \leqslant \epsilon_{l_j}^{3/D}. \tag{33}$$

In the last part of the analytic approximation in equation (8), taking a telescoping sum from j to $+\infty$, we get

$$\|f(\theta) - f_{l_j}(\theta)\|_0 \leqslant \frac{c}{(2\|A\|)^{\tilde{D}} l_1^{D\tau+1/2} l_j^{k-1}}, \tag{34}$$

$$\|f(\theta)\|_0 + \|f_{l_j}(\theta)\|_0 \leqslant \frac{c}{(2\|A\|)^{\tilde{D}} M^{D\tau+1/2}} + \frac{\tilde{c}}{(2\|A\|)^{\tilde{D}} M^{D\tau+1/2}}. \tag{35}$$

Thus, by equations (31)–(35), we have

$$\begin{aligned}\|\overline{f}_{l_j}(\theta)\|_0 &\leqslant \|f'_{l_j}(\theta)\|_0 + \|A_{l_j}^{-1}\|\|B_{l_j}(\theta+\alpha)(Ae^{f(\theta)} - Ae^{f_{l_j}(\theta)})B_{l_j}^{-1}(\theta)\|_0 \\ &\leqslant \epsilon_{l_j}^{2-8/D} + 2\|A\| \times \epsilon_{l_j}^{-\sigma} \times \frac{c}{(2\|A\|)^{\tilde{D}} l_1^{D\tau+1/2} l_j^{k-1}} \\ &\leqslant \epsilon_{l_j}^{1+s}.\end{aligned} \tag{36}$$

Since $\rho(\alpha, Ae^f) \in \mathrm{DC}_d^{\alpha}(\gamma, \tau)$, for any $m \in \mathbb{Z}^d$, we have

$$\begin{aligned}&\|2\rho(\alpha, A_{l_j}e^{\overline{f}_{l_j}(\theta)}) - \langle m, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} \\ &\quad = \|2\rho(\alpha, Ae^{f(\theta)}) - \langle \deg B_{l_j}, \alpha\rangle - \langle m, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}}\end{aligned}$$

https://doi.org/10.1017/etds.2024.88 Published online by Cambridge University Press

$$\geqslant \frac{\gamma}{(|m+\deg B_{l_j}|+1)^\tau}$$
$$\geqslant \frac{\gamma(1+|\deg B_{l_j}|)^{-\tau}}{(|m|+1)^\tau},$$

which implies $\rho(\alpha, A_{l_j}e^{\overline{f}_{l_j}(\theta)}) \in \mathrm{DC}_d^\alpha(\gamma(1+|\deg B_{l_j}|)^{-\tau}, \tau)$.

By equation (32), we can obtain

$$\epsilon_{l_j}^{((1+s)\sigma)/2}(1+|\deg B_{l_j}|)^\tau \leqslant \left[\frac{c}{(2\|A\|)^{\tilde{D}} l_j{}^{D\tau+1/2}}\right]^{((1+s)\sigma)/2}\left(1+4l_j \ln \frac{1}{\epsilon_{l_j}}\right)^\tau.$$

Obviously, let $j \to \infty$ and then the right-hand side of the inequality goes to zero. Therefore, for any given $\gamma > 0$, there exists $j' = j'(\gamma) \in \mathbb{N}$ such that

$$\epsilon_{l_{j'}}^{((1+s)\sigma)/2}(1+|\deg B_{l_{j'}}|)^\tau \leqslant \gamma$$

and then

$$\gamma(1+|\deg B_{l_{j'}}|)^{-\tau} \geqslant \epsilon_{l_{j'}}^{((1+s)\sigma)/2} = \epsilon_{l_{j'+1}}^{\sigma/2}. \tag{37}$$

However, for $0 < |m| \leqslant N_{l_{j'+1}} = (2/(1/l_{j'+1} - 1/l_{j'+2}))|\ln \epsilon_{l_{j'+1}}|$, we have

$$\frac{1}{(|m|+1)^\tau} \geqslant \frac{1}{((2/(1/l_{j'+1}-1/l_{j'+2}))\ln(1/\epsilon_{l_{j'+1}})+1)^\tau} \geqslant 2\epsilon_{l_{j'+1}}^{\sigma/2}. \tag{38}$$

Then by equations (37) and (38), we can get

$$\begin{aligned}
&\|2\rho(\alpha, A_{l_{j'}}) - \langle m, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} \\
&\geqslant \|2\rho(\alpha, A_{l_{j'}}e^{\overline{f}_{l_{j'}}(\theta)}) - \langle m, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} - |2\rho(\alpha, A_{l_{j'}}e^{\overline{f}_{l_{j'}}(\theta)}) - 2\rho(\alpha, A_{l_{j'}})| \\
&\geqslant \frac{\gamma(1+|\deg B_{l_{j'}}|)^{-\tau}}{(|m|+1)^\tau} - 2\epsilon_{l_{j'}}^{(1+s)/2} \\
&\geqslant \epsilon_{l_{j'+1}}^{\sigma/2}\cdot 2\epsilon_{l_{j'+1}}^{\sigma/2} - 2\epsilon_{l_{j'+1}}^{1/2} \\
&\geqslant \epsilon_{l_{j'+1}}^{\sigma},
\end{aligned}$$

which means the $(j'+1)$th step is non-resonant with

$$B_{l_{j'+1}} = \widetilde{B}_{l_{j'}} \circ B_{l_{j'}}, \quad |\widetilde{B}_{l_{j'}}(\theta) - \mathrm{Id}|_{1/l_{j'+2}} \leqslant \epsilon_{l_{j'+1}}^{1-8/D}, \quad \deg B_{l_{j'+1}} = \deg B_{l_{j'}}.$$

Assume that for l_n, $j'+1 \leqslant n \leqslant j_0$, we have

$$B_{l_n}(\theta+\alpha)(Ae^{f_{l_n}(\theta)})B_{l_n}^{-1}(\theta) = A_{l_n}e^{f'_{l_n}(\theta)},$$

which is equivalent to

$$B_{l_n}(\theta+\alpha)(Ae^{f(\theta)})B_{l_n}^{-1}(\theta) = A_{l_n}e^{f'_{l_n}(\theta)} + B_{l_n}(\theta+\alpha)(Ae^{f(\theta)} - Ae^{f_{l_n}(\theta)})B_{l_n}^{-1}(\theta),$$

rewrite that

$$B_{l_n}(\theta+\alpha)(Ae^{f(\theta)})B_{l_n}^{-1}(\theta) = A_{l_n}e^{\overline{f}_{l_n}(\theta)}, \tag{39}$$

with estimates

$$B_{l_n} = \widetilde{B}_{l_{n-1}} \circ B_{l_{n-1}}, \quad |\widetilde{B}_{l_{n-1}}(\theta) - \mathrm{Id}|_{1/l_{n+1}} \leqslant \epsilon_{l_n}^{1-8/D}, \quad \deg B_{l_{n-1}} = \deg B_{l_{j'}}. \tag{40}$$

Therefore, by equation (40) and the Diophantine condition on $\rho(\alpha, Ae^f)$, for $0 < |m| \leqslant (2/(1/l_{n+1} - 1/l_{n+2}))|\ln \epsilon_{l_{n+1}}|$, we obtain

$$\begin{aligned}
&\|2\rho(\alpha, A_{l_n}) - \langle m, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} \\
&\quad \geqslant \|2\rho(\alpha, A_{l_n} e^{\overline{f}_{l_n}(\theta)}) - \langle m, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} - |2\rho(\alpha, A_{l_n} e^{\overline{f}_{l_n}(\theta)}) - 2\rho(\alpha, A_{l_n})| \\
&\quad \geqslant \frac{\gamma(1 + |\deg B_{l_{j'}}|)^{-\tau}}{(|m| + 1)^{\tau}} - 2\epsilon_{l_n}^{(1+s)/2} \\
&\quad \geqslant \epsilon_{l_{j'+1}}^{\sigma/2} \cdot 2\epsilon_{l_{n+1}}^{\sigma/2} - 2\epsilon_{l_{n+1}}^{1/2} \\
&\quad \geqslant \epsilon_{l_{n+1}}^{\sigma} \qquad \text{for all } j' + 1 \leqslant n \leqslant j_0.
\end{aligned}$$

This means the $(j_0 + 1)$th step is still non-resonant with estimates

$$B_{l_{j_0+1}} = \widetilde{B}_{l_{j_0}} \circ B_{l_{j_0}}, \quad |\widetilde{B}_{l_{j_0}}(\theta) - \mathrm{Id}|_{1/l_{j_0+2}} \leqslant \epsilon_{l_{j_0+1}}^{1-8/D}, \quad \deg B_{l_{j_0}} = \deg B_{l_{j'}}.$$

In conclusion, we know that there are at most finitely many resonant steps in the iteration process. Assume that n_q is the last resonant step, then for all $j > n_q$, we have

$$B_{l_j}(\theta + \alpha)(Ae^{f_{l_j}(\theta)})B_{l_j}^{-1}(\theta) = A_{l_j} e^{f'_{l_j}(\theta)},$$

with estimates

$$B_{l_j} = \widetilde{B}_{l_{j-1}} \circ B_{l_{j-1}}, \quad |\widetilde{B}_{l_{j-1}}(\theta) - \mathrm{Id}|_{1/l_{j+1}} \leqslant \epsilon_{l_j}^{1-8/D}, \quad \deg B_{l_{j-1}} = \deg B_{l_{n_q}}. \tag{41}$$

Denote $B_1 = \lim_{j\to\infty} B_{l_j}$, $\tilde{A}_1 = \lim_{j\to\infty} A_{l_j} \in SL(2, \mathbb{R})$. Notice that $\rho(\alpha, \tilde{A}_1) \neq 0$, otherwise it will contradict $\rho(\alpha, Ae^f) \in \mathrm{DC}_d^{\alpha}(\gamma, \tau)$. Thus, $\tilde{A}_1$ can only be standard rotation in $SL(2, \mathbb{R})$, which is the case of equation (27).

By equation (41) and Cauchy estimates, for all $j > n_q$, we can calculate

$$\begin{aligned}
\|\widetilde{B}_{l_j} - \mathrm{Id}\|_{k_0} &\leqslant \sup_{|l|\leqslant k_0, \theta\in\mathbb{T}^d} \|(\partial_{\theta_1}^{l_1} \cdots \partial_{\theta_d}^{l_d})(\widetilde{B}_{l_j} - \mathrm{Id})\| \\
&\leqslant (k_0)!\, (l_{j+2})^{k_0} |\widetilde{B}_{l_j} - \mathrm{Id}|_{1/l_{j+2}} \\
&\leqslant (k_0)!\, (l_{j+1})^{(1+s)k_0} \epsilon_{l_{j+1}}^{1-8/D} \\
&\leqslant \frac{C}{l_{j+1}^{(1-8/D)(D\tau+1/2)-(1+s)k_0}},
\end{aligned}$$

where C does not depend on j.

Note that while we write the assumption as $k > (D + 2)\tau + 2$, in the actual operational process, we simply choose $k = [(D + 2)\tau + 2] + 1$. Therefore, the value of k is entirely determined by the parameter D.

https://doi.org/10.1017/etds.2024.88 Published online by Cambridge University Press

Thus, if we pick $k_0 = [k - 10\tau - 3]$, we have

$$\|\widetilde{B}_{l_j}\|_{k_0} \leqslant 1 + \frac{C}{l_{j+1}^{1/6}}.$$

Since we pick $l_1 = M$ sufficiently large, then

$$\begin{aligned}\|B_1\|_{k_0} &\leqslant \Big\|\prod_{j=n_q+1}^{\infty} \widetilde{B}_{l_j}\Big\|_{k_0}\|B_{l_{n_q}}\|_{k_0} \\ &\leqslant \bigg(\prod_{j=n_q+1}^{\infty}\bigg(1+\frac{C}{l_{j+1}^{1/6}}\bigg)\bigg) \sup_{|l|\leqslant k_0,\theta\in\mathbb{T}^d} \|(\partial_{\theta_1}^{l_1}\cdots\partial_{\theta_d}^{l_d})(B_{l_{n_q}}(\theta))\| \\ &\leqslant 2(k_0)!\,(l_{n_q+1})^{k_0}|B_{l_{n_q}}|_{1/l_{n_q+1}} \\ &\leqslant {l_{n_q}}^{(\sigma/2+s)(D\tau+1/2)+(1+s)k_0}.\end{aligned}$$

The estimate of deg B_1 is clearly valid, which gives equation (28).

(Rational case) By equation (26) and Proposition 3.5, take $\epsilon_0 = \bar{\epsilon}$ as in Proposition 3.5 and apply it to cocycle $(\alpha, Ae^{f(\theta)})$. Then there exists $B_{l_j} \in C^{\omega}_{1/l_{j+1}}(2\mathbb{T}^d, SL(2,\mathbb{R}))$, $A_{l_j} \in SL(2,\mathbb{R})$, and $\overline{f}_{l_j} \in C^{k_0}(\mathbb{T}^d, sl(2,\mathbb{R}))$ with $k_0 \in \mathbb{N}$, $k_0 \leqslant (k - 10\tau - 3)/(1+s)$ such that

$$B_{l_j}(\theta+\alpha)(Ae^{f(\theta)})B_{l_j}^{-1}(\theta) = A_{l_j}e^{\overline{f}_{l_j}(\theta)}, \tag{42}$$

with estimates

$$|B_{l_j}(\theta)|_{1/l_{j+1}} \leqslant \epsilon_{l_j}^{-\sigma/2-s}, \quad \|B_{l_j}(\theta)\|_0 \leqslant \epsilon_{l_j}^{-\sigma/2}, \quad |\deg B_{l_j}| \leqslant 4l_j \ln\frac{1}{\epsilon_{l_j}}, \tag{43}$$

$$\|A_{l_j}\| \leqslant 2\|A\|, \quad \|\overline{f}_{l_j}(\theta)\|_{k_0} \leqslant \epsilon_{l_j}^{3/D}. \tag{44}$$

Since $\rho(\alpha, Ae^{f}) = \langle m_0, \alpha\rangle/2 \bmod \mathbb{Z}/2$, $m_0 \in \mathbb{Z}^d$, we have

$$\begin{aligned}\rho(\alpha, A_{l_j}e^{\overline{f}_{l_j}(\theta)}) &= \rho(\alpha, Ae^{f(\theta)}) - \frac{\langle \deg B_{l_j}, \alpha\rangle}{2} \mod \frac{\mathbb{Z}}{2} \\ &= \frac{\langle m_0 - \deg B_{l_j}, \alpha\rangle}{2} \mod \frac{\mathbb{Z}}{2}.\end{aligned}$$

From now on, we omit 'mod $(\mathbb{Z}/2)$' for simplicity. By equation (6) and the proof of Proposition 3.5, we know the rotation number is invariant in the non-resonant case.

Let $m_{l_{n_i}} \in \mathbb{Z}^d$, $i = 1, 2, 3, \ldots$ represent resonant sites of the (n_i)th step with $0 < |m_{l_{n_i}}| \leqslant N_{l_{n_i}} = 2/(1/l_{n_i}) - (1/l_{n_i+1})|\ln \epsilon_{l_{n_i}}|$. By in [10, Claim 1], we have $N_{l_{n_{i+1}}} \gg 4N_{l_{n_i}}$, $i = 1, 2, 3, \ldots$. So there must exist $j \in \mathbb{Z}$ sufficiently large, provided that there are $q-1$ resonant steps before the jth step, such that

$$m_0 - (m_{l_{n_1}} + m_{l_{n_2}} + \cdots + m_{l_{n_{q-1}}}) = m',$$

where $m' \in \mathbb{Z}^d$ with $0 < |m'| \leqslant N_{l_j} = (2/(1/l_j - 1/l_{j+1}))|\ln \epsilon_{l_j}|$ and $N_{l_j} \gg 2N_{l_{n_{q-1}}} \gg m_{l_{n_1}} + m_{l_{n_2}} + \cdots + m_{l_{n_{q-1}}}$. Then,

$$\begin{aligned}\rho(\alpha, A_{l_j}e^{\overline{f}_{l_j}(\theta)}) &= \frac{\langle m_0 - \deg B_{l_j}, \alpha\rangle}{2}\\ &= \frac{\langle m_0 - (m_{l_{n_1}} + m_{l_{n_2}} + \cdots + m_{l_{n_{q-1}}} + m'), \alpha\rangle}{2}\\ &= 0.\end{aligned} \tag{45}$$

Since all the steps between the (n_{q-1})th step and jth step are non-resonant and by equations (6), (36), and (45), we have

$$\begin{aligned}&\|2\rho(\alpha, A_{l_{j-1}}) - \langle m', \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}}\\ &\quad\leqslant \|2\rho(\alpha, A_{l_{j-1}}e^{\overline{f}_{l_{j-1}}(\theta)}) - \langle m', \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} + |2\rho(\alpha, A_{l_{j-1}}e^{\overline{f}_{l_{j-1}}(\theta)}) - 2\rho(\alpha, A_{l_{j-1}})|\\ &\quad\leqslant |2\rho(\alpha, A_{l_j}e^{\overline{f}_{l_j}(\theta)})| + 2\epsilon_{l_{j-1}}^{(1+s)/2}\\ &\quad= 0 + 2\epsilon_{l_j}^{1/2}\\ &\quad< \epsilon_{l_j}^{\sigma}.\end{aligned}$$

Thus, the jth step is the (n_q)th resonant step and m' is the unique resonant site $m_{l_{n_q}}$ with $0 < |m_{l_{n_q}}| \leqslant N_{l_{n_q}} = (2/(1/l_{n_q} - 1/l_{n_q+1}))|\ln \epsilon_{l_{n_q}}|$.

Now we apply Proposition 3.5 to cocycle $(\alpha, A_{l_{n_q}}e^{\widetilde{f}_{l_{n_q}}(\theta)})$, then we can get

$$\widetilde{B}_{l_{n_q}}(\theta+\alpha)(A_{l_{n_q}}e^{\widetilde{f}_{l_{n_q}}(\theta)})\widetilde{B}_{l_{n_q}}^{-1}(\theta) = A_{l_{n_q+1}}e^{f'_{l_{n_q+1}}(\theta)},$$

which gives

$$B_{l_{n_q+1}}(\theta+\alpha)(Ae^{f_{l_{n_q+1}}(\theta)})B_{l_{n_q+1}}^{-1}(\theta) = A_{l_{n_q+1}}e^{f'_{l_{n_q+1}}(\theta)},$$

where $B_{l_{n_q+1}} = \widetilde{B}_{l_{n_q}} \circ B_{l_{n_q}} \in C^{\omega}_{1/l_{n_q+2}}(2\mathbb{T}^d, SL(2,\mathbb{R}))$.

Since $\rho(\alpha, A_{l_{n_q}}e^{\overline{f}_{l_{n_q}}(\theta)}) = \rho(\alpha, Ae^{\overline{f}_{l_{n_q}}(\theta)}) = 0$, for $m \in \mathbb{Z}^d$ with $0 < |m| \leqslant N_{l_{n_q+1}} = (2/(1/l_{n_q+1} - 1/l_{n_q+2}))|\ln \epsilon_{l_{n_q+1}}|$, we have

$$\begin{aligned}&\|2\rho(\alpha, A_{l_{n_q}}) - \langle m, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}}\\ &\quad\geqslant \|2\rho(\alpha, A_{l_{n_q}}e^{\overline{f}_{l_{n_q}}(\theta)}) - \langle m, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} - |2\rho(\alpha, A_{l_{n_q}}e^{\overline{f}_{l_{n_q}}(\theta)}) - 2\rho(\alpha, A_{l_{n_q}})|\\ &\quad\geqslant \frac{\kappa}{|m|^{\tau}} - 2\epsilon_{l_{n_q}}^{(1+s)/2}\\ &\quad\geqslant \frac{\kappa}{((2/(1/l_{n_q}+1 - 1/l_{n_q}+2))\ln(1/\epsilon_{l_{n_q}+1}))^{\tau}} - 2\epsilon_{l_{n_q+1}}^{1/2}\\ &\quad\geqslant 2\epsilon_{l_{n_q+1}}^{\sigma/2} - 2\epsilon_{l_{n_q+1}}^{1/2}\\ &\quad\geqslant \epsilon_{l_{n_q+1}}^{\sigma},\end{aligned}$$

which means the (n_q+1)th step is non-resonant with

$$\deg B_{l_{n_q+1}} = \deg \widetilde{B}_{l_{n_q}} + \deg B_{l_{n_q}} = m_0.$$

https://doi.org/10.1017/etds.2024.88 Published online by Cambridge University Press

Assume that for l_j, $n_q+1 \leqslant j \leqslant j_0$, we already have

$$B_{l_j}(\theta+\alpha)(Ae^{f_{l_j}(\theta)})B_{l_j}^{-1}(\theta)=A_{l_j}e^{f'_{l_j}(\theta)},$$

which is equivalent to

$$B_{l_j}(\theta+\alpha)(Ae^{f(\theta)})B_{l_j}^{-1}(\theta)=A_{l_j}e^{f'_{l_j}(\theta)}+B_{l_j}(\theta+\alpha)(Ae^{f(\theta)}-Ae^{f_{l_j}})B_{l_j}^{-1}(\theta),$$

rewrite that

$$B_{l_j}(\theta+\alpha)(Ae^{f(\theta)})B_{l_j}^{-1}(\theta)=A_{l_j}e^{\overline{f}_{l_j}(\theta)}, \tag{46}$$

with estimates

$$B_{l_j}=\widetilde{B}_{l_{j-1}}\circ B_{l_{j-1}}, \quad |\widetilde{B}_{l_{j-1}}(\theta)-\mathrm{Id}|_{1/l_{j+1}}\leqslant \epsilon_{l_j}^{1-8/D}, \quad \deg B_{l_{j-1}}=m_0. \tag{47}$$

Note that equation (46) gives

$$\begin{aligned}&\left|\rho(\alpha, Ae^{f(\theta)})-\frac{\langle \deg B_{l_j},\alpha\rangle}{2}-\rho(\alpha, A_{l_j})\right|\\ &\quad=|\rho(\alpha, A_{l_j}e^{\overline{f}_{l_j}(\theta)})-\rho(\alpha, A_{l_j})|\\ &\quad\leqslant \epsilon_{l_{j+1}}^{1/2}\end{aligned}$$

and equation (47) implies

$$\deg B_{l_j}=\deg B_{l_{n_q}}=m_0.$$

Therefore, we have

$$|\rho(\alpha, A_{l_{n_q}}e^{\overline{f}_{l_{n_q}}(\theta)})-\rho(\alpha, A_{l_j})|\leqslant \epsilon_{l_{j+1}}^{1/2}. \tag{48}$$

By equation (48) and $\rho(\alpha, A_{l_{n_q}}e^{\overline{f}_{l_{n_q}}(\theta)})=0$, for $m\in\mathbb{Z}^d$ with $0<|m|\leqslant N_{l_{j+1}}=(2/(1/l_{j+1}-1/l_{j+2}))|\ln \epsilon_{l_{j+1}}|$, we have

$$\begin{aligned}&\|2\rho(\alpha, A_{l_j})-\langle m,\alpha\rangle\|_{\mathbb{R}/\mathbb{Z}}\\ &\quad\geqslant \|2\rho(\alpha, A_{l_{n_q}}e^{\overline{f}_{l_{n_q}}(\theta)})-\langle m,\alpha\rangle\|_{\mathbb{R}/\mathbb{Z}}-|2\rho(\alpha, A_{l_{n_q}}e^{\overline{f}_{l_{n_q}}(\theta)})-2\rho(\alpha, A_{l_j})|\\ &\quad\geqslant \|\langle m,\alpha\rangle\|_{\mathbb{R}/\mathbb{Z}}-2\epsilon_{l_j}^{(1+s)/2}\\ &\quad\geqslant \frac{\kappa}{|m|^{\tau}}-2\epsilon_{l_{j+1}}^{1/2}\\ &\quad\geqslant \frac{\kappa}{((2/(1/l_{j+1}-1/l_{j+2}))\ln(1/\epsilon_{l_{j+1}}))^{\tau}}-2\epsilon_{l_{j+1}}^{1/2}\\ &\quad\geqslant 2\epsilon_{l_{j+1}}^{\sigma/2}-2\epsilon_{l_{j+1}}^{1/2}\\ &\quad> \epsilon_{l_{j+1}}^{\sigma} \quad \text{for all } n_q+1\leqslant j\leqslant j_0.\end{aligned}$$

This means the (j_0+1)th step is also non-resonant with estimates

$$B_{l_{j_0+1}}=\widetilde{B}_{l_{j_0}}\circ B_{l_{j_0}}, \quad |\widetilde{B}_{l_{j_0}}(\theta)-\mathrm{Id}|_{1/l_{j_0+2}}\leqslant \epsilon_{l_{j_0+1}}^{1-8/D}, \quad \deg B_{l_{j_0}}=m_0.$$

https://doi.org/10.1017/etds.2024.88 Published online by Cambridge University Press

To conclude, for all $j > n_q$, we have

$$B_{l_j}(\theta+\alpha)(Ae^{f_{l_j}(\theta)})B_{l_j}^{-1}(\theta) = A_{l_j}e^{f'_{l_j}(\theta)},$$

with estimates

$$B_{l_j} = \widetilde{B}_{l_{j-1}} \circ B_{l_{j-1}}, \quad |\widetilde{B}_{l_{j-1}}(\theta) - \mathrm{Id}|_{1/l_{j+1}} \leqslant \epsilon_{l_j}^{1-8/D}, \quad \deg B_{l_{j-1}} = m_0. \tag{49}$$

Denote $B_2 = \lim_{j\to\infty} B_{l_j}$, $\tilde{A}_2 = \lim_{j\to\infty} A_{l_j} \in SL(2,\mathbb{R})$. The rest of the process is similar to the above Diophantine case and then equations (29) and (30) hold.

This finishes our proof. □

Remark 3.10. Compared with Cai and Ge [**12**], the reducibility results in this paper are certainly optimal. We have conducted a process parallel to the almost reducibility procedure. In contrast to the previous work, the conclusions in this paper are sharper in terms of regularity loss and the loss of regularity is independent of the parameter k.

4. *Spectral application*

With the above reducibility theorem in hand, we can prove our spectral applications of Theorems 1.3 and 1.6 as stated in the introduction from this. Let us first introduce several definitions and cite some results shown by [**13**, **25**].

For spatial transport properties of a quantum particle on the lattice $\mathbb{Z}$, we are interested in studying the observable quantity associated with the position of the particle, which is represented by the unbounded self-adjoint operator

$$(Xx)_n := (nx)_n, \quad n \in \mathbb{Z},$$

with its natural domain of definition

$$\mathrm{Dom}X = \bigg\{x_n \in \ell^2(\mathbb{Z}) : \sum_{n\in\mathbb{Z}} |n|^2|x_n|^2 < +\infty\bigg\}.$$

Our focus is on the phenomenon of ballistic motion, which informally states that the particle's position grows linearly with time ($X(T) \approx T$). More precisely, we aim to investigate the following limit:

$$\lim_{T\to+\infty} \frac{1}{T} X(T)x, \tag{50}$$

where the initial state at time $x \in \mathrm{Dom}X$. The limit in equation (50) can be regarded as the 'asymptotic velocity' of the state x as time approaches infinity, provided that the limit exists. Then we can define the asymptotic velocity operator as

$$Q = \lim_{T\to+\infty} \frac{1}{T} X(T) = \lim_{T\to+\infty} \frac{1}{T} \int_0^T e^{itH} S e^{-itH}\, dt, \tag{51}$$

where S is bounded and satisfies

$$Sx_n = i(x_{n+1} - x_{n-1}). \tag{52}$$

https://doi.org/10.1017/etds.2024.88 Published online by Cambridge University Press

The Schrödinger operator H is said to demonstrate strong ballistic transport if the strong limit on the right-hand side of equation (51) exists, is defined on the entire $\ell^2(\mathbb{Z})$, and $\ker Q = \{0\}$.

Definition 4.1. [25] Let $\mathcal{K} \subset \mathbb{R}$ be a Borel subset. We say H has strong ballistic transport on $\mathcal{K}$ if there exists a self-adjoint operator Q such that

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T e^{itH} 1_{\mathcal{K}}(H) S 1_{\mathcal{K}}(H) e^{-itH}\, dt = 1_{\mathcal{K}} Q 1_{\mathcal{K}} \tag{53}$$

and $\ker Q = \mathrm{Ran}(1_{\mathcal{K}})^{\perp}$, where $1_{\mathcal{K}}(\cdot)$ denotes the indicator function of $\mathcal{K}$.

Definition 4.2. [25] Cocycle (α, S_E^V) is said to be C^k-reducible in expectation on $\mathcal{K}$ if it is C^k-reducible for each $E \in \mathcal{K}$ and there exists a choice of $L^2(\mathbb{T}^d)$-normalized conjugations $B(E; \cdot)$ such that

$$\int_{\mathcal{K}} \|B(E; \cdot)\|^4_{C^k(\mathbb{T}^d)}\, d\rho(E) < +\infty.$$

LEMMA 4.3. [25] *Let a (Borel) subset $\mathcal{K} \subset \mathbb{R}$, $\{H_{V,\alpha,\theta}\}_{\theta\in\mathbb{T}^d}$ be a quasiperiodic operator family whose cocycles are C^k-reducible in expectation on $\mathcal{K}$ for some $k > 5d/2$. Then the family $\{H_{V,\alpha,\theta}\}_{\theta\in\mathbb{T}^d}$ has strong ballistic transport on $\mathcal{K}$.*

As a direct application of our conclusion, now we can prove Theorem 1.3 in the following.

Proof. We will not consider E in the spectral gap, because cocycle (α, S_E^V) is uniformly hyperbolic in the quasi-periodic Schrödinger case. Therefore, it is always reducible under our conditions and the conclusion is naturally valid.

For $E \in \Sigma_{V,\alpha}$, Theorem 3.8 shows that cocyle (α, S_E^V) is reducible. As $k > 14\tau + 2$ and picking $k_0 = [k - 10\tau - 3]$, we can get $k_0 > [4\tau - 1]$ and $\tau > d$, then condition $k_0 > 5d/2$ is obviously true. Here E, which satisfies neither the Diophantine nor rational condition in the spectrum set, is a set of measure zero. Note that, in Ge and Kachkovskiy [25], under the setting of Zhao [41], we have strong ballistic transport on the whole spectrum. In the case of C^k in this paper, the conclusion still holds. Therefore, by Lemma 4.3, the Schrödinger operator $H_{V,\alpha,\theta}$ has strong ballistic transport for a.e. $E \in \Sigma_{V,\alpha}$.

Readers can also refer to the detailed proof of Ge and Kachkovskiy [25].

This finishes the proof of Theorem 1.3. □

Now, for the application of the reducibility theorem to spectral structures in Theorem 1.6, we give the proof as follows.

Proof. Referring to Cai and Wang [13], we know the homogeneity of the spectrum is connected with polynomial decay of gap length and Hölder continuity of IDS. Then we need the following lemmas.

LEMMA 4.4. [13] *Let $G_m(V) = (E_m^-, E_m^+)$ denote the gap with label m and $\alpha \in \mathrm{DC}_d(\kappa, \tau)$, $V \in C^k(\mathbb{T}^d, \mathbb{R})$ with $k \geqslant D_0\tau$, where D_0 is a numerical constant. There exists $\tilde{\epsilon} = \tilde{\epsilon}(\kappa, \tau, k, d) > 0$ such that if $\|V\|_k \leqslant \tilde{\epsilon}$, then $|G_m(V)|_k \leqslant \tilde{\epsilon}^{1/4}|m|^{-k/9}$.*

LEMMA 4.5. [10] *Let $\alpha \in \mathrm{DC}_d(\kappa, \tau)$, $V \in C^k(\mathbb{T}^d, \mathbb{R})$ with $k > 17\tau + 2$. Then there exists λ_0 depending on V, d, κ, τ, k such that if $\lambda < \lambda_0$, then $N_{\lambda V,\alpha}$ is $\frac{1}{2}$-Hölder continuous:*

$$N(E + \hat{\epsilon}) - N(E - \hat{\epsilon}) \leqslant C_0\hat{\epsilon}^{1/2} \quad \textit{for all } \hat{\epsilon} > 0, \ \textit{for all } E \in \mathbb{R},$$

where C_0 depends only on d, κ, τ, k.

Through the quantitative almost reducibility theory and reducibility theory, we can show both of the above lemmas. Under the preconditions of Theorem 3.8 together with Lemmas 4.4 and 4.5, for any $E \in \Sigma_{V,\alpha}$ with $2\rho(\alpha, S_E^V) = \langle m_0, \alpha\rangle \bmod \mathbb{Z}$ for some $m_0 \in \mathbb{Z}^d$, we will divide E into three cases when considering ν-homogeneous. In each case, by calculating directly, Definition 1.5 is satisfied for every $E \in \Sigma_{V,\alpha}$. Please refer to the detailed proof of Cai and Wang [13].

This finishes the proof of Theorem 1.6. □

REFERENCES

[1] J. Asch and A. Knauf. Motion in periodic potentials. *Nonlinearity* **11** (1997), 175–200.

[2] A. Avila, J. Bochi and D. Damanik. Cantor spectrum for Schrödinger operators with potentials arising from generalized skew-shifts. *Duke Math. J.* **146** (2009), 253–280.

[3] A. Avila, B. Fayad and R. Krikorian. A KAM scheme for $SL(2, \mathbb{R})$ cocycles with Liouvillean frequencies. *Geom. Funct. Anal.* **21** (2011), 1001–1019.

[4] A. Avila and S. Jitomirskaya. Almost localization and almost reducibility. *J. Eur. Math. Soc. (JEMS)* **12** (2010), 93–131.

[5] A. Avila and R. Krikorian. Reducibility or nonuniform hyperbolicity for quasiperiodic Schrödinger cocycles. *Ann. of Math. (2)* **164** (2006), 911–940.

[6] A. Avila, Y. Last, M. Shamis and Q. Zhou. On the abominable properties of the almost Mathieu operator with well-approximated frequencies. *Duke Math. J.* **173**(4) (2024), 603–672.

[7] J. Avron and B. Simon. Almost periodic Schrödinger operators II. The integrated density of states. *Duke Math. J.* **50** (1983), 369–291.

[8] M. Berti and L. Biasco. Forced vibrations of wave equations with non-monotone nonlinearities. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **23** (2006), 439–474.

[9] I. Binder, D. Damanik, M. Goldstein and M. Lukic. Almost periodicity in time of solutions of the KdV equation. *Duke Math. J.* **167** (2018), 2633–2678.

[10] A. Cai. The absolutely continuous spectrum of finitely differentiable quasi-periodic Schrödinger operators. *Ann. Henri Poincaré* **23** (2022), 4195–4226.

[11] A. Cai, C. Chavaudret, J. You and Q. Zhou. Sharp Hölder continuity of the Lyapunov exponent of finitely differentiable quasi-periodic cocycles. *Math. Z.* **291** (2019), 931–958.

[12] A. Cai and L. Ge. Reducibility of finitely differentiable quasi-periodic cocycles and its spectral applications. *J. Dynam. Differential Equations* **34** (2022), 2079–2104.

[13] A. Cai and X. Wang. Polynomial decay of the gap length for C^{K} quasi-periodic Schrödinger operators and spectral application. *J. Funct. Anal.* **281** (2021), 109035.

[14] C. Chavaudret. Strong almost reducibility for analytic and Gevrey quasi-periodic cocycles. *Soc. Math. France* **141** (2013), 47–106.

[15] J.-M. Combes. Connections between quantum dynamics and spectral properties of time-evolution operators. *Differ. Equ. Appl. Math. Phys.* **192** (1993), 59–68.

[16] D. Damanik. Schrödinger operators with dynamically defined potentials. *Ergod. Th. & Dynam. Sys.* **37** (2017), 1681–1764.

[17] D. Damanik and J. Fillman. *One-Dimensional Ergodic Schrödinger Operators: I. General Theory (Graduate Studies in Mathematics, 221)*. American Mathematical Society, Providence, RI, 2022.

[18] D. Damanik, M. Goldstein and M. Lukic. The spectrum of a Schrödinger operator with small quasi-periodic potential is homogeneous. *J. Spectr. Theory* **6** (2016), 415–427.
[19] D. Damanik, M. Goldstein and M. Lukic. The isospectral torus of quasi-periodic Schrödinger operators via periodic approximations. *Invent. Math.* **207** (2017), 895–980.
[20] D. Damanik, M. Lukic and W. Yessen. Quantum dynamics of periodic and limit-periodic Jacobi and block Jacobi matrices with applications to some quantum many body problems. *Comm. Math. Phys.* **337** (2015), 1535–1561.
[21] K. Deimling. Nonlinear functional analysis. *Bull. Amer. Math. Soc. (N.S.)* **20** (1989), 277–280.
[22] E. I. Dinaburg and Y. G. Sinai. The one-dimensional Schrödinger equation with a quasiperiodic potential. *Funct. Anal. Appl.* **9** (1975), 279–289.
[23] L. H. Eliasson. Floquet solutions for the 1-dimensional quasi-periodic Schrödinger equation. *Comm. Math. Phys.* **146** (1992), 447–482.
[24] J. Fillman. Ballistic transport for limit-periodic Jacobi matrices with applications to quantum many-body problems. *Comm. Math. Phys.* **350** (2017), 1275–1297.
[25] L. Ge and I. Kachkovskiy. Ballistic transport for one-dimensional quasiperiodic Schrödinger operators. *Comm. Pure Appl. Math.* **76** (2023), 2577–2612.
[26] X. Hou and J. You. Almost reducibility and non-perturbative reducibility of quasi-periodic linear systems. *Invent. Math.* **190** (2012), 209–260.
[27] W. Jian and Y. Shi. Sharp Hölder continuity of the integrated density of states for extended harper's model with a Liouville frequency. *Acta Math. Sci.* **39** (2019), 1240–1254.
[28] R. Johnson and J. Moser. The rotation number for almost periodic potentials. *Comm. Math. Phys.* **84** (1982), 403–438.
[29] R. A. Johnson. Exponential dichotomy, rotation number and linear differential operators with bounded coefficients. *J. Differential Equations* **61** (1986), 54–78.
[30] Y. Last. Quantum dynamics and decompositions of singular continuous spectra. *J. Funct. Anal.* **142** (1996), 406–445.
[31] M. Leguil, J. You, Z. Zhao and Q. Zhou. Asymptotics of spectral gaps of quasi-periodic Schrödinger operators. *Preprint*, 2017, arXiv:1712.04700.
[32] J. Moser and J. Pöschel. An extension of a result by Dinaburg and Sinai on quasi-periodic potentials. *Comment. Math. Helv.* **59** (1984), 39–85.
[33] J. Puig. A nonperturbative Eliasson's reducibility theorem. *Nonlinearity* **19** (2005), 355.
[34] B. Simon. Absence of ballistic motion. *Comm. Math. Phys.* **134** (1990), 209–212.
[35] M. Sodin and P. Yuditskii. Almost periodic Sturm–Liouville operators with Cantor homogeneous spectrum. *Comment. Math. Helv.* **70** (1995), 639–658.
[36] M. Sodin and P. Yuditskii. Almost periodic Jacobi matrices with homogeneous spectrum, infinite dimensional Jacobi inversion, and hardy spaces of character-automorphic functions. *J. Geom. Anal.* **7** (1997), 387–435.
[37] Y. Wang and Z. Zhang. Cantor spectrum for a class of quasi-periodic Schrödinger operators. *Int. Math. Res. Not. IMRN* **2017** (2017), 2300–2336.
[38] J. You. Quantitative almost reducibility and its applications. *Proc. Int. Congr. Math.* **4** (2018), 2113–2135.
[39] G. Young. Ballistic transport for limit-periodic Schrödinger operators in one dimension. *J. Spectr. Theory* **13** (2021), 451–489.
[40] E. Zehnder. Generalized implicit function theorems with applications to some small divisor problems. *Comm. Pure Appl. Math.* **28** (1975), 91–140.
[41] Z. Zhao. Ballistic motion in one-dimensional quasi-periodic discrete Schrödinger equation. *Comm. Math. Phys.* **347** (2016), 511–549.
[42] Z. Zhao and Z. Zhang. Ballistic transport and absolute continuity of one-frequency Schrodinger operators. *Comm. Math. Phys.* **351** (2017), 877–921.