Walk on a grid

MANIJA SHAHALI and H. A. SHAHALI

There are various combinatorial questions on rectangular arrays consisting of points, numbers, fields or, in general, of symbols such as chessboards, lattices, and graphs. Many such problems in enumerative combinatorics come from other branches of science and technology like physics, chemistry, computer sciences and engineering; for example the following two very challenging problems from chemistry:

Problem 1: Dimer problem (Domino tiling)

In chemistry, a large molecule composed repeatedly from monomers as a long chain is called a polymer and a dimer is composed of two monomers (where: mono = 1, di = 2, poly = many and mer = part).

The problem of covering a region (e.g. a rectangular region, which is the simplest 2-dimensional region) by dimers can be interpreted geometrically as follows:

In how many ways can a rectangular region be covered by dominoes? By ‘domino’ we mean a 1×2 rectangle composed of 2 unit squares, and by ‘covering’ we mean a cover with no gap and no overlap. The existence of at least such a cover is equivalent to the existence of a rectangle of size $m \times n$ with positive integers m and n where mn is even.

In 1961, this problem was independently solved by H. N. V. Temperley and M. E. Fisher [1], and by P. W. Kasteleyn [2], as the following counting function:

$$\prod_{i=1}^{\lceil \frac{m}{2} \rceil} \prod_{j=1}^{\lceil \frac{n}{2} \rceil} \left(4\cos^2 \frac{i\pi}{m+1} + 4\cos^2 \frac{j\pi}{n+1} \right),$$

where $\lceil x \rceil$ is the smallest integer not less than x (the ceiling function of x).

This formula shows, for instance, that a chessboard ($m = n = 8$) can be covered by 32 dominoes in exactly 12,988,816 ways.

Problem 2: Polymer problem (self-avoiding walk)

To develop polymers, chemists design and study long chains of monomers and relations of pairwise them. In 1953, the chemist P. Flory [3] introduced problems concerning non-intersecting chains of polymers suggesting problems in combinatorics as follows:

A *self-avoiding walk* is a path on a lattice that does not intersect itself, and a *self-avoiding polygon* is a closed self-avoiding walk on a lattice.

Also, a self-avoiding walk passing through all the nodes of a lattice region, e.g. through all the nodes of a rectangular lattice, is called a *spanning self-avoiding walk*. Now, for a lattice of even size, any solution (a

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spanning one) to Problem 2 is a solution to Problem 1, but its converse is not true in general.

Very little is known about self-avoiding walks; in particular, the number of n-step self-avoiding walks for a general positive integer n is unknown. For more topics on this subject see [4].

In spite of the very tough problems, Problem 1 and Problem 2, in this Article we discuss some very much simpler matters in relation to self-avoiding walks and self-avoiding polygons using equivalent notations and terminology.

Let one person be walking on an $m \times n$ grid of fields, where m and n are positive integers. Without loss of generality, we consider a grid of squares, in fact, the shape of the fields does not play any role, what matters is the number of fields and relations between them. The person moves from one field to an adjacent field along rows or columns, but never visits a field twice, except when the starting and ending fields are the same. If the person walks to all the fields, it is said that a spanning path (complete tour) has been done; if the person comes ends up on the starting field, it is said that a spanning circuit (complete loop) has been done. Such tours are clearly possible in various ways; e.g. see Figures 1 and 2. Throughout this Article, each path will be shown in a thicker line, and since all walks are restricted to these rules, any corresponding path will be a (closed or open) simple broken line that does not intersect itself. The length of a path is defined as the total number of fields spanned. In an $m \times n$ grid of fields of any size (even or odd total number of fields), there is always at least one spanning path (teeth-like or a flat zigzag pattern) as shown in Figure 1; and also there is always at least one spanning circuit (comb-like) for any even mn as shown in Figure 2.

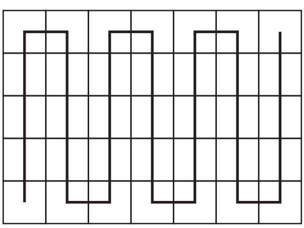

FIGURE 1

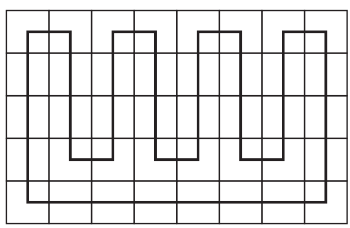

FIGURE 2

One may wonder: if there are obstacles in the grid, i.e. blocked fields, would a spanning path (or circuit) be possible to all the remaining fields?

From there one may ask the following questions:

Question 1: What is the minimum number of fields in an $m \times n$ grid with odd sides that must be excluded to prevent the possibility of a spanning path through the remaining felds?

Question 2: What is the minimum number of fields in an $m \times n$ grid, with m, n odd, that must be excluded to make a spanning circuit possible through the remaining fields?

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The purpose of this Article is to consider and to answer these questions in the theorem below.

We observe (e.g. see the example in Figure 3) that in any circuit (where we call the initial-terminal field the origin) any outbound step from the origin is matched in the circuit by a corresponding inbound step heading back to the origin (and conversely), therefore there must be in total an even number of fields in a circuit. We will show this with a more rigorous argument in the following proposition:

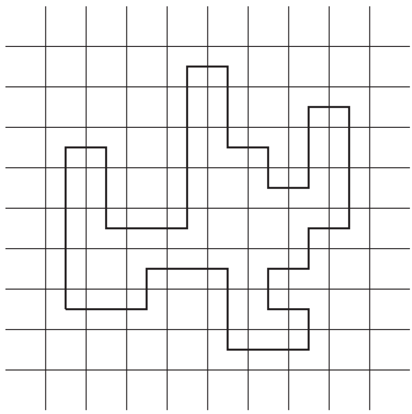

FIGURE 3

Proposition: The length of a circuit on any grid is always even.

Proof: Consider the horizontal walks, and ignore vertical walks temporarily (project the path on a horizontal axis), and let the origin point be the origin of the axis, and assign to any walk (partial path) in the positive direction a positive integer equal to the number of steps involved, and assign to any walk (partial path) in the negative direction a negative integer equal to the number of steps involved. We see that in any circuit we will have a total of zero (the algebraic sum of the outbound and inbound walks vanishes, or we can also say that the sum (resultant) of vectors in a zero translation is zero); in other words

$$w_1 + w_2 + \dots + w_k = 0$$

where any w is the algebraic value of each partial walk. Hence

$$|w_1| + |w_2| + \dots + |w_k|$$

is an even positive integer.

A similar argument for vertical walks holds, therefore the total length of any walk circuit is an even positive integer.

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Corollary: A path with odd length cannot be a circuit. In particular, any grid of fields with odd size has no spanning circuit.

Theorem: Let $m > 1$ and $n > 1$ be any integers.

(i) If mn is odd, then the exclusion (or blockage) of even a single correctly-selected field can prevent a spanning path. The number of such preventative fields is exactly $\frac{1}{2}(mn - 1)$. Also, the exclusion of even one proper field can cause a spanning circuit. The number of such causing fields is exactly $\frac{1}{2}mn + 1$.

(ii) If mn is even, then we must exclude at least two proper fields to prevent a spanning path, and the number 2 is best. The number of ways to choose two such avoiding fields is at least $\frac{mn}{2}\left(\frac{mn - 2}{2}\right)$.

Proof:

(i) When mn is an odd integer:

In this case, we proceed with the proof by contradiction, and use the 'checkerboard-trick' (colouring the alternate fields in black and white in chessboard fashion. Starting with a white field at the upper left corner) we will have $\frac{1}{2}(mn - 1)$ black fields and $\frac{1}{2}(mn + 1)$ white fields, so that their difference $\Delta(B, W) = \Delta = 1$. Figure 4 shows an example for $(m, n) = (5,\ 7)$.

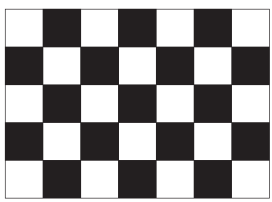

FIGURE 4

Excluding just one arbitrary black field yields $\Delta = 2$. Also, the person always walks through a white field to an adjacent black field (or conversely), and this means that in any walking path, the difference

$$\Delta = \begin{cases} 0, & \text{only when the length of the path is even,} \\ 1, & \text{only when the length of the path is odd.} \end{cases}$$

Therefore, if there were a complete tour, then $\Delta = 0$; a contradiction! Clearly the number of such avoiding fields is exactly $\frac{1}{2}(mn - 1)$, which is the total number of the black fields.

The exclusion of any white field makes a spanning circuit (complete loop) on the total $m \times n$ grid, except the excluded field, means that the person could have a complete tour (actually, a complete loop) from any starting point. We can show this by induction on odd $m \times n$.

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We begin with the first step of the induction, using the minimal values $m = n = 3$; see Figure 5a. Excluding the central field gives the circuit shown in Figure 5b and excluding any one of the four corner fields can provide one spanning circuit, e.g. excluding the upper left field gives the spanning circuit shown in Figure 5c.

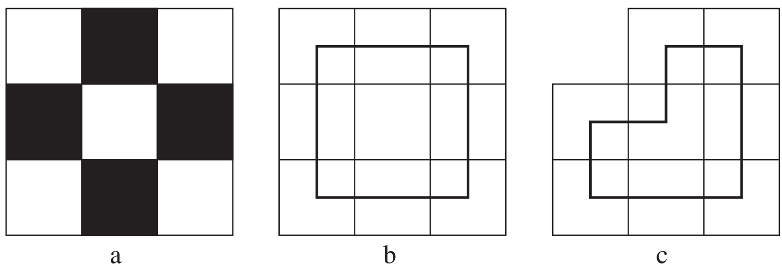

FIGURE 5

Thus the first step of induction holds.

Now let at least one of the odd integers m and n be greater than 3. Without loss of generality, say $m > 3$ (meaning $m \geqslant 5$), so that the excluded field is not in the lower $2 \times n$ grid shown in grey in Figure 6.

Then, by the induction assumption, there is a spanning circuit C_1 in the upper $(m - 2) \times n$ grid (one excluded field), and clearly a spanning circuit C_2 in the lower $2 \times n$ grid.

Obviously (at least) two adjacent fields in the lowest row of the $(m - 2) \times n$ grid (over the lower $2 \times n$ grid) are a part of C_1.

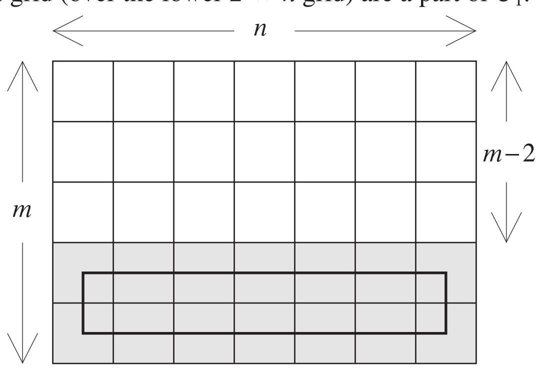

FIGURE 6

Now, disconnecting the C_1 between the two adjacent fields, and disconnecting the C_2 between the two adjacent fields just under them, and connecting C_1 and C_2 in the corresponding fields as shown in Figure 7, we achieve a spanning circuit C on the $m \times n$ grid (excluded in a field), where any field is represented by a node.

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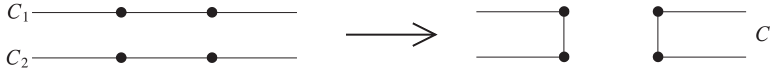

FIGURE 7

Clearly the number of such causing fields is exactly $\frac{1}{2}(mn + 1)$, which is the total number of the white fields.

This completes the proof by induction.

(ii) When mn is an even integer:

In this case, at least one of m and n is even; without loss of generality, we assume that n is even. There does always exist at least one spanning circuit, for example one spanning comb-like circuit as shown in Figure 2 above. This means that excluding no field can avoid the existence of a complete tour on the notched grid. Therefore at least two excluded fields are needed to avoid a complete tour on the notched grid (consisting of $mn - 2$ fields). In this case, exclusion of two fields of the same colour yields the non-existence of a complete tour, since we will have $\Delta = 2$ (according to the above argument there is no path where $\Delta = 2$ holds). Therefore to avoid a complete tour, exclusion of at least two similarly-coloured fields is necessary.

The number of black (or white) fields is $\frac{1}{2}mn$, and consequently the total number of choices when avoiding two fields is at least

$$2\binom{\frac{1}{2}mn}{2} = \frac{mn}{2}\left(\frac{mn}{2} - 1\right),$$

and this completes the proof of the Theorem.

Comments

(1) The case mn with $m = 1$ or $n = 1$ was not considered above, since it is trivial, because the cases for $m, n \in \{1, 2\}$ are completely trivial, and if $m \geqslant 3$ and $n = 1$ (or $m = 1$ and $n \geqslant 3$), there exists no spanning circuit, and there exists a spanning path only when the start (also the end) point is chosen from the two ending fields. The exclusion of any mid-fields causes the non-existence of a spanning path, and the number of such avoiding fields is $m - 2$ (or $n - 2$).

(2) The usefulness of a spanning circuit with respect to an open spanning path is that the person can start walking from each field, but the start field in any open spanning path must be one of the two ending fields.

Exclusion of two fields with different colours for the case (ii) in the Theorem does not have a definite outcome, sometimes it causes the possibility of a complete tour and sometimes not; hence we conclude this Article with the following problem for the reader:

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Problem

If mn is an even positive integer, which excluded two fields give rise to the non-existence of a complete tour?

And for which fields is this not true?

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MANIJA SHAHALI
16869 SW 65th Avenue, #318
Lake Oswego, OR 97035 USA
e-mail: *manija@addbits.com*
H. A. SHAHALI
15332 Antioch St.
Pacific Palisades,
CA 90272 USA
e-mail: *h_a_shahali@hotmail.com*